UPKAR'S

GATE Mathematics

Graduate Aptitude Test in Engineering

Dr. N. K. Singh

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By

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PREFACE

The Graduate Aptitude Test in Engineering (GATE) is an All India Examination conducted by the Six Indian Institutes of Technology and Indian Institute of Science, Bangalore, on behalf of the National Coordinating Board-GATE, Department of Education, Ministry of Human Resources Development (MHRD), Government of India. Admission to Postgraduate Courses, with MHRD Scholarship/ Assistantship, in Engineering/Technology/Architecture/Pharmacy at Colleges/Institutes will be open only to those who qualify in GATE.

The following categories of candidates are eligible to appear for GATE.

1. Bachelor's degree holders in Engineering/Technology/Architecture/Pharmacy and those who are in the final or Pre-final year of such Programmes.
2. Master's degree holders in any branch of Science/Mathematics/Statistics or equivalent and those who are in the final or pre-final year of such Programmes.
3. Candidates in the second or higher year of the (Post B.Sc.) Four-year integrated Master's degree Programme in Engineering/Technology or in the third or higher year of the Five-year integrated Master's degree/Dual degree Programme in Engineering/Technology.
4. Candidate with qualification obtained through examination conducted by Professional Societies recognised by UPSC/AICTE as equivalent to B.E./B. Tech. Those who have completed section A or equivalent of such professional courses are also eligible.

It is very much hoped the subject matter will create a confidence among the candidates and the book will help them like an ideal teacher.

Important Dates :—

1. Availability of GATE forms : October, Ist week.
2. Last date for submitting forms : November, Ist week.
3. Examination Date : 2nd Sunday of February.

Result : March 31st.

I am confident that the readers will find the present form of the book, best suited to their needs. As there is always scope for improvement, valuable suggestions from worthy teachers and dear students will be gratefully received.

I would like to thank Miss Pinki Gole, who came up with the idea of having such a book among our titles.

—Dr. N. K. Singh

Contents

Syllabus

Linear Algebra : Finite Dimensional vector spaces; Linear transformations and their matrix representations, rank; systems of linear equations, eigen values and eigen vectors, minimal polynomial, Cayley-Hamilton Theorem, diagonalisation, Hermitian, Skew-Hermitian and unitary matrices; Finite dimensional inner product spaces, Gram-Schmidt orthonormalization process, self-adjoint operators.

Complex Analysis : Analytic functions, conformal mappings, bilinear transformations; complex integration; Cauchy's integral theorem and formula; Liouville's theorem, maximum modulus principle; Taylor and Laurent's series; residue theorem and applications for evaluating real integrals.

Real Analysis : Sequences and series of functions, uniform convergence, power series, Fourier series, functions of several variables, maxima, minima; Riemann integration, multiple integrals, line, surface and volume integrals, theorems of Green, Stokes and Gauss; metric spaces, completeness, Weierstrass approximation theorem, compactness; Lebesgue measure, measurable functions; Lebesgue integral, Fatou's lemma, dominated convergence theorem.

Ordinary Differential Equations : First order ordinary differential equations, existence and uniqueness theorems, systems of linear first order ordinary differential equations, linear ordinary differential equations of higher order with constant coefficients; linear second order ordinary differential equations with variable coefficients; method of Laplace transforms for solving ordinary differential equations, series solutions; Legendre and Bessel functions and their orthogonality.

Algebra : Normal subgroups and homomorphism theorems, automorphisms; Group actions, Sylow's theorems and their applications; Euclidean domains, Principle ideal domains and unique factorization domains. Prime ideals and maximal ideals in commutative rings; Fields, finite fields.

Functional Analysis : Banach spaces, Hahn-Banach extension theorem, open mapping and closed graph theorems, principle of uniform boundedness; Hilbert spaces, orthonormal bases, Riesz representation theorem, bounded linear operators.

Numerical Analysis : Numerical solution of algebraic and transcendental equations; bisection, secant method, Newton-Raphson method, fixed point iteration; interpolation; error of polynomial interpolation, Lagrange, Newton interpolations; numerical differentiation; numerical integration : Trapezoidal and Simpson rules, Gauss Legendre quadrature, method of undetermined parameters; least square polynomial approximation; numerical solution of systems of linear equations; direct methods (Gauss elimination, LU decomposition); iterative methods (Jacobi and Gauss-Seidel); matrix eigenvalue problems : power method, numerical solution of ordinary differential equations : initial value problems : Taylor series methods, Euler's method, Runge-Kutta methods.

Partial Differential Equations : Linear and quasilinear first order partial differential equations, method of characteristics; second order linear equations in two variables and their classification;

Cauchy, Dirichlet and Neumann problems; solutions of Laplace, wave and diffusion equations in two variables; Fourier series and Fourier transform and Laplace transform methods of solutions for the above equations.

Mechanics : Virtual work, Lagrange's equations for holonomic systems, Hamiltonian equations.

Topology : Basic concepts of topology, product topology, connectedness, compactness, countability and separation axioms, Urysohn's Lemma.

Probability and Statistics : Probability space, conditional probability, Bayes theorem, independence, random variables, joint and conditional distributions, standard probability distributions and their properties, expectation, conditional expectation, moments; Weak and strong law of large numbers, central limit theorem; Sampling distributions, UMVU estimators, maximum likelihood estimators, Testing of hypotheses, standard parametric tests based on normal, χ^2, t, F–distributions; Linear regression; Interval estimation.

Linear Programming : Linear programming problem and its formulation, convex sets and their properties, graphical method, basic feasible solution, simplex method, big-M and two phase methods; infeasible and unbounded LPP's, alternate optima; Dual problem and duality theorems, dual simplex method and its application in post optimality analysis; Balanced and unbalanced transportation problems, u -v method for solving transportation problems; Hungarian method for solving assignment problems.

Calculus of Variation and Integral Equations : Variation problems with fixed boundaries; sufficient conditions for extremum, linear integral equations of Fredholm and Volterra type, their iterative solutions.

MATHEMATICS

Linear Algebra

Finite Dimensional Vector Space—The vector space V(F) is said to be finite dimensional or finitely generated, if there exists a finite subset S of V such that

$$V = L(S)$$

Note—A vector space which is not finitely generated may be referred as an infinite dimensional space.

Some Important Theorems

1. There exists a basis for each finite dimensional vector space.
2. If V(F) is a finite dimensional vector space, then any two basis of V have the same number of elements.
3. Every linearly independent subset of a finitely generated vector space V(F) is either a basis of V or can be extended to form a basis of V.
4. Each subspace ω of a finite dimensional vector space V(F) of dimension n is a finite dimensional space with dim $m \leq n$ also $V = \omega$, iff dim V = dim ω.
5. If ω_1 and ω_2 are two subspaces of a finite dimensional vector space V(F), then
$$\dim(\omega_1 + \omega_2) = \dim \omega_1 + \dim \omega_2 - \dim(\omega_1 \cap \omega_2)$$
6. Each set of $(n + 1)$ or more vectors of a finite dimensional vector space V(F) of dimension n is linearly dependent.
7. If V(F) is a finite dimensional vector space of dimension n, then any set of n linearly independent vector in V forms a basis of V.
8. If a sets of n vectors of a finite dimensional vector space V(F) of dimension n generates V(F), then S is a basis of V.
9. If ω is a subspace of finite dimensional vector space V(F), every linearly independent subset of ω is finite and is part of a finite basis for ω.

Linear Transformation—Let U(F) and V(F) be to vector spaces over the same field F.

Then, a function T from U into V such that

$$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta),$$

$\forall\ \alpha, \beta \in$ U and $a, b \in$ F is called linear transformation from U into V.

Properties of linear Transformations—Let T be a linear Transformation from U(F) into V(F), then

1. T (0) = 0, where 0 on the left hand side is a zero vector of U and 0 on the right hand side is zero vector of V.

2. $T(-\alpha) = -T(\alpha),\ \forall\ \alpha \in U$

3. $T(a_1\alpha_1 + a_2\alpha_2 + \ldots\ldots + a_n\alpha_n)$
$$= a_1T(\alpha_1) + a_2T(\alpha_2) + \ldots\ldots + a_nT(\alpha_n)$$
where $\alpha_1, \ldots\ldots \alpha_n \in$ U and $a_1 \ldots, a_n \in$ F.

Rank of a Matrix—A number r is defined as the rank of a $m \times n$ matrix A if,

1. A has at least one minor of order r which is not equal to zero.
2. There is no minor of order $(r + 1)$ which is not equal to zero.

The rank of the matrix A is denoted by $\rho(A)$.

Note (1) : The rank of a null matrix is defined as zero *i.e.*, $\rho(0) = 0$.

Note (2) : If I_n is a unit matrix of order n, then its rank I = n. *i.e.* $\rho(I_n) = n$

Note (3) : From the definition of the rank of a matrix, we concluded that

(a) If a matrix A does not possesses any minor of order $(r + 1)$, then $\rho(A) \leq r$.

(b) If at least one minor of order r of the matrix is not equal to zero, then $\rho(A) \geq r$.

Note (4) : If every $(r + 1)^{th}$ order minor of A is zero, then any higher order minor will also be zero.

Note (5) : If A is $n \times n$ non-singular matrix, then $\rho(A) = n$.

System of linear equation—If $b_1 = b_2 = b_3$ ...... $= b_m = 0$, the set of equation is said to be homogeneous. Thus $AX = 0$

For such a system the rank of the matrix A and augmented matrix [A : B] are equal. Hence a system of homogeneous linear equations is always consistent.

If $x = 0$ (zero) *i.e.* $x_1 = x_2 \ldots x_n = 0$, then the solution is a trival solution. Thus condition for a trival solution of the system of linear equation is $\rho(A) = n$.

If $\rho(A) < n$, then the solution will be non-trival.

> **Note**—A homogeneous system of linear equation is n unknowns whose determinant of coefficient does not equal to zero, has only the trival solution.

Eigen values and eigen vectors—If V is a vector space over the field F and T is a linear operator on V. An eigen value of T is a scalar c in F such that there is a non-zero vector $\alpha \in V$ with $T\alpha = C\alpha$.

If c is an eigen value of T, then

(a) Any α such that $T\alpha = c\alpha$ is called eigen vector of T associated with the eigen value c;

(b) The collection of all c such that $T\alpha = c\alpha$ is called the eigen space associated with c.

Eigen value of matrix A over F—If A is an $n \times n$ matrix over the field F, an eigen value of A over F is a scalar c is F such that the matrix $(A - cI)$ is singular (not invertible)

Eigen Polynomial—

$$f(c) = |A - cI|.$$

Diagonalisation—If T is a linear operator on the finite dimensional space V. The T is diagonalisation if there is a basis for V each vector of which is an eigen vector of T.

Some Important Theorems

1. If T is a linear operator on a finite dimensional space V and c is any scalar. Then following are equivalent—

   (a) c is an eigen value of T

   (b) The operator $(T - cI)$ is singular (not invertible)

   (c) $\det(T - cI) = 0$

2. Similar matrices have the same eigen polynomial.

3. If $T\alpha = c\alpha$ and F is any polynomial, then $F(T)\alpha = F(c)\alpha$.

4. Suppose T is a linear operator on the finite dimensional space V, c_1 ...... c_k are K-distinct eigen values of T and ω_1 is the space of eigen vector associated with the eigen value c_i,

   If $\quad \omega = \omega_1 + \omega_2 + \ldots\ldots + \omega_k$, then

   $$\dim \omega = \dim \omega_1 + \dim \omega_2 + \ldots + \dim \omega_k.$$

   In fact, if B_i is an ordered basis for ω_i then $B = (B_1 \ldots\ldots B_k)$ is an ordered basis for ω.

5. If T is a linear operator on a finite dimensional space V, and c_1 ...... c_k are k characteristic values of T and w_i is a null space of $(T - c_i I)$. Then the following are equivalent.

   (i) T is diagonalisation

   (ii) The eigen polynomial for T is

   $$F = (x - c_i)^{di} \ldots\ldots (x - c_k)^{dk}$$

   with $\dim \omega_1 = d_i$

   $$i = 1 \ldots\ldots, k$$

   (iii) $\dim V = \dim \omega_1 + \dim \omega_2 + \ldots + \dim \omega_K$.

Minimal Polynomials—If T is a linear operator on a finite dimensional vector space V over the field F. The minimal polynomial for T is the (unique) monic generator of the ideal of polynomials over F which annihilate T.

Theorem—If T is a linear operator on an n dimensional vector space V. Then eigen and minimal polynomials for T have the same roots, except for multiplicities.

Cayley-Hamilton theorem—If T is a linear operator on a finite dimensional vector space V and f is the eigen polynomial for T, then

$$f(T) = 0$$

i.e., the minimal polynomial divides the eigen polynomial for T.

Hermitian Matrix—A square matrix A is said to be a hermitian matrix if the transpose of the conjugate matrix is equal to the matrix itself

i.e. $\quad A^* = A$

$$\Rightarrow \quad \overline{a_{ij}} = a_{ji}$$

where $\quad A = [a_{ij}]_{n \times n}$

> **Note**—In hermitian matrix, the elements on the principal diagonal must be all real numbers.
>
> *i.e.* $\overline{a_{ii}} = a_{ii}$

Important Facts

1. If A is a hermitian matrix, then KA is also hermitian for any real number K.
2. If A and B are hermitian matrices of same order, then $\lambda_1 A + \lambda_2 B$ also hermitian for any real number as λ_1, λ_2 etc.
3. If A be any square matrix, then AA^* and A^*A are also hermitian.
4. If A and B are hermitian, then AB is also hermitian, iff AB = BA.
5. If A is a hermitian matrix, then $\overline{A}$ is also hermitian.
6. If A and B are hermitian matrices of same order, then AB + BA is also hermitian.
7. If A is a square matrix then $A + A^*$ is a hermitian matrix.
8. Any square matrix can be uniquely expressed as A + *i*B, where, A and B are hermitian matrices.
9. If A is any hermitian matrix, then all positive integral powers of A are hermitian.

Skew-Hermitian Matrix—A square matrix A is said to be skew-hermitian, its

$$A^* = -A \Rightarrow \overline{a_{ij}} = -a_{ji}$$

Note—The elements on the principal diagonal must be purely imaginary number or zero.

Important Facts

1. If A is a skew-hermitian matrix, then KA is also skew-hermitian for any real number K.
2. If A and B are skew-hermitian matrices of same orders, then $\lambda_1 A + \lambda_2 B$ is also skew-hermitian for any real number as λ_1, λ_2 etc.
3. If A and B are hermitian matrices of same order, then AB – BA is skew-hermitian.
4. If A is any square matrix, then $A - A^*$ is a skew-hermitian matrix.
5. Every square matrix can be uniquely represented as the sum of a hermitian and a skew-hermitian matrices.
6. If A is a skew-hermitian matrix, then *i*A is a hermitian.
7. If A is a skew-hermitian matrix, then $\overline{A}$ is also skew-hermitian.

Unitary Matrix—A square matrix A is said to be unitary matrix, iff

$$AA^* = I = A^*A.$$

Important Facts

1. If A is a unitary matrix, then A' is also unitary.
2. For any two unitary matrices A and B. AB and BA are also unitary matrices.
3. If A is a unitary matrix, then A^{-1} is also unitary.

Finite dimensional inner product spaces

Inner product—Let F be the field of real or complex numbers and V vector space over F. An inner product on V is a function which assigns to each ordered pair of vectors $\alpha, \beta \in V$ a scalar $(\alpha/\beta) \in F$, defined as

For $\alpha, \beta, \gamma \in V$, and all scalar

(a) $(\alpha + \beta/\gamma) = (\alpha/\gamma) + (\beta/\gamma)$

(b) $(c\,\alpha/\beta) = c\,(\alpha/\beta)$

(c) $(\beta/\alpha) = \overline{(\alpha/\beta)}$ (a complex conjugate)

(d) $(\alpha/\alpha) = 0$ if $\alpha \neq 0$

Inner product spaces—An inner product sapce is a real or complex space with a specified inner product on that space.

Norm—If $\|\alpha\|^2 = (\alpha/\alpha)$, then $\|\alpha\|$ is called norm of (α/α).

Euclidean space—A finite dimensional real inner product space.

Unitary space—A complex inner product space.

Orthogonal set—Let α and β be vectors in an inner product space V. Then α is orthogonal to β if $(\alpha/\beta) = 0$

Orthogonal set—Let V be a vector space $S \subset V$. The set S is orthogonal set if $\forall\ \alpha, \beta \in S$

$$\Rightarrow \quad (\alpha/\beta) = 0, \alpha \neq \beta.$$

Orthonormal set—The orthogonal set S with property

$$\|\alpha\| = 1, \forall\ \alpha \in S.$$

Orthogonal Complement—Let V be a inner product space and $S \subset V$. The orthogonal complement of S is the set $S^{\perp}$ of all vectors in V which are orthogonal to every vector in S.

Standered inner product—On F^n, $\alpha = (x_1 \ldots\ldots x_n)$, $\beta = (y_1 \ldots\ldots y_n)$ and $(\alpha/\beta) = \sum_i x_i \overline{y_i}$

when F = R,

$$(\alpha/\beta) = \sum_i x_i\, y_i = \alpha\beta$$

(the dot product)

Some Important Theorems

1. If V is an inner product space, then for any α, $\beta \in$ V and c any scalar.
   (i) $\| c\alpha \| = |c|\,\|\alpha\|$
   (ii) $\|\alpha\| > 0$ for $\alpha \neq 0$
   (iii) $|(\alpha/\beta)| \leq \|\alpha\|\,\|\beta\|$ (Cauchy-Schwarz-inequality)
   (iv) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$.
2. An orthogonal set of non-zero vectors is linearly independent.
3. If a vector β is a linear combination of an orthogonal sequence of non-zero vectors α_1 …… α_n, then β is the particular combination
   $$\beta = \sum_{i=1}^{n} \frac{\left(\frac{\beta}{\alpha}\right)}{\|\alpha\|^2}\alpha_i.$$
4. Let V be an inner product space and let β_1 … β_n be any independent vectors in V. then one may construct orthogonal vectors α_1 …… α_n $\in$ V such that each K = 1, 2, ……, n the set $\{\alpha_1 \ldots\ldots \alpha_k\}$ is a basis for the subspace spanned by β_1 …… β_k.
5. Every finite dimensional inner product space has an orthonormal basis.
6. Let V be an inner product space, W is finite dimensional subspace and E the orthogonal projection of V on W. Then the mapping $\beta \to \beta - E\beta$ is the orthogonal projection of V on W^1.
7. Let W be a finite dimensional subspace of an inner product space V and E be the orthogonal projection of V on W, then E is an idempotent linear transformation of V onto W, $W^{\perp}$ is the null space of E and.
   $$V = W \oplus W^{\perp}.$$
8. Let $\{\alpha_1 \ldots\ldots \alpha_n\}$ be an orthogonal set of non-zero vectors in an inner product space V. If $\beta \in$ V, then
   $$\sum_K = \frac{|(\beta/\alpha_K)|^2}{\|\alpha\|^2} \leq \|\beta\|^2.$$
   and the equality holds iff
   $$\beta = \sum_K \frac{(\beta/\alpha_K)}{\|\alpha_K\|^2}\alpha_K$$

- Orthogonal complement of inner product space V is zero subspace {0} and $\{0\}^{\perp}$ = V
- If $S \subset V$, then $S^{\perp}$ is always a subspace of V.

Gram-Schmidt Orthonormalization Process

Let S = $\{x_1, x_2, \ldots\ldots\}$ be a linearly independent sequence in an inner product space. Then there exists an orthonormal sequence

$$T = \{y_1, y_2, \ldots\ldots\}$$

such that span (S) = span (T)

Since S is linearly independent $x_K \neq 0$ for each K. Define

$$y_1 = \frac{x_1}{\|x_1\|} \text{ so that } \|y_1\| = 1$$

Define $\quad V = x_2 - (x_2, y_1)y_1$

Then $V \perp y_1$ and $V \neq 0$, since $\{x_1, x_2\}$ is linearly independent.

Hence $y_2 = \dfrac{V}{\|V\|}$ is orthogonal to y_1 and $\|y_2\|$ = 1. We now inductively define.

$$V = x_n - \sum_{K=1}^{n-1} (x_n, y_K)\, y_K$$

and $$y_n = \frac{V}{\|V\|}$$

It is clear from the construction that span (S) = span (T).

Self adjoint operators

Adjoint—Let T be a linear operator on an inner product space V. Then we say T has an adjoint on V if there exists a linear operator T^* on V such that

$$\left(T\frac{\alpha}{\beta}\right) = \left(\frac{\alpha}{T^*}\beta\right) \text{ for all } \alpha, \beta \in V.$$

Some Important Theorem

1. Let V be a finite dimensional inner product space and f a linear functional on V. Then there is a unique vector $\beta \in$ V such that
   $$f(\alpha) = \left(\frac{\alpha}{\beta}\right) \text{ for } \alpha \in V.$$
2. Let V be a finite dimensional inner product space and let B = $\{\alpha_1 \ldots\ldots \alpha_n\}$ be an orthonormal basis for V. Let T be a inner operator on V and let A be the matrix of T in theordared basis B. Then $A_{Kj} = \left(\dfrac{T\alpha_j}{\alpha_K}\right)$
3. Let V be a finite dimensional inner product space, and let T be a linear operator on V. In any orthonormal basis for V, the matrix of T^* is the conjugate transpose of the matrix of T.

4. Let V be a finite dimensional inner product space. If T and U are linear operators on V and c is a scalar.
   (i) $(T+U)^* = T^* + U^*$
   (ii) $(cT)^* = c(T)^*$
   (iii) $(TU)^* = U^*T^*$
   (iv) $(T^*)^* = T.$

Some Solved Examples

Example 1. Find the dimension of V where V $= \{a_0 + a_1x + a_2x^2 + a_3x_i^3, x \in R\}$.

Solution : $S = \{1, x, x^2, x^3\}$ is a basis and V has therefore, dimension.

Example 2. Standard basis of c^n is an orthonormal set with respect to standard inner product.

Solution :

$$S = \{\varepsilon_1, \varepsilon_2, \ldots\ldots \varepsilon_n\}$$

is a standard basis of c^n, where

$$\varepsilon_i = \{0, \ldots\ldots 1, 0, 0\ldots\ldots 0)$$

For standard inner product $(a)\ \forall \left(\frac{\varepsilon_i}{\varepsilon_j}\right) = 0$

$$i \neq i,\ i = 1, \ldots\ldots n$$

and $\left(\frac{\varepsilon_i}{\varepsilon_j}\right) = 1\ \forall\, i = 1 \ldots\ldots n$

$\therefore$ Set $S = \{\varepsilon_1 \ldots\ldots, \varepsilon_n\}$ is an orthonormal set.

Example 3. Find the dimension of the vector space $c(R)$ of the complex number over real numbers ?

Solution : Set $\{i, j\} \subset c$ forms a basis for c

$\because a.1\ (+)\ b.\ i = 0 \Rightarrow a = 0, b = 0, a, b \in R$

$\therefore$ 1, i are linearly independent

$\forall\ a + b_i \in c, a, b \in R \Rightarrow \{1, i\}$ span $c(R)$.

Example 4. Find the condition $\{\alpha + i\beta, a + ib\}$, $\alpha, \beta, a, b \in R$ is a basis for vector space c over R.

Solution : $(\alpha + i\beta)$ and $(a + ib)$ to be linearly inde-pendent

$$\Rightarrow \quad \theta_1(a + ib) + \theta_2(\alpha + i\beta) = 0$$

$$\theta_1 a + \theta_2 \alpha = 0$$

$$\theta_1 b + \theta_2 \beta = 0$$

$$\Rightarrow \quad \theta_1 (a\beta - b\alpha) = 0$$

$$\theta_2(a\beta - b\alpha) = 0$$

$\Rightarrow \theta_1 = 0 = \theta_2 \Leftrightarrow a\beta - b\alpha = 0$

$\therefore$ If $a\beta = b\alpha$, then $\{\alpha + i\beta, a + ib\}$ is a basis.

Example 5. Let V be a finite dimensional vector space, $W_1, \ldots\ldots W_K$ be subspaces of V such that

$$V = W_1 + \ldots\ldots + W_K$$

and $\dim V = \dim W_1 + \ldots\ldots + \dim W_K$

Prove that

$$V = W_1 \oplus \ldots\ldots \oplus W_K.$$

Solution :

$$V = W_1 + \ldots\ldots\ W_K$$

$$\Rightarrow \quad \dim V \leq \dim W_1 + \ldots\ldots + \dim W_K$$

This inequality converts into equality if $W_1 \ldots\ldots W_K$ are linearly independent

$\Rightarrow W = W_1 \oplus W_2 \oplus \ldots\ldots \oplus W_K$ is the direct sum of $W_1 \ldots\ldots W_K$.

OBJECTIVE TYPE QUESTIONS

1. Let V be a vector space, T is a linear transform on V into V such that $T\alpha = 0$, $\forall\ \alpha \in V$—
   (A) T is identity transform
   (B) T is zero transform
   (C) T is invertible
   (D) T is orthogonal

2. Let dimension of a vector space V be dim $V = n$. If any set $S \subset V$ and shave m elements, $m > n$, then—
   (A) S is linearly independent
   (B) S is linearly dependent
   (C) S is zero subspace
   (D) None of these

3. If A is a matrix of order n, then A is invertible iff—
   (A) $A \neq 0$ (B) $A^{-1} = 0$
   (C) $|A| \neq 0$ (D) $|A| = 0$

4. Let S be an orthonormal set, $\alpha \in S$, then—
   (A) $\left(\frac{\alpha}{\alpha}\right) = 1$ (B) $\left(\frac{\alpha}{\alpha}\right) = 0$
   (C) $\left(\frac{\alpha}{\alpha}\right) > 0$ (D) $\left(\frac{\alpha}{\alpha}\right) < 1$

5. Let S be an orthonormal set, then for $\alpha \in S$—
(A) $\|\alpha\| = 0$ (B) $\|\alpha\| > 0$
(C) $\|\alpha\| = 1$ (D) $\|\alpha\| < 1$

6. If V is a vector space, f is a linear functional on V, then the vector space V^* is dual space of V if V^* is—
(A) A collection of all linear functional f on V.
(B) A collection of all vector spaces on which f is defined
(C) Collection of all linear operators
(D) None of these

7. Let A and B, C are two matrices of order n, then—
(A) $|AB| = |A|\,|B|$ (B) $|AB| \neq |A|\,|B|$
(C) $|AB| > |A|\,|B|$ (D) $|AB| < |A|\,|B|$

8. If A is square matrix of order n, then—
(A) $|\text{adj } A| \neq A$ (B) $|\text{adj } A| = |A|$
(C) $|A| \neq |A^T|$ (D) None of these

9. The orthogonal complement of inner product space V is—
(A) Zero subspace {0}
(B) V itself
(C) Any subspace
(D) None of these

10. If {0} is a zero subspace of inner product space V, then $\{0\}^{\perp}$ is equal to—
(A) {0} (B) V
(C) ϕ (D) None of these

11. The zero subspace of inner product space consist—
(A) Zero element only
(B) Non-zero elements
(C) Identity
(D) None of these

12. If V is an inner product space, then for α β $\in$ V—
(A) $\left|\left(\frac{\alpha}{\beta}\right)\right| = \|\alpha\|\,\|\beta\|$
(B) $\left|\left(\frac{\alpha}{\beta}\right)\right| \geq \|\alpha\|\,\|\beta\|$
(C) $\left|\left(\frac{\alpha}{\beta}\right)\right| \leq \|\alpha\|\,\|\beta\|$
(D) None of these

13. If V is an inner product space, then for $\alpha, \beta \in V$—
(A) $\|(\alpha + \beta)\| \geq \|\alpha\|\,\|\beta\|$
(B) $\|(\alpha + \beta)\| = \|\alpha\|\,\|\beta\|$
(C) $\|(\alpha + \beta)\| \leq \|\alpha\|\,\|\beta\|$
(D) None of these

14. If V is an inner product space, then for $\alpha \in V$—
(A) $\|\alpha\| \leq 0$ for $\alpha \neq 0$
(B) $\|\alpha\| \geq 0$ for $\alpha \neq 0$
(C) $\|\alpha\| = 0$ for $\alpha \neq 0$
(D) $\|\alpha\| > 0$ for $\alpha \neq 0$

15. The Cauchy-Schwarz inequality states—
(A) $\left|\left(\frac{\alpha}{\beta}\right)\right| \geq \|\alpha\|\,\|\beta\|$
(B) $\left|\left(\frac{\alpha}{\beta}\right)\right| \leq \|\alpha\|\,\|\beta\|$
(C) $\|\alpha + \beta\| \geq \|\alpha\| + \|\beta\|$
(D) None of these

16. If V is an inner product space $\alpha, \beta \in V$ and c any scalar, then—
(A) $\|c\alpha\| = \|c\|\,\|\alpha\|$ (B) $\|c\alpha\| \geq \|c\|\,\|\alpha\|$
(C) $\|c\alpha\| \leq \|c\|\,\|\alpha\|$ (D) None of these

17. For vectors $\alpha, \beta \in V$ and scalar c, the inner product $\left(\frac{\alpha}{\alpha}\right)$ is—
(A) Greater than zero, for $\alpha \neq 0$
(B) Equals to zero, for $\alpha \neq 0$
(C) Less than zero, for $\alpha \neq 0$
(D) None of these

18. For vector $\alpha, \beta, \gamma \in V(F)$, the inner product of $(\alpha + \beta/\gamma))$ is equal to—
(A) $\left(\frac{\alpha}{\gamma}\right)\beta$ (B) $\alpha\gamma + \beta\gamma$
(C) $(\alpha/\gamma) + (\beta/\gamma)$ (D) None of these

19. For the vectors $\alpha, \beta, \gamma \in V(F)$ and non zero scalar, the inner product of $(\alpha/c\beta + \gamma)$ is equal to—
(A) $\bar{c}\left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha}{\gamma}\right)$ (B) $\bar{c}\left(\frac{\beta}{\alpha}\right) + \left(\frac{\gamma}{\alpha}\right)$
(C) $\bar{c}\,(\alpha) + \beta$ (D) $\bar{c}\,\alpha\gamma + \alpha\beta$

20. The norm of α with respect to inner product (α/α) is—
(A) $\|\alpha\| = \left(\frac{\alpha}{\alpha}\right)$ (B) $\|\alpha\|^2 = \left(\frac{\alpha}{\alpha}\right)$
(C) $\|\alpha\| = \left(\frac{\alpha}{\alpha}\right)^2$ (D) None of these

21. The equality in Cauchy-Schwarz inequality $(\alpha/\beta) \le \|\alpha\|\,\|\beta\|$ occurs, when—
(A) α and β are linear independent
(B) α and β are linear dependent
(C) α and β are non zero
(D) None of these

22. An orthogonal set of non-zero vectors—
(A) Linearly independent
(B) Linearly dependent
(C) Constant
(D) None of these

23. Let V be an inner product space and $\alpha, \beta \in$ V, then α and β are the orthogonal to each other if—
(A) $\left(\frac{\alpha}{\beta}\right) > 0$ (B) $\left(\frac{\alpha}{\beta}\right) < 0$
(C) $\left(\frac{\alpha}{\beta}\right) = 0$ (D) None of these

24. If V is a inner product space then for every $\alpha \in$ V—
(A) Zero vector is orthogonal to α
(B) Zero vector is not orthogonal to α
(C) Zero vector does not exist
(D) None of these

25. Let A and B are two matrix A and B are similar matrix, then—
(A) A, B have same characteristic polynomial
(B) A,B may have different characteristic polynomial
(C) A, B have same value
(D) A, B have different value

26. Let V be a vector space and T a linear operator on V. If W is a subspace of V, W is invariant under T if—
(A) $T(W) \subset W$
(B) $W \subset T(W)$
(C) $T(W) = W$
(D) None of these

27. The vectors $\alpha_1 \ldots\ldots \alpha_n$ are linearly dependent if for scalars $c_1 \ldots\ldots c_n$, $c_1\alpha_1 + c_2\alpha_2 + \ldots\ldots + c_n\alpha_n = 0$ implies—
(A) $c_1 \ldots\ldots c_n$ are not zero
(B) $c_1 = c_2 = \ldots\ldots = c_n = 0$
(C) $c_1 = c_2 = \ldots\ldots = c_n$
(D) None of these

28. If B and B' are two basis of vector space V, then—
(A) B and B' have some numbers of elements
(B) B and B' have distinct number of elements
(C) B = B'
(D) None of these

29. If V is a finite dimensional vector space, then if W is a subspace of V, then—
(A) W is finite dimensional
(B) W is infinite dimensional
(C) The dimensional of W is greater than V
(D) None of these

30. If A is a matrix, then—
(A) Row rank (A) = column rank (B)
(B) Row rank (A) ≠ column rank (B)
(C) Row rank (A) > column rank (B)
(D) None of these

31. If w_1 and w_2 are subspaces of V, then following is false—
(A) $w_1 \cup w_2$ is a subspace of V
(B) $w_1 \cap w_2$ is a subspace of V
(C) $w_1 + w_2$ is a subspace of V
(D) $w_1 \cup w_2$ is not a subspace ofV.

32. If dim W = m, dim V = n, and $W \subset V$ dim $\left(\frac{V}{W}\right)$ is—
(A) $m + n$ (B) $n - m$
(C) $m - n$ (D) None of these

33. A linear transformation T from V onto W, is non-singular if—
(A) $T\alpha = 0 \Rightarrow \alpha = 0$
(B) $T\alpha = 0 \Rightarrow \alpha \neq 0$
(C) $T\alpha = c \Rightarrow \alpha = 0$
(D) None of these

34. If T is a linear transformation from vector space V into vector space W. Let dim V = m, dim W = n, then rank of T is—
(A) m (B) n
(C) $m - n$ (D) $m + n$

35. Let 0 be a zero vector in vector space V, then {0} is—
(A) Zero subspace of V
(B) Null space of V
(C) Identity space of V
(D) None of these

36. The dimension of the vector space is—
(A) Number of elements in vector space
(B) Number of elements in basis of the vector space
(C) Subspace of vector space
(D) None of these

37. If w_1 and w_2 are finite dimensional subspaces of vector space V, then—
(A) $\dim (w_1 + w_2) = \dim w_1 + \dim w_2$
(B) $\dim (w_1 + w_2) = \dim w_1 + \dim w_2 + \dim (w_1 \cup w_2)$
(C) $\dim (w_1 + w_2) = \dim w_1 + \dim w_2 - \dim (w_1 \cap w_2)$
(D) $\dim (w_1 + w_2) = \dim w_1 + \dim w_2 + \dim (w_1 \cap w_2)$

38. Let V and w be finite dimensional vector spaces dim V = n and dim w = n, then dim L(V, w) is—
(A) mn (B) $m + n$
(C) m/n (D) $m - n$

39. If V is a vector space with dim V = n, then dimension of hyper space of V is—
(A) n (B) $n - 1$
(C) $n + 1$ (D) 0

40. If V is a finite dimensional vector space and let w is a subspace of V then—
(A) $\dim w + \dim w^0 = \dim V$
(B) $\dim w - \dim w^0 = \dim V$
(C) $(\dim w)(\dim w^0) = \dim V$
(D) $\dfrac{\dim w}{\dim w^0} = \dim V$

41. If w_1 and w_2 are subspaces of a finite dimensional vector space. Then $w_1 = w_2$ iff—
(A) $w_1^0 = w_1^0$
(B) $w_1^0 - w_1^0 = \{0\}$
(C) $w_1^0 \neq w_1^0$
(D) None of these

42. If w_1 and w_2 are subspaces of finite dimensional space, then $w_1^0 = w_2^0$ iff—
(A) $w_1 = w_2$ (B) $w_1 \neq w_2$
(C) $w_1 \subset w_2$ (D) $w_1 \supset w_2$

43. If w is the proper subspace of a vector space v, then—
(A) $\dim v < \dim w$ (B) $\dim w < \dim v$
(C) $\dim w = \dim v$ (D) None of these

44. If v is a vector space, then dimension of v is equal to—
(A) Number of element of vector space v
(B) Number of element in a basis for v
(C) Number of non zero elements of v
(D) None of these

45. If w_1 and w_2 are two subspaces of vector space, then—
(A) $\dim (w_1 + w_2) \leq \dim w_1 + \dim w_2$
(B) $\dim (w_1 + w_2) \geq \dim w_1 + \dim w_2$
(C) $\dim (w_1 + w_2) = \dim w_1 + \dim w_2$
(D) None of these

46. If w_1 and w_2 are disjoint subspaces of vector space, then—
(A) $w_1 \cap w_2 = 0$ (B) $w_1 \cap w_2 = \{0\}$
(C) $w_1 \cup w_2 = \{0\}$ (D) None of these

47. If v is a vector space, a hyper space in v is—
(A) Maximal subset of v
(B) Maximal subspace of v
(C) Minimal subspace of v
(D) Minimal subset of v

48. If v is n dimensional vector space and w is m dimensional vector space over the same field then space L(v, w) has the dimension—
(A) $m + n$ (B) $m - n$
(C) mn (D) m/n

49. If T is linear operator on v, then—
(A) $T^3 = T.T.T.$ (B) $T^3 = T + T + T$
(C) $T^3 = T^2 + T$ (D) None of these

50. For identity transformation I, on finite dimensional vector space, the rank of I is—
(A) dim V (B) 0
(C) 1 (D) None of these

51. Let V be finite dimensional vector space, T is a zero transformation on V, then null space of T is—
(A) V (B) {0}
(C) ϕ (D) None of these

52. Let V be a finite dimensional space. T is zero transformation on V. Then range of T is—
(A) {0} (B) V
(C) ϕ (D) None of these

53. If T is a zero transformation on finite vector space V. The rank of T is—
(A) 0 (B) 1
(C) dim V (D) None of these

54. If T is a zero transformation on finite vector space V. Then nullity of T is—
(A) dim V (B) Zero
(C) 1 (D) None of these

55. Let T be a linear transformation on finite vector space V. Then—
(A) Rank T < dim V
(B) Rank T = dim V
(C) Rank T > dim V
(D) Rank T = nullity (V)

56. Let V be a vector space and (1) an inner product on V, then $(0/\beta)$ is equal to, for all $\beta \in$ V—
(A) Zero (B) Greater than zero
(C) Less than zero (D) None of these

57. Let V be a vector space, (1) an inner product on V, then If $(\alpha/\beta) = 0$ for all $\beta \in$ V—
(A) $\alpha \neq 0$ (B) $\alpha = 0$
(C) $\alpha > 0$ (D) $\alpha < 0$

58. Let V be a vector space over F.
(a) The sum of two inner product is an inner product.
(b) The difference of two inner product is an inner product.
(A) (a) and (b) both false
(B) (a) and (b) both true
(C) (a) is true, (b) is false
(D) (a) is false, (b) is true

59. Let (1) an standard inner product on R^2. If $(\alpha/\gamma) = -1$ and $(\beta/\gamma) = 3$, given $\alpha = (1, 2)$, $\beta = (-1, 1)$ the value of γ is—
(A) (0, 1) (B) (0, 3)
(C) $\left(0, \frac{2}{3}\right)$ (D) (0, 0)

60. If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then characteristic polynomial for A is—
(A) $x^2 + 1$ (B) $x + 1$
(C) $x - 1$ (D) $x^2 - 1$

61. Let $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$, then the characteristic polynomial for A is—
(A) $x^3 + 5x^2 + 8x + 4$ (B) $x^2 + 5x$
(C) $x^3 - 5x^2 + 8x - 4$ (D) None of these

62. Let T be a linear operator on a finite dimensional space V and c_1 …… c_K be the distinct characteristic values of T. Let w_i be null spaces of $(T - c_iI)$. If T is diagonal *lizable* then—
(A) (dim w_1) (dim w_2) (dim w_3) …… (dim w_K) = dim V
(B) dim w_1 + dim w_2 + … + dim w_K = dim V
(C) dim w_1 – dim w_2 + … – dim w_K = dim V
(D) None of these

63. A linear operator E on vector space V is projection if—
(A) $E(\alpha) = \alpha$ (B) $E^2(\alpha) = E(\alpha)$
(C) $E^3(\alpha) = E(\alpha)$ (D) None of these

64. The zero subspace have the dimension—
(A) One (B) Two
(C) Three (D) Zero

65. The vector (x, y) and $(-y, x)$ with respect to standard inner product are—
(A) Orthonormal (B) Continuous
(C) Orthogonal (D) None of these

66. Let V be a finite dimensional vector space. If V^* is the dual of V, then—
(A) dim V = dim V^*
(B) dim V > dim V^*
(C) dim V < dim V^*
(D) None of these

67. If two vectors α and β are linearly dependent then for some scalar c—
(A) $\alpha = c\beta$ (B) $\alpha = c + \beta$
(C) $\alpha = c - \beta$ (D) None of these

68. If $\{\alpha_1 \ldots\ldots \alpha_n\}$ is a basis of V and if $\{\beta_1 \ldots \beta_n\}$ are linearly dependent in V then—
(A) $m = n$ (B) $m \geq n$
(C) $m \leq n$ (D) None of these

69. Let A be a $m \times n$ matrices with row rank = r= column rank the dimension of the space of solutions of the system of linear equations AX = 0 is—
(A) r (B) $n - r$
(C) $m - r$ (D) $\min(m, n) - r$

70. A matrix M has eigen values 1 and 4 with corresponding eigen vectors $(1, -1)^T$ and $(2, 1)^T$, respectively. then M is—
(A) $\begin{bmatrix} -4 & -8 \\ 5 & 9 \end{bmatrix}$ (B) $\begin{bmatrix} 9 & -8 \\ 5 & -4 \end{bmatrix}$
(C) $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ (D) $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

71. Let PID, ED, UFD denote the set of all principal ideal domains Euclidean domains, unique factorization domain respectively then—
(A) UFD $\subset$ ED $\subset$ PID
(B) PID $\subset$ ED $\subset$ UFD
(C) ED $\subset$ PID $\subset$ UFD
(D) PID $\subset$ UFD $\subset$ ED

72. The Hermite interpolating polynomial for the function $f(x) = x^6$ based on -1, 0 and 1 is—
(A) $x^4 - 2x^2$ (B) $2x^4 - x^2$
(C) $x^4 + 2x^2$ (D) $2x^4 + x$

73. The system of equations

$$\begin{aligned} 3x + 2y &= 4{\cdot}5 \\ 2x + 3y - z &= 5{\cdot}0 \\ -y + 2z &= -0{\cdot}5 \end{aligned}$$

is to be solved by successive over relaxation method. The ortimal relaxation factor w_{opt}, rounded upto two decimal places is given by—
(A) 1·23 (B) 0·78
(C) 1·56 (D) 0·63

74. In a matric space (x, d)—
(A) Every infinite set E has a limit point in E
(B) Every closed subset of a compact set is compact
(C) Every closed and bounded set is compact
(D) Every subset of a compact set is closed

75. Let (x, d) be a complete matric space and $f : x \to x$ satisfies $d\, f(x), f(y) \leq \alpha\, (x, y)$ for some α, $0 \leq \alpha < 1$ for all $x, y, \in x$, then—
(A) f is bonnded function an x
(B) f need not be continuous on x
(C) $\{f(x_n)\}_{n=1}^{\infty}$ may not be a couchy sequence even though $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in x
(D) $f(P) = P$ for some $P \in x$.

76. Let $A \in c^{m \times n}$ and A', A* denote respectively the transpose and conjugate transpose of A, then—
(A) Rank (AA^*A) = rank (A)
(B) Rank (A) = rank (A^2)
(C) Rank (A) = rank (A'A)
(D) Rank (A^2) – rank (A) = rank (A^3) – rank (A^2)

77. Consider 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
if $a + d = 1 = ad - bc$, then A^3 equals—
(A) 0 (B) – I
(C) 31 (D) None of these

78. The sequence $\{x_n\}$ of $m \times m$ matrices difined by the iterations
$x_{n+1} = 2x_n - x_n, Ax_n, n = 0, 1, 2$
when x_0 = I the identity matrix converges to A^{-1}, if and only if each eigen value λ of A satisfies—
(A) $|\lambda^{\circ}| < 1$ (B) $|\lambda - 1| < 1$
(C) $|\lambda + 1| < 1$ (D) None of the above

79. Let P be a matrix of order $m \times n$ and Q be a matrix of order $n \times p$, $n \neq p$. if rank (P) = n and rank (Q) = P, then rank (PQ) is—
(A) n (B) p
(C) np (D) $n + p$

80. Let P and Q be square matrices such that PQ = I, the identity matrix. Then zero is an eigen value of—
(A) P but not of Q (B) Q but not of P
(C) both P and Q (D) Neither P nor Q

81. Let W be the space spanned by $f = \sin x$ and $g = \cos x$, then for any real of θ, $f_1 = \sin(x + \theta)$ and $g_1 = \cos(x + \theta)$—
   (A) are vectors in W
   (B) are linearly independent
   (C) do not form a basis for W
   (D) Form a basis for W

82. Consider the basis $S = \{v_1, v_2, v_3\}$ for R^3 where $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 0, 0)$ and let $T : R^3 \to R^2$ be a linear transformation such that
   $Tv_1 = (1, 0)$, $Tv_2 = (2, -1)$, $Tv_3 = (4, 3)$. Then $T(2, -3, 5)$ is—
   (A) $(-1, 5)$ (B) $(3, 4)$
   (C) $(0, 0)$ (D) $(9, 23)$

83. For $0 < \theta < \pi$ the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
   (A) Has no real eigen value
   (B) Is orthogonal
   (C) Is symmetric
   (D) Is skew symmetric

84. The eigen values of a 3×3 real matrix P are $1, -2, 3$. Then—
   (A) $P^{-1} = \frac{1}{6}(5I + 2P - P^2)$
   (B) $P^{-1} \frac{1}{6}(5I - 2P + P^2)$
   (C) $P^{-1} = \frac{1}{6}(5I - 2P - P^2)$
   (D) $P^{-1} = \frac{1}{6}(5I + 2P + P^2)$

85. Let $T : c^n \to c^n$ be a linear operator having n distinct eigen values then—
   (A) T is inversible
   (B) T is inversible as well is digonalizable
   (C) T is not diagonalizable
   (D) T is diagonalizable

86. Let U be a 3×3 complex Hermitian matrix which is unitary then the distinct eigen values of U are—
   (A) $\pm i$ (B) $1 \pm i$
   (C) ± 1 (D) $\frac{1}{2}(1 \pm i)$

87. The polynomial $f(x) = x^5 + 5$ is—
   (A) Irreducible over C
   (B) Irreducible over R
   (C) Irreduciable over Q
   (D) Not irreducible over Q
   where Q denotes the field of rational number.

88. Let T be the matrix (occurring in a typical transportation problem) given by
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$
   Then—
   (A) Rank T = 4 and T is unimodular
   (B) Rank T = 4 and T is not unimodular
   (C) Rank T = 3 and T is unimodular
   (D) Rank T = 3 and T is not unimodular

89. Let A be an $n \times n$ complex matrix whose characteristic polynomial is given by
   $f(t) = t^n + c_{n-1}t^{n-1} + \ldots\ldots + c_1 t + c_0$, then
   (A) $\det(A) = c_{n-1}$
   (B) $\det(A) = c_0$
   (C) $\det(A) = (-1)^n c_{n-1}$
   (D) $\det(A) = (-1)^n c_0$

90. Let A be any $n \times n$ non singular complex matrix and let $B = (\bar{A})^t A$, where $(\bar{A})^t$ is the conjugate transpose of A. If λ is an eigen value of B, then—
   (A) λ is real and $\lambda < 0$
   (B) λ is real and $\lambda \le 0$
   (C) λ is real and $\lambda \ge 0$
   (D) λ is real and $\lambda > 0$

91. Let $T : c^n \to c^n$ be a linear operator of rank $n - z$ then—
   (A) 0 is not an eigen value of T
   (B) 0 must be an eigen value of T
   (C) 1 can never be an eigen value of T
   (D) 1 must be an eigen value of T

92. The dimension of the vector space of all 3×3 real symmetric matrices is—
   (A) 3 (B) 9
   (C) 6 (D) 4

93. Let A be a non zero upper triangular matrix all of whose eigen values are 0, then I + A is—
   (A) Invertible (B) Singular
   (C) Idempotent (D) Nilpotent

94. The eigen values of a skew symmetric matrix are—
(A) Negative
(B) Singular
(C) Of absolute value 1
(D) Purely imaginary or zero

95. Which of the following Banach spaces is not separable ?
(A) $L^1 [0, 1]$ (B) $L^\infty [0, 1]$
(C) $L^2 [0, 1]$ (D) $C [0, 1]$

96. For a subset A of a metric space, which of the following implies the other three ?
(A) A is closed
(B) A is bounded
(C) Closure of B is compact for every $B \subseteq A$
(D) A is compact

97. Let T be an arbitrary linear transformation form R^n which is not one-one, then—
(A) Rank $T > 0$ (B) Rank $T = n$
(C) Rank $T < n$ (D) Rank $T = n - 1$

98. Let T be a linear transformation form $R^3 \to R^2$ defined by $T(x, y, z) = (x + y, y - z)$. Then the matrix of T with respect to the ordered bases $\{(1, 1, 1), (1, -1, 0), (0, 1, 0)\}$ and $\{(1, 1), (1, 0)\}$ is—
(A) $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$
(C) $\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}$

99. Let the characteristics equations of a matrix M be $\lambda^2 - \lambda - 1 = 0$, then—
(A) M^{-1} does not exist
(B) M^{-1} exist but cannot be determined from the data
(C) $M^{-1} = M + 1$
(D) $M^{-1} = M - 1$

100. Consider the matrix $M = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ and let S_M be the set of 3×3 matrices N such that $MN = 0$. Then the dimension of the real vector S_M is equal to—
(A) 0 (B) 1
(C) 2 (D) 3

101. Choose the correct matching from A, B, C and D for the transformation T_1, T_2 and T_3 (mappings from R^2 to R^3) as defind in group 1 with the statements given in group 2.

Group 1

P. $T_1(x, y) = (x, x, 0)$
Q. $T_2(x, y) = (x, x + y, y)$
R. $T_3(x, y) = (x, x + 1, y)$

Group 2

1. Linear transformation of rank 2
2. Not a linear transformation
3. Linear transformation of rank 1

(A) P – 3, Q – 1, R – 2
(B) P – 1, Q – 2, R – 3
(C) P – 3, Q – 2, R – 1
(D) P – 1, Q – 3, R – 2

102. Let $M = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & -4 & 0 & 0 \end{bmatrix}$. Then—
(A) $MM^T = I$ where M^T is the transpose of M and I is the identify matrix
(B) Column vectors of M from an orthogonal system of vectors
(C) Column vectors of M from an orthonormal system of vectors
(D) $(Mx, My) = (x, y)$ for all x, y in R^4 where (,) is the standard inner product on R^4.

103. Let $M = \begin{bmatrix} 1 & 1+i & 2i & 9 \\ 1-i & 3 & 4 & 7-i \\ -2i & 4 & 5 & i \\ 9 & 7+i & -i & 7 \end{bmatrix}$. Then—
(A) M has a purely imaginary eigen values
(B) M is not diagonalizable
(C) M has eigen values which are neither real nor purely imaginary
(D) M has only real eigen values

104. Consider the matrix $M = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$ where a, b and c are non-zero real numbers. Then the matrix has—
(A) Three non-zero real eigen values
(B) Complex eigen values
(C) Two non-zero eigen value
(D) Only one non-zero eigen value

105. The minimal polynomial of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is—

(A) $(x-1)^2 (x-2)$
(B) $(x-1)(x-2)^2$
(C) $(x-1)(x-2)$
(D) $(x-1)^2 (x-2)^2$

106. The set of all $x \in R$ for which the vectors $(1, x, 0)$, $(0, x^2, 1)$ and $(0, 1, x)$ are linearly independent in R^3 is—

(A) $\{x \in R : x = 0\}$
(B) $\{x \in R : x \neq 0\}$
(C) $\{x \in R : x \neq 1\}$
(D) $\{x \in R : x \neq -1\}$

107. Consider the vector space R^3 and the maps f, $g : R^3 \to R^3$ defined by $f(x, y, z) = (x, |y|, z)$ and $g(x, y, z) = (x+1, y-1, z)$. Then—

(A) Both f and g are linear
(B) Neither f nor g is linear
(C) g is linear but not f
(D) f is linear but not g.

108. Let $M = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$. Then—

(A) M is diagonalizable but not M^2
(B) M^2 is diagonalizable but not M
(C) Both M and M^2 are diagonalizable
(D) Neither M nor M^2 is diagonalizable

109. Let M be a skew symmetric, orthogonal real matrix, the only possible eigen values are—

(A) $-1, 1$ (B) $-i, i$
(C) 0 (D) $1, i$

110. Let S and T be two linear operators on R^3 defined by $S(x, y, z) = (x, x+y, x-y-z)$
$T(x, y, z) = (x+2z, y-z, x+y+z)$. Then—

(A) S is invertible but not T
(B) T is invertible but not S
(C) Both S and T are invertible
(D) Neither S nor T is invertible

111. Let V, W and X be three finite dimensional vector spaces such that dim V = dim X. Suppose $S : V \to W$ and $T : W \to X$ are two linear maps such that to $S : V \to X$ is injective. Then—

(A) S and T are surjective
(B) S is surjective and T is injective
(C) S and T are injective
(D) S is injective and T is surjective

112. If a square matrix of order 10 has exactly 4 distinct eigen values, then the degree of its minimal polynomial is—

(A) At least 4 (B) At most 4
(C) At least 6 (D) At most 6

113. Consider the matrix $M = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$.

Then—

(A) M has no real eigen values
(B) All real eigen values of M are positive
(C) All real eigen values of M are negative
(D) M has both positive and negative real eigen values

114. Consider the real inner product space p [0, 1] of all polynomials with the inner product $\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx$. Let M = span {1} the orthogonal projection of x^2 on to M is—

(A) 1 (B) $\frac{1}{2}$
(C) $\frac{1}{3}$ (D) $\frac{1}{4}$

115. The matrix of T^{-1} with respect to the basis $\{1, x, x^2\}$ is—

(A) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

116. The dimension of the eigen space of T^{-1} corresponding to the eigen value 1 is—

(A) 4 (B) 3
(C) 2 (D) 1

Answers with Hints

1. (B)	2. (B)	3. (C)	4. (A)	5. (C)
6. (A)	7. (A)	8. (B)	9. (A)	10. (B)
11. (A)	12. (C)	13. (C)	14. (D)	15. (B)
16. (A)	17. (A)	18. (C)	19. (A)	20. (B)
21. (B)	22. (A)	23. (C)	24. (A)	25. (A)
26. (A)	27. (B)	28. (A)	29. (A)	30. (A)
31. (A)	32. (B)	33. (A)	34. (A)	35. (A)
36. (B)	37. (C)	38. (A)	39. (B)	40. (A)
41. (A)	42. (A)	43. (B)	44. (B)	45. (A)
46. (B)	47. (B)	48. (C)	49. (A)	

50. (A) Rank $T = \dim(\text{range of I}) = \dim V.$

51. (A) Null space of

$$T = \{\alpha : T\alpha = 0, \alpha \in V\} = V$$

52. (A) Range of $T = \{\beta : T\beta, \beta \in V] = \{0\}$

53. (A) Rank of $T = \dim\{\text{range of T}\} = \dim\{0\} = 0$

54. (A) Nullity $T = \dim\{\text{null space of T}\} = \dim(V)$

55. (A) Rank T + nullity

$$T = \dim V$$
$$\Rightarrow \quad \text{Rank } T < \dim V$$

56. (A) $\left(0+\frac{\beta}{\beta}\right) = \left(\frac{0}{\beta}\right)+\left(\frac{\beta}{\beta}\right)$

$$\Rightarrow \left(\frac{\beta}{\beta}\right) = \left(\frac{0}{\beta}\right)+\left(\frac{\beta}{\beta}\right)$$
$$\Rightarrow \left(\frac{0}{\beta}\right) = 0$$

57. (B) $\left(\alpha+\frac{\beta}{\beta}\right) = \left(\frac{\alpha}{\beta}\right)+\left(\frac{\beta}{\beta}\right)$

$$\because \left(\frac{\alpha}{\beta}\right) = 0$$
$$\Rightarrow \left(\alpha+\frac{\beta}{\beta}\right) = \left(\frac{\beta}{\beta}\right)$$
$$\Rightarrow \alpha+\beta = \beta$$
$$\Rightarrow \alpha = 0$$

58. (B) (a) $\left(\frac{\alpha}{\beta}\right)+\left(\frac{\gamma}{\beta}\right) = \left(\alpha+\frac{\gamma}{\beta}\right)$

(b) $\left(\frac{\alpha}{\beta}\right)-\left(\frac{\gamma}{\beta}\right) = \left(\frac{\alpha}{\beta}\right)+\left(-1\frac{\gamma}{\beta}\right) = \left(\alpha-\frac{\gamma}{\beta}\right)$

59. (C) $\left(\frac{\alpha}{\gamma}\right)+\left(\frac{\beta}{\gamma}\right) = \left(\alpha+\frac{\beta}{\gamma}\right)$

$$\Rightarrow -1+3 = \left(\alpha+\frac{\beta}{\gamma}\right)$$
$$\Rightarrow \left(\alpha+\frac{\beta}{\gamma}\right) = 2$$
$$\because \alpha+\beta = (1, 2)+(-1, 1) = (0, 3)$$
$$\therefore \left(\alpha+\frac{\beta}{\gamma}\right) = 0\cdot y_1 + 3y_2 = 3y_2 = 2$$
$$\Rightarrow y_2 = \frac{2}{3} \text{ and } y_1 = 0$$
$$\therefore y = (y_1, y_2) = \left(0, \frac{2}{3}\right)$$

60. (A) $f(x) = \det(xI + A)$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} x - \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$
$$= \begin{vmatrix} x & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$
$$= \begin{vmatrix} x & 1 \\ -1 & x \end{vmatrix} = x^2+1$$

61. (C) $f(x) = |xI - A|$

$$= \begin{vmatrix} x-3 & -1 & -1 \\ 2 & x-2 & 1 \\ -2 & -2 & x \end{vmatrix}$$
$$= x-3\begin{vmatrix} x-2 & 1 \\ -2 & x \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -2 & x \end{vmatrix} - \begin{vmatrix} 2 & x-2 \\ -2 & -2 \end{vmatrix}$$
$$= x^3 - 5x^2 + 8x - 4$$

62. (B) 63. (C)

64. (D) $\because$ $\{0\}$ is a linear dependent set

$\therefore$ It cannot be a basis

$\therefore$ $\phi \subset \{0\}$, and ϕ spans $\{0\}$

$\therefore$ $\dim\{0\} = \dim \phi = 0$

65. (C) (x, y) and $(-y, x)$ are orthogonal to each other to standard inner product.

$\because \quad \left(\frac{(x, y)}{(-y, x)}\right) = 0 = -xy + yx = 0$

66. (A) 67. (A) 68. (C)

69. (B) Let the equation be

$$\sum_{i=1}^{n} a_{ij} x_j = 0, i = 1, 2, \ldots\ldots, m.$$

Since the rank of A which is $m \times n$ matrix is r, the number of solutions (linearly independent) is $n - r$.

This is because you are looking at the annidilator of the subspace W generated by $(a_{i1}, a_{i2}, \ldots\ldots, a_{in})$.

$i = 1, \ldots\ldots, m$ vectors in R^n, which is an r dimensional vector space.

$$\dim A(W) = \dim R^n - \dim W = n - r$$

70. (D) The matrix is $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Given $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\therefore \quad a_{11} - a_{12} = 1 \quad \ldots(i)$

and $\quad a_{12} - a_{22} = -1 \quad \ldots(ii)$

and $\quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$

$\therefore \quad 2a_{11} + a_{12} = 8 \quad \ldots(iii)$

and $\quad 2a_{21} + a_{22} = 4 \quad \ldots(iv)$

solving equations (i), (ii), (iii) and (iv)

We get $\quad a_{11} = 3,$

$a_{12} = 2,$

$a_{21} = 1,$

$a_{22} = 2$

$\therefore$ Matrix is $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

71. (C) Euclidean domain is a PID but the converse is not true.

$\therefore \quad ED \subseteq PID$

PID and ED are unique factorisation domains

$\therefore$ ED $\subseteq$ PID $\subseteq$ UFD, UFD need not be PID or ED.

72. (B) Since $f(x) = x^6$, we take the hermite interpolating polynomial to be even function.

$$P(x) = a_0 + a_1x^2 + a_2x^4$$

$$P(0) = f(0),$$

$$P'(0) = f'(0)$$

$P(1) = f(1)$ and $P'(1) = f'(1)$ yield the equations

$a_0 = 0, a_1 + a_2 = 1, 2a_1 + 4a_2 = 6$

Solving, we get

$$a_1 = -1, a_2 = 2$$

$\therefore \quad P(x) = 2x^4 - x^2.$

73. (A) $\quad A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4{\cdot}5 \\ 5 \\ -0{\cdot}5 \end{bmatrix}$

where, $\quad A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$A = D + L + U$$

where, $\quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H = -D^{-1}(L + U)$$

$$= \begin{bmatrix} 0 & -\frac{2}{3} & 0 \\ -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Characteristic equation is,

$$\lambda^3 - \lambda\frac{11}{18} = 0$$

$\therefore \quad \lambda = 0, \pm\sqrt{\frac{11}{18}}$

$\therefore \quad P(H) = \sqrt{\frac{11}{18}}$

and $\quad \frac{2}{\mu^2}\sqrt{1 - \mu^2} = 1{\cdot}23$ (approx.)

74. (B) 75. (D) 76. (C) 77. (D) 78. (A)

79. (B) By a theorem which states "If A and B be $m \times n$ and $n \times p$ matrices respectively, then rank (AB) $\leq$ min (rank A, rank B)",

We have $\quad$ rank (P) $= n$

rank (Q) $=$ P

Since P in a $m \times n$ matrix with rank n, $n \leq m$ and Q is $n \times p$ matrix with rank P. P $\leq n$

$\therefore \quad$ rank (PQ) $= \min(n, p) =$ P

80. (D) It $\lambda = 0$ is an eigen value of P, then eigen value of a is $\frac{1}{\lambda}$ *i.e.*, ∞. It is not possible.

Hence neither P nor Q.

81. (D)	82. (D)	83. (B)	84. (A)	85. (B)
86. (C)	87. (A)	88. (C)	89. (B)	90. (D)
91. (B)	92. (B)	93. (D)	94. (D)	95. (C)
96. (D)	97. (D)			

98. (B) $f_1 = (1, 1, 1), f_2 = (1, -1, 0), f_3 = (0, 1, 0)$, $g_1 = (1, 1), g_2 (1, 0)$

$$\begin{aligned}(a, b) &= xg_1 + yg_2\\ &= x(1, 1) + y(1, 0)\\ &= (x + y, x + 0.y)\end{aligned}$$

$\left.\begin{array}{l}\therefore\ x + y = a\\ \therefore\ x + 0.y = b\end{array}\right[$ and $x = b$ and $y = a - b$

$$\begin{aligned}\therefore \quad (a, b) &= bg_1\ (a - b)\ g_2\\ T(x, y, z) &= (x + y, y - z)\\ F(f_1) &= F(1, 1, 1) = (2, 0)\\ &= 0\cdot g_1 + 2g_2\\ F(f_2) &= F(1, -1, 0) = (0, -1)\\ &= -g_1 + g_2\\ F(f_3) &= F(0, 1, 0) = (1, 1)\\ &= g_1 + 0\cdot g_2\\ \therefore \quad [F]_f^g &= \begin{bmatrix} 0 & -1 & 1\\ 2 & 1 & 0\end{bmatrix}\end{aligned}$$

99. (D) Given

$$\begin{aligned}\lambda^2 - \lambda - 1 &= 0\\ \Rightarrow \quad 1 &= \lambda^2 - \lambda\\ \Rightarrow \quad \lambda^{-1} &= \lambda - 1\\ \Rightarrow \quad M^{-1} &= M - 1\end{aligned}$$

100. (D)

101. (A) P : $\quad T = \begin{bmatrix} 1 & 0\\ 1 & 0\\ 0 & 0\end{bmatrix}$

$\Rightarrow \quad \text{Rank} = 1$

Q : $\quad T_2(x, y) = (x, x + y, y)$

Linear transformation of rank 2.

102. (B) Let A_1 and A_2 be two complex n-vectors, then A_1 is said to be orthogonal to A_2, if $(A_1, A_2) = 0$

i.e. $\quad A_1^T A_2 = 0$

Here, $\quad A_1 = \begin{bmatrix} 0\\ 2\\ 0\\ 0\end{bmatrix}$, $\quad A_2 = \begin{bmatrix} 0\\ 0\\ 0\\ -4\end{bmatrix}$

$$A_1^T = [0, 2, 0, 0]$$

$$\therefore \quad (A_1 A_2) = [0\ \ 2\ \ 0\ \ 0]\begin{bmatrix} 0\\ 0\\ 0\\ -4\end{bmatrix} = 0$$

Similarly

$$(A_2 A_3) = (A_3 A_4) = 0$$

Thus column vectors from the orthogonal set of system.

103. (D) Since M is Hermitian matrix, so it will have only real eigen values.

104. (A)

105. (C) The monic polynomial of lowest degree that arihilates a matrix A is called the minimal polynomial of A.

$$\text{Here } [A - \lambda I] \begin{vmatrix} 1-\lambda & 0 & 0 & 0\\ 1 & 1-\lambda & 0 & 0\\ 0 & 0 & 2-\lambda & 0\\ 0 & 0 & 0 & 2-\lambda\end{vmatrix}$$

$$= (1 - \lambda)^2 (2 - \lambda)^2$$

$\therefore$ Roots of the equation $|A - \lambda I| = 0$ are 1, 1, 2, 2.

Each characteristic root of A is also a root of minimal polynomial. If $m(x)$ is the minimal polynomial of A, then both $(x - 1)$ and $(x - 2)$ are factors of $m(x)$.

106. (A) $a_1(1, x, 0) + a_2(0, x^2, 1) + a_3(0, 1, x) = 0$

This is for linearly independence

on comparing, we get

$$\begin{aligned} a_1 &= 0 & \dots\text{(i)}\\ a_1 x + a_2 x^2 + a_3 &= 0 & \dots\text{(ii)}\\ a_2 x^2 + a_3 &= 0 & \dots\text{(iii)}\\ \Rightarrow \quad a_2 + a_3 x &= 0 & \end{aligned}$$

from equations (i), (ii), (iii)

$a_1 = a_2 = a_3 = 0$ for linear independent

This is true only for $x = 0$.

107. (C) Let $\quad f = a_1 x + a_2 |y| + a_3 z$

Here f are two functions one for y and other for $-y$.

$$f = a_1 x + a_2 y + a_3 z$$

and $\quad f' = a_1 x - a_2 y + a_3 z$

Now $\quad g = a_1(x+1) + a_2(y-1) + a_3 z$
$= a_1 x + a_2 y + a_3 z + (a_1 - az)$
$= f + (a_1 - a_2)$

i.e., g is linear but not f.

108. (D) $\quad M = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$

Then $\quad M' = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ 3 & 5 & 9 \end{bmatrix}$

$\Rightarrow MM' \neq I$

Now $\quad M^2 = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$

$= \begin{bmatrix} 1 & 15 & 45 \\ 0 & 16 & 45 \\ 0 & 0 & 81 \end{bmatrix}$

$M^2 M'^2 \neq I$

Hence neither M nor M^2 is diagonalizable.

109. (A) Let $\quad M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

But M is skew symmetric matrix

i.e. $\quad M' = -M$

$\Rightarrow \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ -b_1 & -b_2 & -b_3 \\ -c_1 & -c_2 & -c_3 \end{bmatrix}$

On comparing, we get

$a_1 = -a_1 \Rightarrow a_1 = 0$

$a_2 = -b_1$ and so on

Then $\quad M = \begin{bmatrix} 0 & -b_1 & -c_1 \\ -a_2 & 0 & -c_2 \\ -a_3 & -b_3 & 0 \end{bmatrix}$

But M is also orthogonal real matrix

i.e. $\quad MM' = I$

$\Rightarrow \begin{bmatrix} b_1^2 + c_1^2 & c_1 c_2 & b_1 b_3 \\ c_1 c_2 & a_2^2 + c_2^2 & a_2 a_3 \\ b_1 b_3 & a_2 a_3 & a_3^2 + b_3^2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

on comparing, we get

$M = \begin{bmatrix} b_1^2 + c_1^2 & 0 & 0 \\ 0 & a_2^2 + c_2^2 & a_2 a_3 \\ b_1 b_3 & a_1 a_3 & a_3^2 + b_3^2 \end{bmatrix}$

For eigen values

$|M - \lambda I| = 0$

Hence, $\quad \lambda = -1, 1.$

110. (A) Let $\quad \alpha = (a_1, b_1, c_1),$
$\beta = (a_2, b_2, c_2)$

then $\quad S(\alpha) = S(\beta)$

$\Rightarrow (a_1, a_1 + b_1, a_1 - b_1 - c_1)$
$= (a_2, a_2 + b_2, a_2 - b_2 - c_2)$

$\Rightarrow a_1 = a_2,$

$a_1 + b_1 = a_2 + b_2,$

$a_1 - b_2 - c_1 = a_2 - b_2 - c_2$

$\Rightarrow a_1 = a_2, b_1 = b_2, c_1 = c_2$

Hence, S is one-one.

Therefore, S must be onto also and thus S is invertible.

Similarly,

$T(\alpha) = T(\beta)$

$\Rightarrow (a_1 + 2c_1, b_1 - c_1, a_1 + b_1 + c_1)$
$= (a_2 + 2c_2, b_2 - c_2, a_2 + b_2 + c_2)$

$\Rightarrow a_1 \neq a_2, b_1 \neq b_2, c_1 \neq c_2$

$\therefore$ T is not one-one.

Hence, T is not onto and thus T is not invertible.

111. (C) $\quad \dim V = \dim X$

where V, W and X be three finite dimensional vector spaces.

$S : V \rightarrow W,$

$T : W \rightarrow X$

such that $ToS \rightarrow V \rightarrow X$ is one-one

Hence, S and T both are injective.

112. (A) A square matrix of order 10 has exactly 4 distinct eigen values, then the degree of its minimal polynomials must be at least 4.

113. (A) $\quad |M - \lambda I| = \begin{vmatrix} -\lambda & 1 & 2 & 0 \\ 1 & -\lambda & 1 & 0 \\ 2 & 1 & -\lambda & 2 \\ 0 & 0 & 2 & -\lambda \end{vmatrix}$

On expanding

$= -\lambda \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix}$

$-1 \begin{vmatrix} 1 & 1 & 0 \\ 2 & -\lambda & 2 \\ 0 & 2 & -\lambda \end{vmatrix} + 2 \begin{vmatrix} 1 & -\lambda & 0 \\ 2 & 1 & 2 \\ 0 & 0 & -\lambda \end{vmatrix}$

$= -\lambda^4 - 9\lambda^2 - 2\lambda + 4$

$$\Rightarrow \quad |M - \lambda I| = 0$$
$$\Rightarrow \quad \lambda^4 + 9\lambda^2 + 2\lambda - 4 = 0,$$

$\Rightarrow$ M has no real eigen values.

114. (C)

115. (B) $T(x_0, x_1, x_2) = (x_0, x_0 + x_1, x_0 + x_1 + x_2)$

Let basis are (1,0,0) $(0,x,0)$ and $(0,0,x^2)$

Then
$$T(1, 0, 0) = (1, 1, 1)$$
$$T(0, x, 0) = (0, x, x)$$
$$T(0, 0, x^2) = (0, 0, x^2)$$
$$\Rightarrow \quad T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & x & x \\ 0 & 0 & x^2 \end{bmatrix},$$
$$|T| = 1(x^3) = x^3.$$

At $x = 1$,
$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and $|T| = 1.$

Cofactors of T

$T_{11} = 1, T_{12} = 0, T_{13} = 0$

$T_{21} = -1, T_{22} = 1, T_{23} = 0$

$T_{31} = 0, T_{32} = -1, T_{33} = 0$

$\therefore$ adj T = Transpose of Co-factors matrix.
$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $T^{-1} = \frac{1}{|T|}\text{adj } T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

116. (B)
$$|T^{-1} - \lambda I| = \begin{bmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$
$$= (1-\lambda)^3$$

Hence, dimension of eigen space of T^{-1} is 3.

●●

Complex Analysis

Complex Number—A complex number z is an order pair (x, y) of real numbers x and y.

$$z = (x, y) = x + iy$$
$$\text{Re } z = x \text{ (Real part)},$$
$$\text{Im } z = y \text{ (imaginary part)}$$

and $i^2 = -1$, *i.e.* $i = \sqrt{-1}$ (imaginary unit)

(a) $x + iy = a + ib \Leftrightarrow x = a$ and $y = b$.

(b) For $\quad z = x + iy.$

If $x = 0$, then $z = iy$ (pure imaginary)

If $y = 0$, then $z = x$ (pure real)

Addition of complex numbers—

Let $\quad z_1 = x_1 + iy_1$

and $\quad z_2 = x_2 + iy_2$, then

$$z_1 + z_2 = (x_1 + x_2) + i\,(y_1 + y_2)$$

Multiplication of complex numbers—

$$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

Subtraction of complex numbers—

$$z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$$

Division of complex numbers—

$$z = \frac{z_1}{z_2} = x + iy$$

where $\quad x = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2},$

$$y = \frac{x_2y_1 + x_1y_2}{x_2^2 + y_2^2},\ z_2 \neq 0$$

Practical rule—

$$z = \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$
$$= \left(\frac{x_1 + iy_1}{x_2 - iy_2}\right) \times \left(\frac{x_2 - iy_2}{x_2 + iy_2}\right)$$
$$= x + iy$$

Complex Conjugate Number—

The complex conjugate of the number $z = x + iy$ is $\bar{z} = x - iy$

(a) $\quad \text{Re } z = x = \dfrac{1}{2}(z + \bar{z})$

and $\quad \text{Im } z = y = \dfrac{1}{2i}(z - \bar{z})$

(b) When z is real, $z = x$, then $z = \bar{z}$

(c) $\quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2,$

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

(d) $\quad \overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2,$

$$\left(\frac{\bar{z}_1}{\bar{z}_2}\right) = \frac{\bar{z}_1}{\bar{z}_2}$$

Polar Form of Complex Numbers

Let (x, y) be the Co-ordinate in castesian co-ordinate system and (r, θ) be the co-ordinates in polar co-ordinates, then

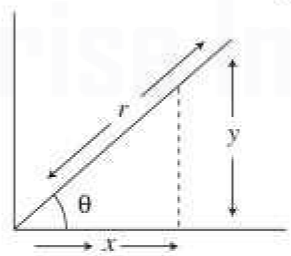
$$x = r\cos\theta,\ y = r\sin\theta$$

$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta)$

Here $\quad r = |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$

(absolute value/modulus of z)

$$\theta = arg\ z = \tan^{-1}\left(\frac{y}{z}\right)$$



Triangle inequality—

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Generalized triangle inequality—

$|z_1 + z_2 + \ldots\ldots + z_n|$

$$\leq |z_1| + |z_2| + \ldots + |z_n|$$

Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

and $z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

then, $z_1 z_2 = r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]$

(a) $|z_1 z_2| = |z_1| \, |z_2|$

(b) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(c) $\arg \dfrac{z_1}{z_2} = \arg z_1 - \arg z_2.$

De Moivre's formula—$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$.

n^{th} root of $z - \sqrt[n]{z} = \sqrt[n]{r}$

$$\left(\cos \frac{\theta + 2K\pi}{n} + i \sin \frac{\theta + 2K\pi}{n}\right)$$

$$K = 0, 1, 2, \ldots\ldots n - 1$$

Some Elementary Functions

Single valued functions

1. $z^n = (x + iy)^n, n \in N \; (z \neq 0 \text{ if } n < 0)$
2. $e^z = e^x e^{iy} = e^x (\cos y + i \sin y)$, period $= 2\pi i$
3. $\cosh z = \dfrac{1}{2} (e^z + e^{-z})$,

   $\sinh z = \dfrac{1}{2} (e^z - e^{-z})$

   $\tanh z = \dfrac{\sin hz}{\cos hz}, z \neq \left(K + \dfrac{1}{2}\right)\pi i$

   $\coth z = \dfrac{\cosh z}{\sinh z}, z \neq K\pi i$
4. $\cos z = \dfrac{1}{2} (e^{iz} + e^{-iz})$

   $\sin z = \dfrac{1}{2i} (e^{iz} - e^{-iz})$

   $\tan z = \dfrac{\sin z}{\cos z}, z \neq \left(K + \dfrac{1}{2}\right)\pi i$

   $\cot z = \dfrac{\cos z}{\sin z}, z \neq K\pi i$

   $\cos iz = \cosh z, \sin iz = i \sinh z$

   $\cosh iz = \cos z, \sinh iz = i \sin z$
5. Multiple – Valued Functions—

   $\log z = \log |z| + i \arg z$

   $= \text{In}\, r + i (\theta + 2n\pi)$

   Principal Branch :

   $\log z = \log r + i\theta, -\pi < \theta < \pi$
6. $a^z = e^{a \log z}$, a non-integer

   If $a = \dfrac{p}{q} \in \theta$ a rational number,

   then z^a is q-valued.

   If $a \notin \theta$ then z^a is as valued

   e.g., $\log 2i = \log |2i| + i \arg 2i$

   $= \log 2 + i\left(\dfrac{\pi}{2} + 2n\pi\right)$

   $(2i)^i = e^{i \log 2i} = e^{-(\pi/2 + 2n\pi) + i \log 2}$

   $= e^{-\pi/2 - 2n\pi}$ $[\cos (\log 2) + i \sin (\log 2)]$

Analytic functions

Differentiability—If G is an open set in $\mathbb{C}$ and $f : G \to \mathbb{C}$.

Then f is differentiable at a point $a \in G$, if

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \text{ exist}$$

If f is differentiable at each point of G, then f is differentiable on G and $f : G \to \mathbb{C}$.

Continuously differentiable—If $f' : G \to \mathbb{C}$ is continuous analytic function (Holomorphic function)—

A function $f : G \to \mathbb{C}$ is analytic if f is continuously differentiable on G.

Branch of logarithm—If G is an open connected set in $\mathbb{C}$ and $f : G \to \mathbb{C}$ is continuous function such that

$$z = \exp f(z)$$

for all $z \in G$, then f is a branch of logarithm.

Cauchy–Riemann equation—

If $\mu = \mu(x, y)$

and $v = v(x, y)$; then

$$\frac{\delta\mu}{\delta x} = \frac{\delta v}{\delta y}$$

and $$\frac{\delta\mu}{\delta y} = -\frac{\delta v}{\delta x}$$

Harmonic function—If $\mu(x, y)$ and $\dfrac{\delta^2\mu}{\delta x^2} + \dfrac{\delta^2\mu}{\delta y^2} = 0$.

Harmonic Conjugate—If $\mu : G \to \mathbb{R}$ is harmonic and $r : G \to \mathbb{R}$ such that $f = \mu + iv$ is analytic in G, then v is the harmoinc conjugate of μ.

Some Important Theorems

1. If $f : G \to \mathbb{C}$ is differentiable at a point $a \in G$. Then f is continuous at a.
2. If f and g are analytic on G and Ω respectively and suppose that f (G) C Ω, then gof is analytic on G and.
$$(gof)'(z) = g'(f(z)) f'(z) \text{ for all } z \in G.$$
3. If f and g analytic in G. Then fg and $f + g$ are also analytic.
4. If f and g are analytic in G and g does not vanish in G, then $f/_g$ is analytic.
5. If $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ have radius of convergence R > 0, then
   (a) For each $K \geq 1$ the series $\sum_{n=0}^{\infty} n(n-1)\ldots(n-K+1)a_n (z-a)^{n-k}$ has radius of convergence R.
   (b) For $n \geq 0$, $a_n = \frac{1}{n!} f(n)(a)$.
6. If the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence R = 0. Then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ is analytic in open boll $\beta(a, R)$
7. If G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in G$, then f is constant.
8. If $G \subset \mathbb{C}$ is open and connected and f is a branch of log z on G, then the totality of branches of log z are the functions $f(z) + 2\pi ki$, $K \in \mathbb{Z}$.
9. If G and Ω are open subsets of $\mathbb{C}$. Suppose f: (F) $\to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ are continuous functions such that $f(G) \subset \Omega$ an $g(f(z)) = z$ for all $z \in G$. If g is differentiable and $g'(z) \neq 0$, f is differentiable and $f'(z) = \frac{1}{g(f(z))}$, then if g is analytic f is analytic too.
10. A branch of the logarithm function is analytic and its derivative is z^{-1}.
11. If μ and v are real valued function defined on a region G and suppose that μ and v have continuous partial derivatives. Then $f : G \to \mathbb{C}$ defined by $f(z) = \mu(z) + iv(z)$ is analytic iff μ and v satisfies Cauchy-Riemann equations.
12. Suppose G is either the whole plane $\mathbb{C}$ or some open disk. If $\mu : G \to \mathbb{R}$ is harmonic function. Then μ has a harmonic conjugate.

The Conformal Mappings Theorem

1. Let G be a region and f be an analytic function on G with zeros, $a_1, a_2, \ldots\ldots a_n$. If y is a closed rectifiable curve in G which does not pass through any point a_k and If $y = 0$, then
$$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} n(y, a_k)$$
2. If G is a region, f is analytic function on G with $a_1, \ldots, a_k \in$ G and y is a closed rectifiable curve in G which does not pass through any point a_k and if $y = 0$ an $f(z) = \alpha$, then
$$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)-\alpha} dz = \sum_{k=1}^{m} n(y, a_k)$$
3. If f is analytic in B(a, R) and let $\alpha = f(a)$. If $f(z) - \alpha$ has a zero of order m at $z = a$, then there is an $\in > 0$ and $\delta > 0$ such that $|\delta - \alpha| < \delta$, then equation $f(z) = y$ has exactly m simple roots in B(a, $\in$).
4. Conformal mapping theorem—Let G be a region and suppose f is a non constant analytic function on G. Then for any conformal set U in G, f(U) is conformal.
5. If $f : G \to \mathbb{C}$ is one analytic and $f(G) = \Omega$, then $f^{-1} : \Omega \to \mathbb{C}$ is analytic and $f^{-1}(w) = [f(z)]^{-1}$ where $w = f(z)$
6. Goursat is theorem—Let G be an conformal set and $f : G \to \mathbb{C}$ be differentiable function; then f is analytic on G.

The Cauchy's Integral Theorem and Formula

1. If y is a rectifiable curve and suppose ϕ is a function defined and continuous on $\{y\}$. For each $m \geq 1$, Let $F_m(z) = \int_y \phi(w)(w-z)^{-m} dw$ for $z \notin y$. Then F_m is analytic on $\mathbb{C} - \{y\}$ and $F_m(z) = mF_{m+1}(z)$ for each m.
2. Cauchy's integral formula (first version)—
   Let G be an open subset of the plane and f : G $\to \mathbb{C}$ an analytic function. If y is a closed rectifiable curve in G. Such that $n(y, w) = 0$ for all $w \in \mathbb{C} \ldots$ G, then for $a \in G - \{y\}$.
$$n(y : a) = \frac{1}{2\pi i}\int_y \frac{f(g)}{z-a} dz.$$

3. Cauchy's integral formula (second version)— Let G be an open subset of the plane and f : G $\to \mathbb{C}$ an analytic function. If $y_1, \ldots\ldots, y_m$ are closed rectifiable curves in G such that $n(y_i, w) + \ldots + n(y_m, w) = 0$ for all $w \in \mathbb{C} \to$ G then for $a \in$ G – {y}

$$f(a) = \sum_{k=1}^{m} n(y_k;a)$$
$$= \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{y_k} \frac{f(z)}{z-a} dz$$

4. Cauchy's theorem (First version)—

If G is an open subset of the plane and f : G $\to \mathbb{C}$ an analytic function. If $y_1, \ldots, y_m$ are closed rectifiable curves in G such that $n(y_1; w) + n(y_2; w) + \ldots\ldots + n(y_m; w) = 0$ for all $w \in \mathbb{C} \to$ G, then

$$\sum_{k=1}^{m} \int_{y_k} f = 0$$

5. Let G be an open subset of the plane and f: G $\to \mathbb{C}$ an analytic function. If $y_1 \ldots\ldots y_n$ are closed rectifiable curves in G such that $n(y_1; w) + n(y_2; w) + \ldots\ldots + n(y_m; w) = 0$ for all $w \in \mathbb{C}$ – G, then $a \in$ G – {y} and K ≥ 1.

6. Let G be an open set and f : G $\to \mathbb{C}$ an analytic function. If y is closed rectifiable curve in G such that $n(y; w) = 0$ for all $w \in \mathbb{C}$ —G, then for $a \in$ G – {y}

$$f^{(K)}(a)\, n(y; a) = \frac{\underline{|K}}{2\pi i} \int_{y} \frac{f(z)}{(z-a)^{k+1}} dz$$

7. Morera's Theorem—

Let G be an region and f : G $\to \mathbb{C}$ be an analytic function such that $\int_T f = 0$ for every T, a Triangular path in G; then f is analytic in G.

Some Important Theorems

1. If G is an open set which is a star shaped, if y_0 is the curve which is constantly equal to a then every closed rectifiable curve in G is homotopic to y_0

2. Cauchy's theorem (Second version)—If f : G $\to \mathbb{C}$ is and analytic function and y is a closed rectifiable curve in G such that y —0,

then $\int_y f = 0$

3. Cauchy's theorem (Third version)—If y_0 and y_1 are two closed rectifiable curves in G and $y_0 \sim y_1$, then $\int_{y_0} f = \int_{y_1} f$ for every function f analytic on G.

4. If y is a closed rectifiable curve in G such that $y \sim 0$ then $n\,(y : w) = 0$ for all $w \in \mathbb{C} \to$ G

5. Cauchy's theorem (Fourth version)—If G is simply connected then $\int_y f = 0$ for every closed rectifiable curve and every analytic function f.

6. If G is simply connected and f : G $\to \mathbb{C}$ is anlytic in G then f has a primitive in G.

7. If G is simply connected and f : G $\to \mathbb{C}$ an analytic function in G such that $f(z) \neq 0$ for any $z \in$ G, then there is an analytic function g : G $\to \mathbb{C}$ such that $f(z) = \exp g(z)$. If $z_0 \in$ G and $e^{w_0} = f(z_0)$, we have $g(z_0) = w_0$.

The Liouville's theorem—If f is a bounded entire function then f is constant.

Given f a bounded function

$\therefore \quad |f(z)| \leq$ M for all $z \in \mathbb{C}$.

By Cauchy's estimate theorem since f is bounded and analytic we have,

$$|f^{(n)}(z)| \leq \frac{LnM}{R}$$

$$\Rightarrow \quad |f'(z)| \leq \frac{M}{R}, \text{ since R is arbitrary,}$$

we have

$$|f'(z)| = 0$$

$\therefore f(z)$ is constant.

The Maximum Modulus Principle

1. Maximum modulus principle (First version)—If f is analytic in a region G and $a \in$ G with $|f(a)| \geq |f(z)|$ for all $z \in$ G, then f must be a constant function.

2. Maximum modulus principle (Second version)—If G is a bounded open set in $\mathbb{C}$ and f is continuous function on $\bar{G}$ which is analytic in G then max $\{|f(z)| : z \in \bar{G}\}$ = max $\{|f(z)| : z \in \delta G\}$

3. Maximum modulus principle (Third version)—If G is a region in $\mathbb{C}$ and f an analytic function on G suppose M is a constant such that $\lim_{z \to a} |f(z)| \leq$ M for all a

$\in \delta_\infty$ G (δ_∞ G is the boundary of G in $\mathbb{C}_\infty$) then $|f(z)| \le$ M for all $z \in$ G.

4. Schwarz's lemma—If D = $\{z : |z| < 1\}$ and suppose f is analytic on D with
(a) $|f(z)| \le i$ for $z \in$ D
(b) $f(0) = 0$
Then $|f'(0)| \le 1$ and $|f(z)| \le |z|$ for all z in the disk D. Moreover if : $f'(0) = 1$ or $|f(z)| = |z|$ for some $z \ne 0$, then there is a constant C, $|c| = 1$, such that
$$f(w) = cw \text{ for } w \in D.$$
5. If $|a| < 1$ then ϕ_a is a one-one map of D = $\{z : |z| < 1\}$ onto itself, the inverse of ϕ_a is ϕ_{-ar}. Further more, ϕ_a maps δD onto δD, $\phi_a(a) = 0$, $\phi'_a(0) = 1 - |a|^2$ and $\phi'_a(a) = (1 - |a|^2)^{-1}$
6. If f : D $\to$ D is one one analytic map of D onto itself and suppose $f(a) = 0$, then there is a complex number C with $|c| = 1$ such that $f = c\,\phi_a$.
7. A function $f : [a, b] \to$ R is convex iff the set A = $\{(x, y) : a \le x \le b$ and $f(x) \le y\}$ is convex.
8. A function $f : [a, b] \to$ R is convex iff for any points $x_1 \ldots\ldots x_n$ in $[a, b]$ and real numbers $t_1, \ldots\ldots t_n \ge 0$ with $\sum_{k=1}^{n} t_k = 1$
$$f\left(\sum_{k=1}^{n} t_k x_k\right) \le \sum_{k=1}^{n} t_k f(x_k)$$
9. A set A $\subset \mathbb{C}$ is convex iff for any points $z_1, \ldots\ldots z_n \in$ A and real numbers $t_1 \ldots\ldots t_n \ge 0$ with
$$\sum_{k=1}^{n} t_k = 1 \text{ and } \sum_{k=1}^{n} t_k z_k \in A$$
10. A differentiable function f on $[a, b]$ is convex iff f is increasing.
11. If $a < b$ and G is the vertical slrip $\{x + iy : a < x < b\}$. Suppose $f : \overline{G} \to \mathbb{C}$ is continuous and f is analytic in G. If M : $[a, b] \to \mathbb{R}$ then M(x) = sup $\{|f(x + iy)| : -\infty < y < \infty\}$ and $|f(z)| <$ B for all $z \in$ G, then log M(x) is a convex function.
12. **Hadamard's three circle theorem**—
If $0 < R_1 < R_2 < \infty$ and suppose f is analytic on ann $(0; R_1, R_2)$. If $R_1 < r < R_2$, define M(r) = max $\{|f(re^{i\theta}| : 0 \le 0 \le 2\pi\}$. Then for $R_1 < r_1 \le r \le r_2 < R_2$,
$$\log M(r) \le \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2)$$
13. **Phragmen–Lindel of theorem**—
If G is simply connected region and f is an analytic function on G. Suppose there is an analytic function ϕ : G $\to \mathbb{C}$ which never vanishes and is bounded on G. If M is a constant and δ_∞G = A $\cup$ B such that
(a) for every $a \in$ A, $\lim_{z \to a} \sup |f(z)| \le$ M.
(b) for every $b \in$ B and $n > 0$; $\lim_{z \to a} \sup |f(z)|\, |\phi(z)|^n \le$ M, then $|f(z)| \le$ M for all $z \in$ G.
14. If $a \ge \frac{1}{2}$ and G = $\{z : |arg\, z| < \frac{\pi}{2a}\}$. Suppose f is analytic on G and there is a constant M such that $\lim_{z \to w} \sup |f(z)| \le$ M for all $w \in \delta$G. If there are positive constant P and $b < a$ such that $|f(z)| \le$ P exp ($|z|\ b$) for all z with $|z|$ sufficiently large then $|f(z)| \le$ M for all $z \in$ G.

Taylor and Laurent Series—

Taylor Series—A taylor series of a function $f(z)$ is
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
where $$a_n = \frac{1}{\underline{|n}} f(n)(z_0)$$
or $$a_n = \frac{1}{2\pi i} \phi_c \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$
with positive orientation around a simple closed contour c that contains z_0 in its interior and $f(z)$ is analytic on and everywhere inside c.

Taylor's series with reminder R_n—
$$f(z) = \sum_{k=0}^{n} \frac{f(k)(z_0)}{\underline{|K}} (z - z_0)^K + R_n(z)$$
where $$R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \oint_c \frac{f(\xi)}{(\xi - z_0)^{n+1}} (\xi - z)\, d\xi$$

Maclaurin series—A Taylor series with centre $z_0 = 0$.

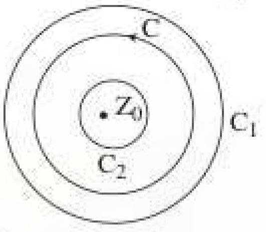
Laurent Series—If $f(z)$ is analytical on two co-centric circles c_1 and c_2 with centre z_0 and in the annulus between them then $f(z)$ can be represented by the laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z-z_0)^n}$$

$$= \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

Consisting of non-negative powers and negative power (principle part). The coefficients of this series are the integrals.

$$c_n = a_n = \frac{1}{2\pi i} \oint_c \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$$



and $$c_{-n} = b_n = \frac{1}{2\pi i} \oint_c (\xi - z_0)^{n-1} f(\xi) d\xi$$

with positive orientation around any simple closed contoure, that lies in the annulus and encircles the inner circle.

Th.—The Laurent series of a given analytic function in its annulus of convergence is unique. However $f(z)$ may have different laurent series in two annuli with the same centre.

Residue theorem and applications for evaluating real integrals—

Residue—If $f(z)$ is analytic and have non removable singularity at $z = z_0$

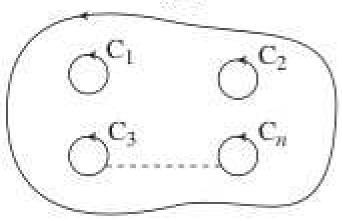
Then $f(z)$ has the laurent series representation

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-z_0)^n$$

The coefficient c_{-1} of $\frac{1}{(z-z_0)}$ is called the residue of f at z_0, *i.e.* $\operatorname{Res}_{z=z_0} f(z) = c_{-1}$

Cauchy's Residue theorem—Let D be a simple connected domain, and let c be a simple closed positively oriented contour that lies in D. If f is analytic inside c and on c, except at the points $z_1, z_2, \ldots z_n$ that lies inside c, then

$$\oint_c f(z) dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res} f(z)\ z = z_0$$



Residues at Singularities

1. If $f(z)$ has a removable singularity at z_0, then $a_n = 0$ for $n = 1, 2, \ldots$ and $\operatorname{Res}_{z=z_0} f(z) = 0$
2. If $f(z)$ has a simple pole at z_0,
$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} f(z)(z-z_0)$$
3. If $f(z)$ has a pole of order 2 at z_0,
$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} \frac{d}{dz}(z-z_0)^2 f(z)$$
4. If $f(z)$ has a pole of order K at z_0,
$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{\lfloor K-1} \lim_{z \to z_0} \frac{d^{K-1}}{dz^{K-1}} (z-z_0)^K f(z)$$
5. If $f(z)$ and $g(z)$ have an isolated singularity at z_0, then
$$\operatorname{Res}_{z=z_0} [(f+g)(z)] = \operatorname{Res}_{z=z_0} f(z) + \operatorname{Res}_{z=z_0} g(z)$$
6. If f and g are analytic at z_0, $f(z_0) \neq 0$, $g(z_0)$ has simple zero, then
$$\operatorname{Res}_{z=z_0} [(f/g)(z)] = \frac{f(z_0)}{g'(z_0)}.$$

Some Solved Examples

Example 1. Let $\mu - v = (x-y)(x^2 + 4xy + y^2)$ and $f(z) = \mu + iv$ is an analytic function. Find $f(z)$ in terms of z.

Solution : Given

$$f(z) = \mu + iv,$$

then $$if(z) = i\mu - v$$

$$\Rightarrow \quad (1+i) f(z) = \mu - v + i(\mu + v)$$
$$= U + iv \text{ (Say)}$$

since $$U = \mu - v$$
$$= (x-y)(x^2 + 4xy + y^2)$$

$$\Rightarrow \quad \frac{\delta U}{\delta x} = \frac{\delta \mu}{\delta x} - \frac{\delta v}{\delta x}$$
$$= x^2 + 4xy + y^2 + (x-y)(2x+4y)$$
$$= 3x^2 + 6xy - 3y^2 = \phi_1(x, y)$$

and $$\frac{\delta U}{\delta y} = \frac{\delta \mu}{\delta y} - \frac{\delta v}{\delta y}$$
$$= -(x^2 + 4xy + y^2) + (x-y)(4x+2y)$$
$$= 3x^2 - 6xy - 3y^2 = \phi_2(x, y)$$

Since, $(1+i)f(z)$

$$= \int [\phi_1(z,0) - i\,\phi_2(z,0)]dz + c$$

$$= \int (3z^2 - i\,3z^2)dz + c$$

$$= \int 3(1-i)z^2 dz + c$$

$$= (1-i)z^3 + c$$

$$\therefore \quad f(z) = \frac{1-i}{1+i}z^3 + \frac{c}{1+i}$$

$$= -iz^3 + d.$$

Example 2. Prove that if $f : G \to \mathbb{C}$ is analytic and $a \in G$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

for $|z-a| < R$ where $R = d(a, \delta G)$

Solution : Since $R = d(a, \delta G)$ we have an open ball centered at $a \in G$. such that $B(a, R) \subset G$.

If f is analytic in open ball $B(a, R)$ then it can be expressed as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

for $|z-a| < R$ where

$$a_n = \frac{1}{\underline{|n}} f(n)(a).$$

Example 3. Evaluate the following integral $\int_c \tan z\, dz$ where c is the circle $|z| = 2$

Solution : The poles of $f(z) = \frac{\sin z}{\cos z}$ are $\cos z = 0$, *i.e.* $z = (2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$, $z = \frac{\pi}{2}$ and $-\frac{\pi}{2}$ only poles that are within the given circle.

$$\therefore \text{Res}\, f(\pi/2) = \lim_{z \to \frac{\pi}{2}} \frac{\sin z}{\frac{d}{dz}(\cos z)}$$

$$= \lim_{z \to \frac{\pi}{2}} \left(\frac{\sin z}{-\sin z}\right) = -1$$

Similarly

$$\text{Res}\, f\left(-\frac{\pi}{2}\right) = \lim_{z \to -\frac{\pi}{2}} \frac{\sin z}{\frac{d}{dz}(\cos z)} = -1$$

Hence, by residue theorem.

$$\int_c f(z)\, dz = 2\pi i \left\{\text{Res}\, f\left(\frac{\pi}{2}\right) + \left\{\text{Res}.\, f\left(-\frac{\pi}{2}\right)\right\}\right.$$

$$= 2\pi i(-1-1) = -4\pi i.$$

Example 4. Prove that $\int \frac{ds}{y}$ is an invariant with respect to the transformation $z = \frac{az+b}{cz+d}$ where a, b, c, d satisfies $ad - bc = 1$ and $ds = \sqrt{(dx^2 + dy^2)}$

Solution : From

$$z = \frac{az+b}{cz+d} \text{ we have}$$

$$z = \frac{b-dz}{cz-a} \qquad \ldots(1)$$

Differentiating (1), we get

$$dz = \frac{ad-bc}{(cz-a)^2} dz$$

$$\Rightarrow \qquad dz = \frac{dz}{(cz-a)^2} \qquad [\because ad - bc = 1]$$

$$\text{And so,} \qquad ds = \sqrt{(dx^2 + dy^2)}$$

$$= |dz| = \frac{|dz|}{|cz-a|^2}$$

$$= \frac{d\sigma}{|cz-a|^2} \qquad \ldots(2)$$

$$\text{Also,} \qquad 2iy = z - \bar{z}$$

$$= \frac{b-dz}{cz-a} - \frac{b-d\bar{z}}{c\bar{z}-a}$$

$$= \frac{(ad-bc)(z-\bar{z})}{|cz-a|^2}$$

$$\text{or} \qquad 2iy = \frac{2iy}{|cz-a|^2}$$

$$[\because ad - bc = 1, z = x + iy]$$

$$\text{or} \qquad \frac{v}{y} = \frac{1}{|cz-a|^2}$$

$\therefore$ (2) given,

$$d\sigma = \frac{v}{y} d\sigma$$

$$\text{or} \qquad \int \frac{ds}{y} = \int \frac{d\sigma}{y}$$

Thus $\int \frac{d\,s}{y}$ is invariant under the given transformation.

Example 5. Evaluate $\int_c \frac{dz}{z-a}$ where c represents the circle $|z-a| = r$.

Solution : Here for the circle $|z-a| = r$

$$z = a + re^{\theta}$$

where $0 \le \theta \le 2\pi$

$$\therefore \quad dz = ire^{i\theta}\,d\theta$$

Hence $\int_\gamma \frac{dz}{z-a} = \int_\theta^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\,d\theta = 2\pi i$

Example 6. Evaluate the residue of $\frac{z^2}{z^2+a^2}$ at $z = ia$.

Solution :

Here $f(z) = \frac{z^2}{z^2+a^2} = \frac{z^2}{(z+ia)(z-ia)}$

$z = ia$ is a simple pole of $f(z)$

Residue at $z = ia$ is $\lim\limits_{z \to ia} (z-ia) f(z)$

$$= \lim_{z \to ia} (z-ia)\frac{z^2}{(z-ia)(z+ia)}$$

$$= \lim_{z \to ia} \frac{z^2}{z+ia} = \frac{-a^2}{2ia} = \frac{1}{2}ia.$$

Example 7. Evaluate the analytic function

$$f(z) = \mu + iv, \text{ if}$$

$$\mu - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cosh y)}$$

and $f\left(\frac{\pi}{2}\right) = 0$

Solution :

Here $\mu - v = \frac{\cos x + \sin x - e^{-y}}{2(\cos x - \cos hy)}$

$$\therefore \frac{\delta\mu}{\delta x} - \frac{\delta v}{\delta x}$$

$$= \frac{(\sin x - \cos x)\cosh y + 1 - e^{-y}\sin x}{2(\cos x - \sin hy)^2} \quad \ldots(1)$$

and $\frac{\delta\mu}{\delta y} - \frac{\delta v}{\delta y} = \frac{(\cos x - \cosh y)e^{-y} + (\cos x + \sin x - e^{-y})\sinh y}{2(\cos x - \cosh y)^2}$

Since, $f(z) = \mu + iv$ is analytic

$$\therefore \quad \frac{\delta\mu}{\delta x} = +\frac{\delta v}{\delta y}$$

and $\frac{\delta v}{\delta x} = -\frac{\delta\mu}{\delta y}$ gives

$$-\frac{\delta v}{\delta x} - \frac{\delta\mu}{\delta x}$$

$$= +\frac{e^{-y}(\cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2} \quad \ldots(2)$$

Subtructing (2) from (1), we get

$$2\frac{\delta\mu}{\delta x} = \frac{(\sin x - \cos x)(\sin x + \cos x)\sinh y + 1 - e^{-y}(\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

Adding (1) and (2), we have

$$\frac{(\sin x - \cos x)\cosh y - (\sin x + \cos x)\sinh y + 1 + e^{-y}(-\sin x + \cos x - \cosh y - \sinh y)}{2(\cos x - \cosh y)^2}$$

$$\Rightarrow \quad f'(z) = \frac{\delta\mu}{\delta x} + i\frac{\delta v}{\delta x} = \frac{1 - \cos z}{2(1-\cos z)^2}$$

[Putting $x = z$ and $y = 0$]

$$= \frac{1}{2(1-\cos z)} = \frac{1}{4\sin^2\frac{z}{2}}$$

$$= \frac{1}{4}\operatorname{cosec}^2\frac{z}{2}$$

Integrating,

$$f(z) = -\frac{1}{2}\cot\frac{z}{2} + c$$

Since $f\left(\frac{\pi}{2}\right) = 0$,

$$0 = -\frac{1}{2}\cot\frac{\pi}{4} + c,$$

where $c = \frac{1}{2}$

$$\therefore \quad f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right)$$

Example 8. Evaluate $\int_c \frac{z^2 - z + 1}{z-1}\,dx$ where C is the circle (i) $|z| = 1$, (ii) $|z| = \frac{1}{2}$.

Solution :

(i) $\because \int_C \frac{f(z)dz}{z-a} = \int_C \frac{z^2-z+1}{z-1}\,dz$ and

$$f(z) = z^2 - z + 1 \text{ and } a = 1$$

Since $f(z)$ is analytic within and on circle C : $|z| = 1$ and $a = 1$ lies on C.

$\therefore$ By Cauchy's integral formula.

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a} = f(a) = 1,$$

i.e. $\int_C \frac{z^2-z+1}{z-1}dz = 2\pi i$

(ii) Here $a = 1$ lies out side the circle C : $|z| = y_2$. So $\frac{z^2-z+1}{z-1}$ is analytic everywhere within C.

$\therefore$ By Cauchy's theorem $\int_C \frac{z^2-z+1}{z-1}dz = 0$

Example 9. Prove that the transformation $w = \frac{iz+2}{4z+i}$ maps the real axis in the z-plane into a circle in the w-plane.

Solution : The given transformation

$$w = \frac{iz+2}{4z+i}$$

$$\Rightarrow \qquad z = \frac{2-iw}{4w-i}$$

The equation of the real axis in the plane is $z - \bar{z} = 0$ substituting for z and $\Sigma$the transformation equation is

$$\frac{2-iw}{4w-i} - \frac{2-i\bar{w}}{4\bar{w}+i} = 0$$

or $8\bar{w} + 2i - 4iw\bar{w} + w - 8w + 2i - 4iw\bar{w} - \bar{w} = 0$

or $8iw\bar{w} + 7(w - \bar{w}) - 4i = 0$

or $8i(\mu^2 + v^2) + 14iv - 4i = 0$

or $\mu^2 + v^2 + \frac{7}{4}v - \frac{1}{2} = 0$

which is the equation of a circle in the w-plane.

Example 10. If $0 < |z-1| < 2$, expand

$$f(z) = \frac{z}{(z-1)(z-3)}$$

Solution :

Here $\quad f(z) = \frac{z}{(z-1)(z-3)}$

$$= -\frac{1}{2(z-1)} + \frac{3}{2}\frac{1}{(z-3)}$$

Putting $z - 1 = \mu$, we have $0 < |\mu| < 2$ and

$$f(z) = -\frac{1}{2\mu} + \frac{3}{2(\mu-2)}$$

$$= -\frac{1}{2\mu} - \frac{3}{4}\left(1 - \frac{\mu}{2}\right)^{-1}$$

$$= -\frac{1}{2\mu} - \frac{3}{4}\left(1 + \frac{\mu}{2} + \frac{\mu^2}{2^2} + \frac{\mu^3}{2^3} + \dots\right)$$

$$= -\frac{1}{2\mu} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{\mu}{2}\right)^n$$

$$= -\frac{1}{2(z-1)} \frac{-3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

Example 11. Evaluate $\int_y \frac{z+2}{z}dz$, where y is the semicircle $z = 2e^{it}, 0 \le t \le \pi$

Solution :

Here $\qquad z = 2e^{it}, 0 \le t \le \pi$

$\Rightarrow \qquad dz = 2ie^{it}dt$

Hence $\int_y \frac{z+2}{z}dz = \int_0^{\pi}\frac{(2e^{it}+2)}{2e^{it}}2ie^{it}dt$

$$= 2i\int_0^{\pi}(1 + e^{-it})e^{it}dt$$

$$= 2i\int_0^{\pi}(e^{it}+1)dt$$

$$= 2i\left[\frac{e^{it}}{i} + t\right]_0^{\pi}$$

$$= 2i\left(\frac{e^{i\pi}}{i} - \frac{1}{i} + \pi\right)$$

$$= 2e^{i\pi} - 2 + 2i\pi$$

$$= -2 + 4i\pi$$

Example 12. Prove that for $0 < |z| < 4, \frac{1}{4z-z^2} = \sum_{\mu=0}^{\infty}\frac{z^{n-1}}{4^{n+1}}$

Solution : We have $0 < |z| < 4 \Rightarrow \frac{|z|}{4} < 1$

$$\therefore \frac{1}{4z-z^2} = \frac{1}{4z\left(1-\frac{z}{4}\right)}$$

$$= \frac{1}{4z}\left(1-\frac{z}{4}\right)^{-1} = \frac{1}{4z}\left[1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + \left(\frac{z}{4}\right)^3 + \dots\right]$$

$$= \frac{z^{-1}}{4} + \frac{1}{4^2} + \frac{z}{4^3} + \frac{z^2}{4^4} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}.$$

Example 13. Evaluate the integral

$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2\ (z-2)} dz$, where C is the circle $|z| = 3$.

Soltuion :

Here $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 (z-2)}$

is analytic within the circle $|z| = 3$ except at the poles $z = 1$ and $z = 2$.

Since $z = 1$ is a pole of order 2.

$$\therefore \text{Res.} f(1) = \frac{1}{1!}\left[\frac{d}{dz}(z-1)^2 f(z)\right]_{z=1}$$

$$= \left[\frac{d}{dz}\left(\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right)\right]_{z=1}$$

$$= \left[\frac{(z-2)(2\pi z \cos \pi z^2 - 2\pi z \sin \pi z^2) - (\sin \pi z^2 + \cos \pi z^2)}{(z-2)^2}\right]_{z=1}$$

$$= (-1)(-2\pi) - (-1) = 2\pi + 1$$

and $\text{Res} f(2) = \lim_{z \to 2} [(z-2) f(z)]$

$$= \lim_{z \to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

By residue theorem

$$\int_C f(z) dz = 2\pi i\, [\text{Res} f(1) + \text{Res} f(2)]$$

$$= 2\pi i\, (2\pi + 1 + 1)$$

$$= 4\pi\, (\pi + 1)^i.$$

Example 14. If $\mu = \frac{\sin 2x}{\cos 2y + \cos 2x}$, find the corresponding analytic function

$$f(z) = \mu + iv$$

Solution :

Here $\mu = \frac{\sin 2x}{\cos 2y + \cos 2x}$

$$\therefore \frac{\delta\mu}{\delta x} = \frac{2 \cos 2x (\cosh 2y + \cos 2x) - \sin 2x (-2\sin 2x)}{(\cos 2y + \cos 2x)^2}$$

$$= \frac{2 + 2 \cos 2x \cos 2y}{(\cosh 2y + \cos 2x)^2} = g_1(x,y)$$

and $\frac{\delta\mu}{\delta y} = \frac{-2 \sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2} = g_2(x,y)$

The function $f(z)$ is given by

$$f(z) = \int [g_1(z, 0) - i g_2(z, 0)] dz + c$$

$$= \int \left[\frac{2 + 2 \cos 2z}{(1 + \cos 2z)^2} - i0\right] dz + c$$

$$= \int \frac{2dz}{1 + \cos 2z} + c$$

$$= \int \sec^2 z\, dz + c$$

$$= \tan z + c$$

Example 15. Find the bilinear transformation which maps the points $z_1 = 2$, $z_2 = i$ and $z_3 = -2$, into the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$ respectively.

Solution : The bilinear transformation which transforms z_1, z_2, z_3 respectively into w_1, w_2, w_3 is given by

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

Substituting $z_1 = 2$, $z_2 = i$, $z_3 = -2$, $w_1 = 1$, $w_2 = i$, $w_3 = -1$ we have

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

$$\text{or} -\frac{(w-1)(1+i)^2}{(w+1)(1-i)(1+i)} = -\frac{(z-2)(2+i)^2}{(z+2)(2-i)(2+i)}$$

$$\text{or} \quad \frac{(w-1).2i}{(w+1).2} = \frac{(z-2)(3+4i)}{(z+2).5}$$

$$\text{or} \quad \frac{w-1}{w+1} = \frac{(z-2)(4-3i)}{5(z+2)}$$

$$\text{or} \quad \frac{2w}{2} = \frac{(z-2)(4-3i) + 5(z+2)}{5(z+2) - (z-2)(4-3i)}$$

$$\text{or} \quad w = \frac{3(3-i)z + 2(1+3i)}{(1+3i)z + 6(3-i)}$$

$$= \frac{3z + \left[\frac{2(1+3i)}{(3-i)}\right]}{\left[\frac{(1+3i)}{(3-i)}\right] z + 6}$$

$$\text{or} \quad w = \frac{3z + 2i}{iz + 6}.$$

The required transformation.

Example 16. Let $f(z) = \mu + iv$ is an analytic function of $z = x + iv$ and $\mu - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ find $f(z)$, given

$$f\left(\frac{\pi}{2}\right) = \frac{3+i}{2}$$

Solution :

Given $f(z) = \mu + iv$,

then if $(z) = i\mu + v$

Adding $f(z) + if(z)$, we have

$$f(z) + if(z) = \mu - v + i(\mu + v)$$
$$= \mu - iv \text{ (say)}$$

Since $U = \mu - v$

$$= \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$
$$= \frac{\cosh y + \sinh y - \cos x + \sin x}{\cosh y - \cos x}$$
$$= 1 + \frac{\sinh y + \sin x}{\cosh y - \cos x}$$
$$\frac{\delta U}{\delta x} = -\frac{(\sinh y + \sin x)\sin x}{(\cosh y - \cos x)^2}$$
$$= g_1(x, y)$$
$$\Rightarrow g_1(z, 0) = \frac{\cos z(1 - \cos z) - \sin^2 z}{(1 - \cos z)^2}$$
$$= \frac{\cos z - 1}{(1 - \cos z)^2} = \frac{-1}{1 - \cos z}$$
$$= -\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}z.$$

and
$$\frac{\delta U}{\delta y} = \frac{\cosh y(\cosh y - \cos x) - \sinh y(\sinh y + \sin x)}{(\cosh y - \cos x)^2}$$
$$= g_2(x, y)$$
$$\Rightarrow \phi_2(z, 0) = \frac{1 - \cos z}{(1 - \cos z)^2} = \frac{1}{1 - \cos z}$$
$$= -\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}z.$$

Since $(1 + i)f(z)$

$$= \int [g_1(z, 0) - ig_2(z, 0)]dz + c$$
$$= \int\left(-\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}z - i\frac{1}{2}\operatorname{cosec}^2\frac{1}{2}\ z\right)dz + c$$
$$= -\frac{1}{2}(1 + i)\int \operatorname{cosec}^2\frac{1}{2}zdz + c$$
$$= (1 + i)\cot\frac{1}{2}z + c.$$
$$\Rightarrow \quad f(z) = \cot\frac{1}{2}z + \frac{c}{1 + i}$$
$$\therefore \quad f\left(\frac{\pi}{2}\right) = \frac{3 - i}{2}$$
$$\Rightarrow \quad \frac{c}{1 + i} = f\left(\frac{\pi}{2}\right) - \cot\frac{\pi}{4}$$
$$= \frac{3 - i}{2} - 1 = \frac{1 - i}{2}$$

which gives $c = \frac{1}{2}(1 - i)$

Hence $f(z) = \cot\frac{1}{2}z + \frac{1}{2}(1 - i)$.

Example 17. Find the bilinear transformation which maps the points $z = 1, i = -1$ onto the points $w = i, 0 = -i$

Hence find the image of $|z| < 1$.

Solution : Let the points $z_1 = 1, z_2 = i, z_3 = -1$ and $z_4 = z$ map onto the points $w_1 = i, w_2 = 0, w_3 = -i$ and $w_4 = w$.

Since the cross-ratio remains unchanged under a bilinear transformation

$$\therefore \frac{(1 - i)(-1 - z)}{(1 - z)(-1 - i)} = \frac{(i - 0)(-i - w)}{(i - w)(-i - 0)}$$

or
$$\frac{w + i}{w - i} = \frac{(z + 1)(1 - i)}{(z - 1)(1 + i)}$$

By Componendo, dividendo, we get

$$\frac{2w}{2i} = \frac{(z + 1)(1 - i) + (z - 1)(1 + i)}{(z + 1)(1 - i) - (z - 1)(1 + i)}$$

or
$$w = \frac{1 + iz}{1 - iz}$$

which is the required bilinear transformation

Since
$$z = i\frac{1 - w}{1 + w}$$
$$\therefore \left|\frac{i(1 - w)}{1 + w}\right| = |z| < 1$$

or $\quad |i|\,|1 - w| < |1 + w|$

or $\quad |1 - \mu - iv| < |1 + \mu + iv| \qquad [\because |i| = 1]$

or $\quad (1 - \mu)^2 + v^2 < (1 + \mu)^2 + v^2$

which reduces to $\mu > 0$.

Hence the interior of the circle $x^2 + y^2 = 1$ in the z-plane is maped onto the entire half of the w-plane to the right of the imaginary axis.

Example 18. Prove that if w_1, w_2, w_3, w_4 is the images of the four distinct points z_1, z_2, z_3, z_4 in the plane under a bilinear transformation.

Then $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$

Solution : Let the bilinear transformation be

$$w = T(z)$$
$$= \frac{az + b}{cz + d}(ad - bc \neq 0) \qquad \ldots(1)$$

Since w_1, w_2, w_3, w_4 are the images of z_1, z_2, z_3, z_4 resp. we have

$$w_1 = \frac{az_1+b}{cz_1+d},$$

$$w_2 = \frac{az_2+b}{cz_2+d}$$

$$\Rightarrow w_1 - w_2 = \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}$$

Similarly, we have

$$w_2 - w_3 = \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

$$w_3 - w_4 = \frac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)}$$

$$w_4 - w_1 = \frac{(ad-bc)(z_4-z_1)}{(cz_4+d)(cz_1+d)}$$

$$\Rightarrow \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

$$\Rightarrow (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4).$$

Example 19. Expand $\dfrac{(z-2)(z+2)}{(z+1)(z+4)}$ for

(i) $|z| < 1$, (ii) $1 < |z| < 4$.

Solution :

Here $$f(z) = \frac{(z-2)(z+2)}{(z+1)(z+4)}$$

$$= 1 - \frac{5z+8}{(z+1)(z+4)}$$

$$= 1 - \frac{1}{z+1} - \frac{4}{z+4}$$

(i) $|z| < 1$, we have

$$f(z) = 1 - (1+z)^{-1} - \left(1+\frac{z}{4}\right)^{-1}$$

$$= 1 - (1 - z + z^2 - z^3 + \dots)$$

$$- \left[1 - \frac{z}{4} + \left(\frac{z}{4}\right)^2 - \left(\frac{z}{4}\right)^3 + \dots\right]$$

$$= -1 + \left(1+\frac{1}{4}\right)z + \left(-1-\frac{1}{4^2}\right)z^2 + \left(1+\frac{1}{4^3}\right)z^3 + \dots$$

$$= -1 - \frac{5}{4}z - \frac{17}{16}z^2 - \frac{65}{64}z^3 - \dots\dots$$

The required expansion.

(ii) $1 < |z| < 4$. then $\dfrac{1}{|z|} < 1$ and $\dfrac{|z|}{4} < 1$

$$\therefore \quad f(z) = 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{z}{4}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right]$$

$$- \left[1 - \frac{z}{4} + \frac{z^2}{4^2} - \frac{z^3}{4^3} + \dots\right]$$

$$= -\dots + \frac{1}{z^4} - \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} + \frac{z}{4} - \frac{z^2}{4^2} + \frac{z^3}{4^3} \dots\dots$$

The required expansion.

Example 20. What are the residues of the function $\dfrac{\cot \pi z}{(z-a)^2}$?

Solution :

Here $$f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{\sin \pi z\,(z-a)^2}$$

Poles of $f(z)$ are given by $(z-a)^2 \sin \pi z = 0$

so $(z-a)^2 = 0 \Rightarrow z = a$ is a pole of order two and $\sin \pi z = 0 \Rightarrow \pi z = n\pi \Rightarrow z = n$ where $n \in I$

$z = n$, ($n \in I$) are simple poles of $f(z)$ if n is finite $z = \infty$ is the limit point of these poles so $z = \infty$ is the non-isolated essential singularity.

Residue at $(z = a)$ is $\dfrac{1}{1!}f'(a)$

where $f(z) = \cot \pi z = -\pi \operatorname{cosec}^2 \pi a$

Residue at $(z = n)$ is $\left[\dfrac{f(z)}{g'(z)}\right]_{z=n}$

(where $g(z) = \sin \pi z$)

$$= \left[\frac{\dfrac{\cos \pi z}{(z-a)^2}}{\pi \cot \pi z}\right]_{z=n}$$

$$= \frac{1}{\pi (n-a)^2}.$$

Example 21. Show that both the transformation $w = \dfrac{z+i}{z-i}$ and $w = \dfrac{i+z}{i-z}$ transform $|w| \le 1$ into the lower half plane $I(z) \le 0$.

Solution :

Here $$w = \frac{z+i}{z-i}$$

$$\Rightarrow \quad \overline{w} = \frac{\overline{z}-i}{\overline{z}+i}$$

$$\therefore\ w\bar{w}-1 = \frac{z+i}{z-i}\frac{\bar{z}-i}{\bar{z}+i}-1$$

$$= \frac{(z+i)(\bar{z}-i)-(z-i)(\bar{z}+i)}{(z-i)(\bar{z}+i)}$$

$$= \frac{-2i(z-\bar{z})}{|z-i|^2} = \frac{4I(z)}{|z-i|^2}$$

$$= |\because z-\bar{z} = 2i\,I(z)|$$

Taking $w = \dfrac{i+z}{i-z}$, we have

$$w = \bar{w}-1$$

Hence for both the transformations $|w^2| - 1 \le 0$

i.e., $|w| \le 1$ gives $I(z) \le 0$

i.e., The boundary of the circle $|w| = 1$ corresponds to real axis in the z-plane and the interior of the circle transforms into the lower half z-plane.

Example 22. Prove that function $f(z) = z^n$, where n is a positive integer is an analytic function.

Solution : We have

$$f(z) = z^n$$

Now $$f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^n + nz^{n-1}\Delta z + \frac{1}{2} n(n-1) z^{n-2}(\Delta z)^2 + \ldots + \Delta^n z - z^n}{\Delta z}$$

(by Binomial theorem)

$$= \lim_{\Delta z \to 0} (nz^{n-1} + \frac{1}{2}n(n-1)z^{n-2}\Delta z + \ldots\ldots + \Delta^{n-1}z)$$

$$= nz^{n-1}$$

which exists for all finite values of z.

Hence $f(z)$ is an analytic function.

Example 23. If λ is real, a, b are complex numbers such that $|a| > |b|$, show that the bilinear transformation $w = e^{i\lambda}\dfrac{az+b}{\bar{a}+\bar{b}z}$ maps the inside of the circle $|z| = 1$ on the inside of the circle $|w| = 1$

Solution : We have

$$w\bar{w}-1 = e^{i\lambda}\frac{az+b}{\bar{a}+\bar{b}z}\,e^{-i\lambda}\frac{\bar{a}\bar{z}+\bar{b}}{a+b\bar{z}}-1$$

$$= \frac{(az+b)(\bar{a}\bar{z}+\bar{b}) - (\bar{a}+\bar{b}z)(a+b\bar{z})}{(\bar{a}+\bar{b}z)(a+b\bar{z})}$$

$$= \frac{(a\bar{a}-b\bar{b})-(z\bar{z}-1)}{|a+b\bar{z}|^2}$$

$$= \frac{(|a|^2-|b|^2)(|z|^2-1)}{|a+b\bar{z}|^2}$$

If $|z| < 1$, then $w\bar{w} - 1 < 0$, because $|a| > |b|$.

Hence $|z| < 1$ corresponds to $|w| < 1$. *i.e.*, the interiors of the two circles correspond.

Example 24. Show that the function e^{-1/z^2} has no singularities.

Solution : We have $f(z) = e^{-1/z^2}$

Zeros of $f(z)$ are given by

$$e^{-1/z^2} = 0$$

or $$z^2 = 0$$

$\therefore$ $z = 0$ is a zero of order two

Since zeros have no limit point,

$\therefore$ There is no singularity of $f(z)$.

Here the poles of $f(z)$ are given by

$e^{-1/z^2} = 0$ which is not possible.

$\therefore$ There exist no poles

Hence e^{-1/z^2} has no singularities.

Example 25. If $w = \left(\dfrac{z-c}{z+c}\right)^2$ where $c > 0$, find the area of the z-plane of which the upper half of the w-plane is the conformal representation.

Solution : Let $w = \mu + iv$ and $z = x + iv$, then

$$\mu = \frac{(z^2+y^2-c^2)^2 - 4c^2y^2}{\{(x+c)^2+y^2\}^2} \quad \ldots(1)$$

and $$v = \frac{4cy(x^2+y^2-c^2)}{\{(x+c)^2+y^2\}^2} \quad \ldots(2)$$

from (2) $v < 0$ if

(i) y and $x^2 + y^2 - c^2$ are both positive, *i.e.*, for the points in the upper half z-plane and exterior of the circle $|z| = c$, or

(ii) y and $x^2 + y^2 - c^2$ are both negative, *i.e.*, for the points in the lower half z-plane and interior of the circle $|z| = c$.

Thus interior of the circle $|z| = c$ in the lower half and its exterior in the upper half both separately correspond to the upper half of the w-plane.

Examples 26. Prove that the function $\mu = x^3 - 3xy^2$ is harmonic and find the corresponding analytic function.

Solution : We have $\mu = x^3 - 3xy^2$

$$\frac{\delta\mu}{\delta x} = 3x^2 - 3y^2,$$

$$\frac{\delta^2\mu}{\delta x^2} = 6x.$$

$$\frac{\delta\mu}{\delta x} = -6xy \text{ and } \frac{\delta^2\mu}{\delta y^2} = -6x.$$

Now $\frac{\delta^2\mu}{\delta x^2} + \frac{\delta^2\mu}{\delta y^2} = 0$, so that μ satisfies Laplace's equation.

Also since first and second order partial derivatives of μ are continuous functions of x and y.

Thus μ is a harmonic function.

Example 27. Prove that $f(z) = \sin x (\cosh y) + i \cos (\sinh y)$ is continuous as well as analytic every where.

Solution : Let $\mu (x, y) = \sin x (\cosh y)$ and $v (x, y) = \cos x (\sinh y)$.

Here μ and v are both rational functions of x and y, μ and v are both continuous every where.

Hence $f(z)$ is continuous every where.

Here $$\frac{\delta\mu}{\delta v} = \cos x (\cosh y)$$

$$\frac{\delta\mu}{\delta y} = \sin x (\sinh y)$$

and $$\frac{\delta v}{\delta x} = -\sin x (\sinh y)$$

$$\frac{\delta v}{\delta y} = \cos x (\cosh y).$$

$$\therefore \quad \frac{\delta\mu}{\delta n} = \frac{\delta v}{\delta y}, \frac{\delta v}{\delta x} = -\frac{\delta\mu}{\delta y}$$

$\therefore$ μ and v satisfy Cauchy-Riemann equations.

Thus $f(z)$ is analytic every where.

Example 28. Show that the function $f(z) = xy + iy$ is everywhere continuous but is not analytic.

Solution : Let $f(z) = \mu(x, y) + iv (x, y)$

Then $\mu(x, y) = xy$

and $v (x, y) = y$

Since μ and v are polynomials in x and y, therefore, they are continuous at each point.

Hence $f(z)$ is every where continuous

Here $$\frac{\delta\mu}{\delta x} = y,$$

$$\frac{\delta\mu}{\delta y} = x,$$

$$\frac{\delta v}{\delta x} = 0,$$

$$\frac{\delta v}{\delta y} = 1$$

and $$\frac{\delta\mu}{\delta x} \neq \frac{\delta v}{\delta y},$$

$$\frac{\delta\mu}{\delta y} \neq -\frac{\delta v}{\delta x},$$

so that Cauchy-Riemann equations are not satisfied.

Hence $f(z)$ is not analytic.

OBJECTIVE TYPE QUESTIONS

1. $(1 + i)^{10} + (1 - i)^{10} =$

(A) -1 (B) 1
(C) 0 (D) 2

2. If $z = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{35} + \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right)^{200}$, then—

(A) $\text{Re}(z) < 0$ (B) $\text{Re}(z) = 0$
(C) $\text{Re}(z) > 0$ (D) None of these

3. The product of two complex numbers (a, b) and (c, d) is—

(A) $(ac + bd, ad + bc)$
(B) $(ac - bd, ad + bc)$
(C) $(ac + bd, ad - bc)$
(D) $(ad + bc, ac - bd)$

4. Suppose a $a'b$ and b' are real numbers, then $\frac{(a + ib)}{(a' + ib')}$ will also be real number, if—

(A) $ab - a'b' = 0$ (B) $ab' - a'b = 0$
(C) $aa' - bb' = 0$ (D) $ab' - a'b = 0$

5. The conjugate of $(1+i)^2$ is given by—
(A) $(1-i)^{-2}$ (B) $(1+i)^{-1}$
(C) $-2i$ (D) $2i$

6. Which of the following is true for complex number $\mathbb{C}$—
(A) $\alpha + i\beta = \gamma + i\delta$, if $\alpha = \gamma$ and $\beta = \delta$
(B) $\alpha + i\beta > 0 + i\beta$, if $\alpha > 0$ and $\beta < 0$
(C) Transitivity low holds in $\mathbb{C}$
(D) Trichotomy low holds in $\mathbb{C}$

7. Which of the following is false ?
(A) $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$
(B) $|z_1 + z_2| \geq |z_1| + |z_2|$
(C) $|z_1 z_2| = |z_1|\,|z_2|$
(D) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
(E) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\{|z_1|^2 + |z_2|^2\}$

8. If $x = -2 - \sqrt{3i}$, then the value of $2x^4 + 5x^3 + 7x^2 + 41$ is—
(A) $4 + \sqrt{3}i$ (B) $4 - \sqrt{3}i$
(C) $\sqrt{3} + 4i$ (D) $\sqrt{3} - 4i$

9. The real part of exp (exp $i\theta$) is—
(A) $e^{\cos\theta}$ (B) $e^{\cos\theta}\sin(\sin\theta)$
(C) $e^{\cos\theta}\cos(\sin\theta)$ (D) $e^{\cos\theta}\cos(\cos\theta)$

10. The correct polar form of the complex number $1 - i$ is—
(A) $\sqrt{2}\,e^{\pi/4i}$ (B) $e^{-\pi/4i}$
(C) $\sqrt{2}\,e^{-\pi/4i}$ (D) $e^{\pi/4i}$

11. Let c be any complex numbers. Let for any $z = (x, y)$ in c, $z \cdot \bar{z} = z$, then $\bar{z}$ is equal to—
(A) (0, 0) (B) (1, 0)
(C) (0, 1) (D) (1, 1)

12. The value of $\left(\dfrac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}\right)^4$ is—
(A) 1 (B) 0
(C) $\cos 4\theta - i\sin 4\theta$ (D) $\cos 8\theta + i\sin 8\theta$

13. $(\sin\theta + i\cos\theta)^6$—
(A) $\sin 6\theta + i\cos 6\theta$
(B) $\cos 6\theta - i\sin 6\theta$
(C) $-\cos 6\theta + i\sin 6\theta$
(D) $\sin 6\theta - i\cos 6\theta$

14. If 1, w, w^2 are the cube roots of unity then $(x-y)(x-wy)(x-w^2y)$ is equal to—
(A) $x - y$ (B) $x^2 - y^2$
(C) $x^3 - y^3$ (D) $x^3 + y^3$

15. The reciprocal $a + ib$ is equal to—
(A) $\dfrac{1}{a^2+b^2} - ib$
(B) $\dfrac{ib}{a+ib} - a$
(C) $\dfrac{a^2}{a^2+b^2} - \dfrac{b^2 i}{a^2+b^2}$
(D) $\dfrac{a}{a^2+b^2} - \dfrac{bi}{a^2+b^2}$

16. arg $(-1 + \sqrt{3i})$ equals—
(A) $\dfrac{\pi}{3}$ (B) $\dfrac{\pi}{6}$
(C) $\dfrac{2\pi}{3}$ (D) $\dfrac{5\pi}{6}$

17. The necessary condition that the points A, B, C representing numbers z_1, z_2, z_3 respectively on the Argand plane be the vertices of an equilateral triangle in that—
(A) $\dfrac{1}{z_1 - z_2} = \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_2 - z_3}$
(B) $\dfrac{1}{z_1 - z_2} = \dfrac{1}{z_1 - z_3} + \dfrac{1}{z_2 - z_3}$
(C) $\dfrac{1}{z_1 - z_2} = \dfrac{1}{z_1 - z_3} + \dfrac{1}{z_3 - z_2}$
(D) None of these

18. If 1, w, w^2 are the cube root of unity, then the roots of $(x-1)^3 + 8 = 0$ are—
(A) $-1, -1, -1$
(B) $-1, 1 + 2w, 1 + 2w^2$
(C) $1, w, 2w$
(D) $-1, 1 - 2w, 1 - 2w^2$

19. Complex form of $\sqrt{3+4i}$ is—
(A) $\sqrt{3+i}$ (B) $2 - i$
(C) $2 + i$ (D) $\sqrt{3} - i$

20. If cube root of $a + ib$ is $x + iy$, then $H(x^2 - y^2)$ is—
(A) $\dfrac{a}{x} + \dfrac{b}{y}$ (B) $\dfrac{x}{a} + \dfrac{y}{b}$
(C) $ax + by$ (D) $ax - by$

21. If w be an imaginary cube root of unity, then $(1 - w + w^2)^5 + (1 + w - w^2)^5$ is equal to—
    (A) 64 (B) 32
    (C) 16 (D) 8

22. The complex numbers $z_1 = 1 + 2i$, $z_2 = 4 - 2i$ and $z_3 = 1 - 6i$ form the vertices of a—
    (A) Right angled triangle
    (B) Isosceles triangle
    (C) Equilateral triangle
    (D) Scalene triangle

23. If m and n are integers, then the value of the complex number $\log_i i$ is given by—
    (A) $\frac{4m+1}{4n+1}$ (B) $e^{\frac{4m+1}{4n+1}}$
    (C) $\log\frac{4m+1}{4n+1}$ (D) 1

24. The amplitude of the complex number $\left(\frac{1}{1-2i}+\frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$ is given by—
    (A) $\tan^{-1}6$ (B) $\tan^{-1}9$
    (C) $\tan^{-1}3$ (D) $\tan^{-1}\frac{3}{2}$

25. The real part of the complex number $(1 + i)^n$ is—
    (A) $2^{n/2}\cos\frac{n\pi}{4}$ (B) $2^n\cos\frac{n\pi}{2}$
    (C) $2^{-n/2}\cos n\pi$ (D) $2^{-n}\cos\frac{n\pi}{2}$

26. Let z_1 and z_2 are two complex numbers with α and β as their principal arguments such that $\alpha + \beta > \pi$, then $\text{Arg}(z_1 \cdot z_2)$ is given by—
    (A) $\alpha + \beta$ (B) $\alpha + \beta - \pi$
    (C) $\alpha + \beta + \pi$ (D) $\alpha + \beta - 2\pi$

27. The locus of the complex number satisfying $\arg\frac{z-1}{z+1} = \frac{\pi}{3}$ is a—
    (A) Straight line (B) Circle
    (C) Parabola (D) Hyperbola

28. If w is an imaginary cube root of unity, $x = a + b$, $y = aw + bw^2$ and $z = aw^2 + bw$, then xyz equals—
    (A) $a + b$ (B) $a^2 + b^2$
    (C) $a^4 + b^4$ (D) $a^3 + b^3$

29. Principal value of argument of (cos 1200° + i sin 1200°) is—
    (A) 300° (B) 120°
    (C) – 150° (D) 180°

30. If z_1 and z_2 are complex numbers, then amp $(z_1.z_2)$ is equal to—
    (A) amp (z_1) + amp (z_2)
    (B) amp (z_1) amp (z_2)
    (C) amp (z_1) – amp (z_2)
    (D) $\frac{\text{amp }(z_1)}{\text{amp }(z_2)}$

31. The value of $\arg(z) + \arg(\bar{z})$, $(z \neq 0)$ is—
    (A) 0 (B) π
    (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

32. If $z = x + iy$, then the number of solutions of the equation $z^2 = \bar{z}$ is—
    (A) One (B) Two
    (C) Four (D) Infinite

33. If $\frac{4+3i}{3-4i} = x + iy$, then $\frac{x}{y}$ is equal to—
    (A) 0 (B) 1
    (C) $\frac{4}{3}$ (D) $\frac{4}{5}$

34. If $|z - 1| = 2$, then the value of $z\bar{z} - z - \bar{z}$ is—
    (A) 4 (B) 2
    (C) 1 (D) 3

35. The solution of the equation $|z| - z = 1 + 2i$ is—
    (A) $1 - 2i$ (B) $2 - \frac{3}{2}i$
    (C) $\frac{3}{2} + 2i$ (D) $\frac{3}{2} - 2i$

36. If $|z| = |z - 1|$, then—
    (A) $\text{Re}(z) = 1$ (B) $\text{Re}(z) = \frac{1}{2}$
    (C) $\text{Im}(z) = 1$ (D) $\text{Im}(z) = \frac{1}{2}$

37. If $\left|\frac{z-5i}{z+5i}\right| = 1$, then $z = x + iy$ lie on—
    (A) The real axis

(B) The straight line $x = 5$
(C) The straight line $y = 5$
(D) A circle passing through origin

38. If $2\cos\alpha_1 = a + \frac{1}{a}$, $2\cos\alpha_2 = b + \frac{1}{b}$ etc, then $abc + \frac{1}{abc}$ ...... will be given by—
(A) $\cos(2\alpha_1 + 2\alpha_2 + ...)$
(B) $2\cos(\alpha_1 + \alpha_2 + ...)$
(C) $2\sin(\alpha_1 + \alpha_2 + ...)$
(D) $\sin(2\alpha_1 + 2\alpha_2 + ...)$

39. If α, β are the roots of the equation $x^2 - 2x + 4 = 0$, $\alpha^n + \beta^n$ is equal to—
(A) $2^{n+1}\cos\left(\frac{n\pi}{3}\right)$
(B) $2^n\cos\left(\frac{n\pi}{3}\right)$
(C) $2^{n+1}\sin\left(\frac{n\pi}{3}\right)$
(D) $2^n\sin\left(\frac{n\pi}{3}\right)$

40. If the imaginary part of $\frac{2z+1}{iz+1}$ is -2, then the locus of a point representing z, is a—
(A) Circle (B) Straight line
(C) Parabola (D) None of these

41. For complex number z,
$|z + 5|^2 + |z - 5|^2 = 75$ represents—
(A) A circle (B) An ellipse
(C) A triangle (D) A straight line

42. If θ is a positive acute angle, then real and imaginary parts of $\cos^{-1}(\cos\theta + i\sin\theta)$ are—
(A) $\sin^{-1}(\sqrt{\sin\theta}) + i\log\{\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\}$
(B) $\cos^{-1}(\sqrt{\sin\theta}) + i\log\{\sqrt{1+\cos\theta} - \sqrt{\cos\theta}\}$
(C) $\sin^{-1}(\sqrt{\cos\theta}) + i\log\{\sqrt{1+\sin\theta} - \sqrt{\cos\theta}\}$
(D) None of these

43. If $(1 + i\tan\alpha)^{1 + i\tan\beta}$ has only real values, one of them is given by—
(A) $(\sec\beta)^{\sec^2\alpha}$ (B) $(\sec\alpha)^{\sec\beta}$
(C) $(\sec\alpha)^{\sec^2\beta}$ (D) None of these

44. If y_0 and y_1 are two rectifiable curves in G and y_0 and y_1 are fixed-end-point homotopic, then for any analytic function f in G—
(A) $\int_{y_0} f - \int_{y_1} f = 0$ (B) $\int_{y_0} f + \int_{y_1} f = 0$
(C) $\int_{y_0} f \int_{y_1} f = 0$ (D) None of these

45. If G is simply connected, then for every curve $y \in G$, $\int_y f = 0$—
(A) For every function f
(B) For every non-analytic function f
(C) For every analytic function f
(D) None of these

46. If G is simply connected, then $\int_y f = 0$ for—
(A) Every rectifiable curve
(B) Every closed rectifiable curve y
(C) Every function f
(D) None of these

47. If f and g are continuous functions on $[a, b]$ and y and σ are the function of bounded variation on $[a, b]$, then—
(A) $\int_a^b (f+g)dy = \int_a^b f\,dy$
(B) $\int_a^b (f+g)\,dy = \int_a^b g\,dy$
(C) $\int_a^b (f+g)dy = \int_a^b f\,dy = \int_a^b g\,dy$
(D) None of these

48. If y piecewise smooth and $f : [a, b] \to \mathbb{C}$ is continuous, then—
(A) $\int_a^b f\,dy = \int_a^b f(t)y'(t)dt$
(B) $\int_a^b f\,dy = \int_a^b f(t)\,y(t)dt$
(C) $\int_a^b f\,dy = \int_a^b f'(t)\,y(t)dt$
(D) None of these

49. If y is a rectifiable curve and f is continuous function on $\{y\}$, then—
(A) $\int_y f = -\int_y f$
(B) $\int_y f\,dt = -\int_{-y} f\,dt$
(C) $\int_y f\,dt = -\int_y f\,dt$
(D) $\int_y f\,dt = -\int_{-y} f\,dt$

50. If y is a rectifiable curve and f is continuous function on $\{y\}$, then if $c \in \mathbb{C}$—
(A) $\int_y f(z)dz = \int_{y+c} f(z-c)dz$
(B) $\int_y f(z)dz = \int_y f(z-c)dz$
(C) $\int_y f(z)dz = \int_y f(c)dz$
(D) None of these

51. If G is simply connected and $f : G \to \mathbb{C}$ is analytic in G, then—
(A) f has a primitive in G
(B) f has no primitive in G
(C) f is constant in G
(D) None of these

52. If G is an open set then curve y is homologous to zero if for all $w \in \mathbb{C} - G$—
(A) $n(y ; w) = 0$ (B) $n(y ; w) = 1$
(C) $n(y ; w) = 2$ (D) $n(y ; w) = 4$

53. If G is a region and f is non-constant analytic function on G. Then open mapping theorem states, for any open set $\cup \subset G$—
(A) F(U) is closed (B) F(U) is open
(C) F(U) = U (D) None of these

54. If G is an open set and $f : G \to \mathbb{C}$ is differentiable function, then—
(A) f is analytic on G
(B) f is non-analytic on G
(C) f is constant on G
(D) None of these

55. If function $f(z)$ has an isolated singularities at $z = a$, then $z = a$ has removable singularity if—
(A) $\lim_{z \to a} (z - a) = 0$
(B) $\lim_{z \to a} f(z) = 0$
(C) $\lim_{z \to a} (z - a) f(z) = 0$
(D) None of these

56. Series Σa_n converges absolutely if—
(A) $\Sigma |a_n|$ converges (B) Σa_n converges
(C) $\Sigma |a_n|$ diverges (D) None of these

57. If f and g are analytic function, then—
(A) $\frac{f}{g}$ is always analytic
(B) $\frac{f}{g}$ is analytic when ever $g(x) \neq 0$
(C) $\frac{f}{g}$ is analytic whenever $f(x) \neq 0$
(D) None of these

58. A function $f(z + c) = f(z)$, where c is any number, then f is—
(A) A periodic function
(B) Periodic function with period C
(C) Periodic function with period z
(D) None of these

59. If G is open connected set in $\mathbb{C}$ and f: $G \to \mathbb{C}$ is a continuous function. Then f is a branch of logarithm if $z \in G$—
(A) $z = \sin f(z)$ (B) $z = \cos f(z)$
(C) $z = \exp f(z)$ (D) $z = f(z)$

60. If $w' = T_1(z) = \frac{z+2}{z+3}$, then $T_1^{-1}(z)$ is—
(A) $\frac{2-3w}{w+1}$ (B) $\frac{2-3w}{w-1}$
(C) $\frac{1}{w+3}$ (D) None of these

61. What is the radius of convergence for power series $f(z) = \Sigma \frac{1}{n\rho} z^n$?
(A) 1 (B) 2
(C) 0 (D) ∞

62. $f(z) = \frac{\sin z}{(z-\pi)^2}$ have the pole of order—
(A) 1 (B) 2
(C) 3 (D) 0

63. If y is a rectifiable curve and f is a continuous function on $\{y\}$, then—
(A) $\int_y f(z)dz \leq \int_y |f(z)|\,|dz|$
(B) $\int_y f(z)dz \geq \int_y |f(z)|\,|dz|$
(C) $\int_y f(z)dz = \int_y |f(z)|\,|dz|$
(D) None of these

64. G is open set in $\mathbb{C}$, y is a closed rectifiable path in G and $f : G \to \mathbb{C}$ is a continuous function, then—
(A) $\int_y f(z)dz = 0$ (B) $\int_y f(z)dz > 0$
(C) $\int_y f(z)dz < 0$ (D) None of these

65. An analytic function is—
(A) Infinitely differentiable

(B) Finitely differentiable
(C) Not differentiable
(D) None of these

66. If $\phi : [a, b] \times [c, d] \to \mathbb{C}$ is a continuous function and $g : [c, d] \to \mathbb{C}$ such that $g(t) = \int_a^b \phi(s, t)ds$, then—
(A) g is not a continuous function
(B) g is a continuous function
(C) g is an increasing function
(D) g is a decreasing function

67. If is a analytic and $f'(z) \neq 0$, then—
(A) f is non-conformal mapping
(B) f is a conformal mapping
(C) f is a constant function
(D) None of these

68. A mapping $S(z) = \dfrac{az+b}{cz+d}$ which is a linear functionl transformation is mobius transformation, if—
(A) $ad - bc \neq 0$
(B) $ad - bc = 0$
(C) $ab - cd \neq 0$
(D) None of these

69. For any point z_1, if z_2, z_3, z_4 are distinct points and T is any mobius transformation then the cross ratio (z_1, z_2, z_3, z_4) is equal to—
(A) (Tz_1, Tz_2, z_3, z_4)
(B) (Tz_1, Tz_2, Tz_3, z_4)
(C) (Tz_1, Tz_2, Tz_3, Tz_4)
(D) None of these

70. If $z_1 \neq z_2 \neq z_3 \neq z_4$ in $\mathbb{C}_\infty$. Then cross ratio (z_1, z_2, z_3, z_4) is a real number if z_1, z_2, z_3, z_4 lies on—
(A) Triangle
(B) A parabola
(C) A circle
(D) A hyperbola

71. The mobius transform takes—
(A) Circles in to line
(B) Circle into circle
(C) Circle into square
(D) None of these

72. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in ann $(a', 0, R)$. Then if $a_n = 0$ for $n \leq -1$, $z = a$ is—
(A) A pole of order m
(B) An essential singularity
(C) A removable singularity
(D) None of these

73. If G is a region and $f : G \to \mathbb{C}$ is continuous function such that $\int_y f = 0$ for every closed path y in G. Then—
(A) f is analytic in G
(B) f is continuous in G
(C) f is non-analytic in G
(D) f is discontinuous in G

74. If the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence $R > 0$, then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ is—
(A) Analytic in ball B $(a; R)$
(B) Analytic outside ball B$(a; R)$
(C) Non-analytic in ball B$(a; R)$
(D) None of these

75. The following statement is false for complex number z—
(A) Re $z = \dfrac{1}{2}(z + \bar{z})$
(B) Im $z = \dfrac{1}{2i}(z + \bar{z})$
(C) $|\bar{z}| = |z|$
(D) $|z^2| = z$

76. If $z = a$ is an isolated singularity of f, then a is the pole of f. If—
(A) $\lim_{z \to a} |f(z)| = 0$
(B) $\lim_{z \to a} |f(z)| = a$
(C) $\lim_{z \to a} |f(z)| = \infty$
(D) None of these

77. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in *ann* $(a;\ 0;\ R)$. Then $z = a$ is a removable singularity, if—
(A) $a_n = 0, n \leq -1$
(B) $a_n \neq 0, n \leq -1$
(C) $a_n = 0, n \geq -1$
(D) $a_n \neq 0, n \geq -1$

78. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n(z-a)^n$ is its Laurent expansion in *ann* $(a; 0, R)$. Then $z = a$ is a pole of order m, then—
(A) $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq -(m+1)$
(B) $a_{-m} = 0$ and $a_n \neq 0$ for $n \leq -(m+1)$

(C) $a_{-m} = 0$ and $a_n = 0$ for $m \leq -(m+1)$
(D) None of these

79. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in *ann* $(a;\ 0,\ R)$. Then $z = a$ is an essential singularity if—
(A) $a_n \neq 0$ for all integers n
(B) $a_n = 0$ for all integers n
(C) $a_n \neq 0$ for infinitely many negative integers n
(D) $a_n \neq 0$ for infinitely many positive integers n

80. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in *ann* $(a;\ 0,\ R)$. Also if $a_n \neq 0$ for infinitely many negative integers n, then—
(A) $z = a$ is a removable singularity
(B) $z = a$ is a pole of order m
(C) $z = a$ is a an essential singularity
(D) None of these

81. If f have an isolated singularity at $z = a$ and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion about $z = a$. Then residue of f at $z = a$ is—
(A) a_{-1} (B) a_0
(C) a_{-2} (D) a_1

82. If f has a pole of order m at $z = a$ and $g(z) = (z-a)^m f(z)$, then—
(A) $\text{Res}\,(f\,;a) = \frac{1}{|\underline{m-1}}\, g\,(m-1)\,(a)$
(B) $\text{Res}\,(f\,;a) = g^{(m-1)}\,(a)$
(C) $\text{Res}\,(f\,;a) = \frac{1}{|\underline{m-1}}\, g(a)$
(D) None of these

83. If $z = a$ is an isolated singularity of f and $f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n$ is its Laurent expansion in *ann* $(a',\ 0,\ R)$. Also if $a_{-m} \neq 0$ and $a_n = 0$ for $n \leq (m+1)$, then—
(A) $z = a$ is a removable singularity
(B) $z = a$ is a pole of order m
(C) $z = a$ is a an essential singularity
(D) None of these

84. If $T_1(z) = \frac{z+2}{z+3}$ and $T_2(z) = \frac{z}{z+1}$; then $T_2T_z(z)$ is—
(A) $z+2$ (B) $\frac{2}{2z+5}$
(C) $\frac{z+2}{2z+5}$ (D) None of these

85. The radius of convergence of the power series $f(z) = \Sigma \frac{n+1}{(n+2)\,(n+3)} z^n$ is—
(A) 1 (B) 2
(C) 3 (D) 4

86. If $f(z) = \frac{1-e^z}{1+e^z}$, then at $z = \infty$, $f(z)$ have—
(A) Pole
(B) Removable singularity
(C) Isolated singularity
(D) Non-isolated singularity

87. A mapping $S(z)$ is called linear transformation if—
(A) $S(z) = \frac{az}{cz}$ (B) $S(z) = az$
(C) $S(z) = \frac{az+b}{cz+d}$ (D) None of these

88. T is a circle through points z_2, z_3, z_4. The points z, $z^* \in \mathbb{C}_\infty$ are symmetric with respect to T if—
(A) $(z^*, z_2, z_3, z_4) = (z_1, z_2, z_3, z_4)$
(B) $(z^*, z_2, z_3, z_4) = \overline{(z_1, z_2, z_3, z_4)}$
(C) $z^*, z_2 = z, z_3$
(D) None of these

89. If (z_1, z_2, z_3) is an orientation of T, then right side of Γ with respect to (z_1, z_2, z_3) is—
(A) $\{z : \text{Im}\,(z_1, z_2, z_3) < 0\}$
(B) $\{z : \text{Im}(z, z_1, z_2, z_3) > 0\}$
(C) $\{z : \text{Re}\,(z, z_1, z_2, z_3) < 0\}$
(D) None of these

90. If $(1,\ 0,\ \infty)$ is the orientation of $\mathbb{R}$, then the cross ratio—
(A) $(z,\ 1,\ 0,\ \infty) = 1$ (B) $(z,\ 1,\ 0,\ \infty) = 0$
(C) $(z,\ 1,\ 0,\ \infty) = \infty$ (D) $(z,\ 1,\ 0,\ \infty) = z$

91. If f is an entire function, then—
(A) f has a power series expression
(B) f has not a power series expression
(C) f is constant
(D) None of these

92. If f is a bounded entire function, then—
(A) f is constant
(B) f is equal to zero
(C) f is increasing function
(D) f is decreasing function

93. If $y : [0, 1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{y\}$, then $\frac{1}{2\pi i}\int_y \frac{dz}{z-a}$ is—
(A) An integer (B) Rational number
(C) Real number (D) Complex number

94. If y is a closed rectifiable curve in $\mathbb{C}$, then for $a \notin \{y\}$, the index of y with respect to point a is—
(A) $n(y\,;a) = \frac{1}{2\pi i}\int_y z\,dz$
(B) $n(y\,;a) = \frac{1}{2\pi i}\int_y (z-a)^{-1}dz$
(C) $n(y\,;a) = \frac{1}{2\pi i}\int_y a\,dz$
(D) None of these

95. If $G \subset \mathbb{C}$ and G is open and connected. Also if f is a branch of log z on G, then the totality of branches of log z are the functions—
(A) $f(z)$ (B) $f((z) + 2n\pi i$
(C) $f(z)$ + const. (D) None of these

96. A branch of logarithm function is—
(A) Continuous function
(B) Differential function
(C) Analytic function
(D) None of these

97. A derivative of branch of logarithm function is—
(A) z (B) $\frac{1}{z}$
(C) 0 (D) None of these

98. The Cauchy–Riemann equation are if $\mu = \mu(x, y)$ and $v(x, y) = v$
(A) $\frac{\delta\mu}{\delta x} = \frac{\delta v}{\delta y}, \frac{\delta\mu}{\delta y} = -\frac{\delta v}{\delta x}$
(B) $\frac{\delta^2\mu}{\delta x^2} = \frac{\delta^2\mu}{\delta y^2}$
(C) $\frac{\delta^3 v}{\delta x^3} = \frac{\delta^2\mu}{\delta x^2}$
(D) None of these

99. A function $\mu = \mu(x, y)$ is harmonic if—
(A) μ have continuous second derivative
(B) μ have continuous second derivative and $\frac{\delta^2\mu}{\delta x^2} + \frac{\delta^2\mu}{\delta y^2} = 0$
(C) $\frac{\delta^2\mu}{\delta x^2} + \frac{\delta^2\mu}{\delta y^2} = 0$
(D) None of these

100. If y and σ are closed rectifiable curve having same initial points, then for ever $a \notin \{y\}$—
(A) $n(y\,;a) = -n(y\,;a)$
(B) $n(y\,;a) = n(-y\,;a)$
(C) $n(y\,;a) = -n(-y\,;a)$
(D) None of these

101. If y and σ are closed rectifiable curves having same initial points, then for every $a \notin \{y\} \cup \{\sigma\}$—
(A) $m(y+\sigma\,;a) = n(y\,;a)$
(B) $m(y+\sigma; a) = n(\sigma\,;a)$
(C) $m(y+\sigma; a) = n(y\,;a) + n(\sigma\,;a)$
(D) None of these

102. If y_0 and y_1 are two closed rectifiable curves in G and y_0 is homotopic to y_1, then for every analytic function f on G—
(A) $\int_{y_0} f \neq \int_{y_1} f$ (B) $\int_{y_0} f = \int_{y_1} f$
(C) $\int_{y_0} f + \int_{y_1} f = 0$ (D) None of these

103. If y is a closed rectifiable curve in G and y in homotopic to 0, then for all $\omega \in \mathbb{C} - G$—
(A) $n(y\,;\omega) = 1$ (B) $n(y\,;\omega) = 2$
(C) $n(y\,;\omega) = 0$ (D) $n(y\,;\omega) = 3$

104. If $T_1(z) = \frac{z+2}{z+3}$ and $T_2(z) = \frac{z}{z+1}$, then $T_2^{-1}T_1(z)$ is—
(A) $z+3$ (B) $z+2$
(C) $z+6$ (D) $z-3$

105. If $T_1(z) = \frac{z+2}{z+3}$ and $T_2(z) = \frac{z}{z+1}$, then $T_1T_2(z)$ is equal to—

(A) $\frac{3z+2}{4z+2}$ (B) $\frac{2}{2z+1}$
(C) $\frac{3z}{4z-2}$ (D) None of these

106. If $\Sigma \frac{z^n}{n!}$ is a power series, its radius of convergence is—
(A) 0 (B) ∞
(C) 1 (D) n

107. For the power series $\Sigma \frac{n!}{n^n}, z^n$ the radius of convergence is—
(A) e (B) 1
(C) ∞ (D) Zero

108. If (z_1, z_2, z_3) is an orientation of T, then left side of T with respect to (z_1, z_2, z_3) is—
(A) $\{z : \text{Im}\,(z, z_1, z_2, z_3) < 0\}$
(B) $\{z : \text{Im}\,(z, z_1, z_2, z_3) > 0\}$
(C) $\{z : \text{Im}\,(z, z_1, z_2, z_3) = 0\}$
(D) None of these

109. If $y : [a, b] \to \mathbb{C}$ is of bounded variation, then and P, Q are partition of $[a, b]$. If $P \subset Q$ then—
(A) $v(y ; P) \le v(y ; Q)$
(B) $v(y ; Q) \le v(y ; P)$
(C) $v(y ; P) = v(y ; Q)$
(D) None of these

110. If $y : [a, b] \to \mathbb{C}$ and P is any partition of $[a, b]$. Then variation of y, $v(y)$ is equal to—
(A) $\inf \{v(y ; P)\}$
(B) $\sup \{v(y ; P)\}$
(C) $\max \{v(y ; P)\}$
(D) $\min \{v(y ; P)\}$

111. If $y : [a, b] \to \mathbb{C}$ and y is of bounded variation then total variation $v(y)$ is—
(A) $v(y) \le \int_b^i |y'(t)|dt$
(B) $v(y) \ge \int_b^i |y'(t)|dt$
(C) $v(y) = \int_b^i |y'(t)|dt$
(D) None of these

112. If y is a rectifiable curve in $\mathbb{C}$ and F_n and F are continuous function on $\{y\}$. Also if $F = \mu - \lim F_n$ on $\{y\}$, then—
(A) $\int_y F = \lim \int_y F_n$ (B) $\int_y F = \int_y F_n$
(C) $\int_y F \neq \lim \int_y F_n$ (D) $\int_y F \neq \int_y F_n$

113. If $f : G \to \mathbb{C}$ is analytic and $R = d(a, \delta G)$ then for $a \in G$ and $|z - a| < R$—
(A) $f(z) = a_n (z - a)^n$
(B) $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$
(C) $f(z) = \Sigma (z - a)^n$
(D) None of these

114. If f is analytic in a ball $B(a ; R)$ and $|f(z)| \le M$, $\forall a \in R(a ; R)$, then—
(A) $|f^{(n)}(a)| \le \frac{n!.M}{R^n}$ (B) $|f^{(n)}(a)| \ge \frac{n!\,M}{R^n}$
(C) $|f^{(n)}(a)| = n!\,M$ (D) None of these

115. If f is analytic in a disk $B(a ; R)$ and y is closed rectifiable curve in $B(a ; R)$. Then—
(A) $\int_y F = 0$ (B) $\int_y F \neq 0$
(C) $\int_y F = R$ (D) None of these

116. If series Σa_n converges absolutely, then—
(A) Σa_n converges
(B) Σa_n does not converges
(C) Σa_n diverges
(D) None of these

117. If $f : G \to \mathbb{C}$ is differentiable at a point $a \in G$, then—
(A) f is discontinuous at a
(B) f is continuous at a
(C) f is constant at a
(D) None of these

118. If f is analytic function in some domain, then in that domain—
(A) f is continuous only
(B) f is differentiable only
(C) f is continuous and differentiable both
(D) None of these

119. $f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n$ have radius of convergence $R > 0$, then—
(A) For $n \le 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$
(B) For $n \ge 0$, $a_n = 0$

(C) For $n \geq 0$, $a_n = \frac{1}{n!} f^{(n)}(a)$

(D) None of these

120. If $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all $z \in G$ and G is open and connected, then—

(A) f is constant function

(B) f is increasing function

(C) f is decreasing function

(D) None of these

121. A path in same region is—

(A) Continuous function

(B) Discontinuous function

(C) Differentiable function

(D) None of these

122. A path is said to be smooth path if—

(A) It is continuous function

(B) It is a continuous and differentiable function

(C) Differentiable function

(D) None of these

123. A path y is piecewise smooth in interval $[a, b]$ if in a partition P of $[a, b]$—

(A) In each sub-interval y is continuous

(B) In each sub-interval y is smooth path

(C) In each sub-interval y is a path

(D) None of these

Answers with Hints

1. (C) $(1+i)^{10} + (1-i)^{10}$

$$= \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]^{10} + \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]^{10}$$

$$= 2^5\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{10} + 2^5\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{10}$$

$$= 2^5\left[\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right) + \left(\cos\frac{10\pi}{4} - i\sin\frac{10\pi}{4}\right)\right]$$

$$= 2^5\left[2\cos\frac{10\pi}{4}\right]$$

$$= 2^6\cos\left(\frac{5\pi}{2}\right) = 2^6 \times 0 = 0$$

2. (A)

$$z = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^{35} + \left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)^{200}$$

$$= \left(\cos\frac{70\pi}{3} + i\sin\frac{70\pi}{3}\right) + \left(\cos\frac{400\pi}{3} - i\sin\frac{400\pi}{3}\right)$$

$$= \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) + \left(\cos\frac{4\pi}{3} - i\sin\frac{4\pi}{3}\right)$$

$$= 2\cos\frac{4\pi}{3} = -2\cos\frac{\pi}{3}$$

$$= -2 \times \frac{1}{2} = -1.$$

$\therefore$ $\text{Re}(z) < 0$

3. (B)

4. (D) $$\frac{a+ib}{a'+ib'} = \frac{a+ib}{a'+ib'} \times \frac{a'-ib'}{a'-ib'} = \frac{(aa'+bb') + i(a'b - ab')}{(a')^2 + (b')^2}$$

This number will be real, if its imaginary part is zero

$\therefore$ $a'b - ab' = 0$.

5. (C) $(1+i)^2 = 1 + i^2 + 2i = 1 + (-1) + 2i = 2i$

$\therefore$ Conjugate of $(1+i)^2$ = conjugate of $2i = -2i$

6. (A) Since the set of complex number does not possess order relation. Hence (B), (C) and (D) are wrong.

7. (B) The relation $|z_1 + z_2| \geq |z_1| + |z_2|$ is false. The correct relation is $|z_1 + z_2| \leq |z_1| + |z_2|$.

8. (B) Given that

$$x = -2 - \sqrt{3}i$$

$$\therefore \quad x + 2 = -\sqrt{3}i$$

$$\therefore \quad (x+2)^2 = (-\sqrt{3}i)^2$$

$$\Rightarrow \quad x^2 + 4x + 4 = -3$$

$$\Rightarrow \quad x^2 + 4x + 7 = 0$$

Dividing $2x^4 + 5x^3 + 7x^2 + 41$ by $x^2 + 4x + 7$ we get $2x^4 + 5x^3 + 7x^2 + 41$

$= (2x^2 - 3x + 5)(x^2 + 4x + 7) + x + 6$

$= (2x^2 - 3x + 5)(0) + (-2 - \sqrt{3}i) + 6$

$= 4 - \sqrt{3}i$

9. (C) $\exp(\exp i\theta) = e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta}$

$= e^{\cos\theta}.e^{i\sin\theta}$

$= e^{\cos\theta}\{\cos(\sin\theta) + i\sin(\sin\theta)\}$

$\therefore$ Real (Exp (exp $i\,\theta$))

$= e^{\cos\theta}.\cos(\sin\theta)$

10. (C) Let

$$1 - i = r(\cos\theta + i\sin\theta)$$

$\therefore \quad r\cos\theta = 1,$

$r\sin\theta = -1$

$\therefore r^2\cos^2\theta + r^2\sin^2\theta = 1 + 1$

$\therefore \quad r^2 = 2$

$\Rightarrow \quad r = \sqrt{2}$

and $\quad \tan\theta = \dfrac{r\sin\theta}{r\cos\theta} = -1$

$\Rightarrow \quad \theta = -\dfrac{\pi}{4}$

$\therefore \quad 1 - i = \sqrt{2}\left(\cos\dfrac{\pi}{4} - i\sin\dfrac{\pi}{4}\right)$

$= \sqrt{2}\,e^{-i\pi/4}$

11. (B) Given that

$z = (x, y)$

i.e., $\quad z = x + iy$

$\therefore \quad \bar{z} = x - iy$

and $\quad z\bar{z} = z$

$\Rightarrow \quad \bar{z} = 1$

12. (D)

$$\left(\frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}\right)^4$$

$= (\cos\theta + i\sin\theta)^4.(\cos\theta - i\sin\theta)^{-4}$

$= (\cos\theta + i\sin\theta)^4.(\cos\theta + i\sin\theta)^4$

$= (\cos\theta + i\sin\theta)^8$

$= \cos 8\theta + i\sin 8\theta.$

13. (C) $(\sin\theta + i\sin\theta)^6$

$= \left[\cos\left(\dfrac{\pi}{2} - \theta\right) + i\sin\left(\dfrac{\pi}{2} - \theta\right)\right]^6$

$= \cos 6\left(\dfrac{\pi}{2} - \theta\right) + i\sin 6\left(\dfrac{\pi}{2} - \theta\right)$

$= -\cos 6\theta + i\sin 6\theta$

14. (C) $(x - y)(x - wy)(x - w^2y)$

$= x^3 - (1 + w + w^2)x^2y + (1 + w + w^2)xy^2 - w^3y^3$

$= x^3 - (0)x^2y + (0)xy^2 - (1)y^3$

$= x^3 - y^3 \qquad [\because 1 + w + w^2 = 0 \text{ and } w^3 = 1]$

15. (D) The reciprocal of $a + ib$ is $\dfrac{1}{a + ib}$

$\therefore \quad \dfrac{1}{a + ib} = \dfrac{1}{a + ib} \times \dfrac{a - ib}{a - ib}$

$= \dfrac{a - ib}{a^2 + b^2}$

$= \dfrac{a}{a^2 + b^2} - \dfrac{ib}{a^2 + b^2}$

16. (C)

17. (C) Let $\alpha = z_2 - z_3$, $\beta = z_3 - z_1$, $\gamma = z_1 - z_2$ then, $\alpha + \beta + \gamma = 0 \qquad \ldots(1)$

Since ABC is an equilateral triangle.

$\therefore$ AB = BC = CA

$\Rightarrow \quad |z_2 - z_3|^2 = |z_3 - z_1|^2 = |z_1 - z_2|^2$

$\Rightarrow \quad \alpha\bar{\alpha} = \beta\bar{\beta} = \gamma\bar{\gamma} = K \qquad [\because |z|^2 = z\bar{z}]$

From (1)

$\bar{\alpha} + \bar{\beta} + \bar{\gamma} = 0$

$\dfrac{K}{\alpha} + \dfrac{K}{\beta} + \dfrac{K}{\gamma} = 0$

or, $\dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} + \dfrac{1}{z_1 - z_2} = 0$

or, $\quad \dfrac{1}{z_1 - z_2} = \dfrac{1}{z_1 - z_3} + \dfrac{1}{z_3 - z_2}$

18. (D) $\quad (x - 1)^3 + 8 = 0$

$\Rightarrow \quad (x - 1)^3 = -8$

$\therefore \quad x - 1 = (-8)^{1/3}$

$= -2, -2w, -2w^2$

Hence $\quad x = -1, 1 - 2w, 1 - 2w^2$

19. (C) Let

$\sqrt{3 + 4i} = x + i$

Then $\quad 3 + 4i = x^2 - y^2 + 2ixy$

$\therefore \quad x^2 - y^2 = 3$ and $xy = 2$

Solving these, we get

$x = \pm 2, y = \pm 1$

Thus, we have $x = 2, y = 1$ or $x = -2, y = -1$

$\therefore \quad \sqrt{3+4i} = 2 + i$

20. (A) Given that

$$(a + ib)^{1/3} = x + iy$$
$$\Rightarrow \quad (a + ib) = (x + iy)^3$$
$$\therefore \quad a + ib = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Comparing real and imaginary parts, we get

$$\therefore \quad x^3 - 3xy^2 = a;$$
$$3x^2y - y^3 = b$$
$$\text{or,} \quad x(x^2 - 3y^2) = a,$$
$$y(3x^2 - y^2) = b$$
$$\text{or,} \quad x^2 - 3y^2 = \frac{a}{x},$$
$$3x^2 - y^2 = \frac{b}{y}$$
$$\therefore \quad x^2 - 3y^2 + 3x^2 - y^2 = \frac{a}{x} + \frac{b}{y}$$
$$\text{or} \quad 4(x^2 - y^2) = \frac{a}{x} + \frac{b}{y}$$

21. (B) $(1 - w + w^2)^5 + (1 + w - w^2)^5$

$$= (-w - w)^5 + (-w^2 - w^2)^5$$
$$= -32w^5 - 32w^{10} = -32w^2 - 32w$$
$$= -32(w + w^2) = -32.(-1)$$
$$= 32$$

22. (B) $z_1 = (1, 2), z_2 = (4, -2), z_3 = (1, -6)$

Here distance between z_1 and z_2

= distance between z_2 and $z_3 = 5$

But distance between z_1 and $z_3 = 8$

Hence, z_1, z_2 and z_3 forms a isosceles triangle.

23. (D) We know that

$$\log_a a = 1$$
$$\therefore \quad \log_i i = 1$$

24. (B)

$$z = \left(\frac{1}{1-2i} + \frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$$
$$= \frac{(4-5i)}{(1-2i)(1+i)} \times \left(\frac{3+4i}{2-4i}\right)$$
$$= \frac{32+i}{2(1-7i)}$$
$$= \frac{32+i}{2(1-7i)} \times \left(\frac{1+7i}{1+7i}\right)$$
$$= \frac{25+225i}{100} = \frac{1}{4} + \frac{9}{4}i$$

$$\therefore \quad \text{Amp } z = \tan^{-1}\frac{\frac{9}{4}}{\frac{1}{4}} = \tan^{-1} 9$$

25. (A)

$$(1 + i)^n = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n$$
$$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$$

$\therefore$ Real part of $(1 + i)^n = 2^{n/2}\cos\frac{n\pi}{4}$

26. (D)

27. (B) Let $z = x + iy$

$$\therefore \quad \frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy}$$
$$\Rightarrow \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$
$$= \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$
$$\therefore \quad \text{Arg}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$$
$$\Rightarrow \tan^{-1}\frac{2y}{x^2+y^2-1} = \frac{\pi}{3}$$
$$\Rightarrow \frac{2y}{x^2+y^2-1} = \tan\frac{\pi}{3} = \sqrt{3}$$
$$\Rightarrow x^2 + y^2 - \frac{2}{\sqrt{3}}y - 1 = 0$$

which is a circle.

28. (D)

$$xyz = (a + b)(aw + bw^2)(aw^2 + bw)$$
$$= [a^2w + b^2w^2 + ab(w + w^2)](aw^2 + bw)$$
$$= [a^2w + b^2w^2 - ab](aw^2 + bw)$$
$$= a^3w^3 + b^3w^3$$
$$= a^3 + b^3 \qquad [\because w^3 = 1]$$

29. (B) 30. (A)

31. (A) Let

$$z = x + iy$$
$$\Rightarrow \quad \bar{z} = x - iy$$

$\therefore \arg(z) + \arg(\bar{z})$

$$= \tan^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{-y}{x}\right)$$
$$= \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{y}{x}\right) = 0$$

32. (C) $z^2 = \bar{z}$

$\Rightarrow \quad (x + iy)^2 = x - iy$

$\Rightarrow \quad x^2 - y^2 + 2ixy = x - iy$

Comparing real and imaginary parts of both side, we get

$$x^2 - y^2 = x \quad \text{...(i)}$$
$$2xy = -y \quad \text{...(ii)}$$

From (ii) $y = 0$ and $x = -\frac{1}{2}$

$\therefore$ From (i), when $y = 0$, $x = \pm 1$, when $x = \frac{-1}{2}$, $y = \pm\frac{\sqrt{3}}{2}$

$\therefore$ $(0, 1), (0, -1), \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{-1}{2}, -\frac{\sqrt{3}}{2}\right)$

satisfy the *eg.*

(i) and (ii)

$\therefore$ There are four solutions.

33. (A) $\frac{4+3i}{3-4i} = x + iy$

$\therefore \quad 4 + 3i = 3x + 4y + (3y - 4x)i$

$\therefore \quad 3x + 4y = 4 \quad$ ...(i)

and $\quad 3y - 4x = 3 \quad$ ...(ii)

Solving (i) and (ii), we get

$$x = 0, y = 1$$

34. (D) Given that, $|z - 1| = 2$

$\therefore \quad |z - 1|^2 = 4$

$\therefore \quad (z - 1)(\bar{z} - 1) = 4$

$\Rightarrow \quad z\bar{z} - z - \bar{z} + 1 = 4$

$\Rightarrow \quad z\bar{z} - \bar{z} - z = 3$

35. (D)

36. (B) Sol. $|z| = |z - 1|$

$|x + iy| = |(x - 1) + iy|$

$\Rightarrow \quad x^2 + y^2 = (x - 1)^2 + y^2$

$\Rightarrow \quad 2x = 1$

$\Rightarrow \quad x = \frac{1}{2}$

$\Rightarrow$ Real of $z = \frac{1}{2}$

37. (D) $\left|\frac{z - 5i}{z + 5i}\right| = 1$

$\Rightarrow \quad |z - 5i| = |z + 5i|$

$\Rightarrow \quad |x + (y - 5)i| = |x + (y + 5)i|$

$\Rightarrow \quad x^2 + (y - 5)^2 = x^2 + (y + 5)^2$

$\Rightarrow \quad y = 0$

38. (B) $2\cos\alpha_1 = a + \frac{1}{a}$

$\Rightarrow \quad a^2 - 2a\cos\alpha_1 + 1 = 0$

$\Rightarrow \quad a = \frac{2\cos\alpha_1 \pm \sqrt{4\cos^2\alpha_1 - 4}}{2}$

$= \cos\alpha_1 \pm i\sin\alpha_1$

Similarly from $2\cos\alpha_2 = b + \frac{1}{b}$

$b = \cos\alpha_2 \pm i\sin\alpha_2$ etc.

$abc \ldots\ldots + \frac{1}{abc\ldots}$

$= (\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2)$

$\ldots\ldots + \frac{1}{(\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2)}$

$= \{\cos(\alpha_1 + \alpha_2 + \ldots) + i\sin(\alpha_1 + \alpha_2 + \ldots)\}$

$+ \frac{1}{\cos(\alpha_1 + \alpha_2 + \ldots) + i\sin(\alpha_1 + \alpha_2 + \ldots)}$

$= \{\cos(\alpha_1 + \alpha_2 + \ldots) + i\sin(\alpha_1 + \alpha_2 + \ldots)\}$

$+ \{\cos(\alpha_1 + \alpha_2 + \ldots) - i(\sin(\alpha_1 + \alpha_2 + \ldots)$

$= 2\cos(\alpha_1 + \alpha_2 + \ldots)$

39. (A) $x^2 - 2x + 4 = 0$

$\Rightarrow \quad x = \frac{2 \pm \sqrt{4 - 16}}{2}$

$= 2\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)$

$= 2\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right)$

Let $\quad \alpha = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

and $\quad \beta = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$\therefore \quad \alpha^n + \beta^n = 2^n\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n$

$+ 2^n\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n$

$= 2^n\left[\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + \left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)\right]$

40. (B) $z = x + iy$

$$\therefore \frac{2z+1}{iz+1} = \frac{(2x+1)+2iy}{i(x+iy)+1}$$

$$= \frac{(2x+1)+2iy}{(-y+1)+ix}$$

$$= \frac{(2x+1)+2iy}{(-y+1)+ix} \times \frac{(-y+1)-ix}{(-y+1)+ix}$$

$$= \frac{(2x-y+1)-i(2x^2+2y^2+x-2y)}{(-y+1)^2+x^2}$$

Given that $\text{Im}\left(\frac{2z+1}{iz+1}\right) = -2$

$$\Rightarrow \frac{-(2x^2+2y^2+x-2y)}{x^2+y^2-2y+1} = -2$$

$\Rightarrow x + 2y = 2$, which is a straight line.

41. (A) $|z+5|^2 + |z-5|^2 = 75$

$\Rightarrow |(x+5)+iy|^2 + |(x-5)+iy|^2 = 75$

$\Rightarrow \quad 2x^2 + 2y^2 = 25$

which is a circle.

42. (D) Let $\cos^{-1}(\cos\theta + i\sin\theta) = x + iy$

or $\cos\theta + i\sin\theta$

$$= \cos(x+iy)$$
$$= \cos x . \cos iy - \sin x \sin yi$$
$$= \cos x \cosh y - \sin x \sinh y$$

Equating real and imaginary parts on both sides, we get

$$\cos\theta = \cos x \cosh y \quad \ldots\text{(i)}$$
$$\sin\theta = -\sin x \sinh y \quad \ldots\text{(ii)}$$

From (i) and (ii), we get

$$\frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = \cosh^2 y - \sinh^2 y = 1$$

or, $\cos^2\theta \sin^2 x - \sin^2\theta \cos^2 x = \cos^2 x \sin^2 x$

or, $\cos^2\theta \sin^2 x - \sin^2\theta (1 - \sin^2 x)$

$$= (1 - \sin^2 x)\sin^2 x$$

or, $\sin^4 x + \sin^2 x(\cos^2\theta + \sin^2\theta - 1) - \sin^2\theta = 0$

or, $\sin^4 x - \sin^2\theta = 0$

or, $\sin^2 x = \sin\theta$,

or $\sin x = \sqrt{\sin\theta}$

or, $x = \sin^{-1}\sqrt{\sin\theta}$

From (ii) $\sin\theta = -\sin x \sinh y$

or, $\sinh y = -\sqrt{\sin\theta}$

or, $y = \sinh[-\sqrt{\sin\theta}]$

$$= \log_e[-\sqrt{\sin\theta} + \sqrt{(\sin\theta + 1)}]$$

Hence, $\cos^{-1}(\cos\theta + i\sin\theta)$

$$= x + iy$$
$$= \sin^{-1}(\sqrt{\sin\theta}) + i\log_e[-\sqrt{\sin\theta} + \sqrt{\sin\theta + 1}]$$

43. (B)	44. (A)	45. (C)	46. (B)	47. (C)
48. (A)	49. (B)	50. (A)	51. (A)	52. (A)
53. (B)	54. (A)	55. (C)	56. (A)	57. (B)
58. (B)	59. (C)			

60. (B) Here $w = \frac{z+2}{z+3}$

$$\Rightarrow \quad z = \frac{2-3w}{w-1}$$

$$\therefore \quad T_1^{-1}(w) = \frac{2-3w}{w-1}$$

61. (A) Here $a_n = \frac{1}{n^p}$

and $a_{n+1} = \frac{1}{(n+1)^p}$

$\therefore$ Radius of convergence,

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^p}{n^p} = 1$$

62. (C) Let $f(z) = \frac{\sin z}{(z-\pi)^2}$

Then singularities of $f(z)$ are given by

$$(z-\pi)^2 = 0$$

$\Rightarrow z = \pi$ is a pole of order two of $f(z)$.

63. (A)	64. (A)	65. (A)	66. (B)	67. (B)
68. (A)	69. (C)	70. (C)	71. (B)	72. (C)
73. (A)	74. (A)	75. (D)	76. (C)	77. (A)
78. (A)	79. (C)	80. (C)	81. (A)	82. (A)
83. (B)				

84. (C) $T_2T_1(z) = T_2\left(\frac{z+2}{z+3}\right)$

$$= \frac{\frac{z+2}{z+3}}{\frac{z+2}{z+3}+1} = \frac{z+2}{2z+5}$$

85. (A) Here $a_n = \frac{n+1}{(n+2)(n+3)}$

and $a_{n+1} = \frac{n+2}{(n+3)(n+4)}$

$\therefore$ The radius of convergence,

$$R = \lim_{n \to \infty} \frac{a_n}{a_{n+1}}$$

$$= \lim_{n \to \infty} \frac{n+1}{(n+2)(n+3)} \cdot \frac{(n+3)(n+4)}{(n+2)}$$

$$= \lim_{n \to \infty} \frac{(n+1)(n+4)}{(n+2)^2} = 1$$

86. (D) Here $f(z) = \dfrac{1-e^z}{1+e^z}$

Poles of $f(z)$ are obtained by equating to zero the denominator of $f(z)$. *i.e.*

$$1 + e^z = 0$$

$$\Rightarrow \quad e^z = -1 = e^{2\pi ni + \pi i}$$

$$\Sigma = (2n+1)\pi i,$$

where n is any integer

Hence $z = (2n+1)\pi i$ $(n \in I)$ are the simple poles of $f(z)$.

Obviously $z = \infty$ is a limit point of these poles

$\therefore$ $z = \infty$ is a non-isolated essential singularity.

87. (C) 88. (B) 89. (B) 90. (D) 91. (A)
92. (A) 93. (A) 94. (B) 95. (B) 96. (C)
97. (B) 98. (A) 99. (B) 100. (C) 101. (C)
102. (B) 103. (C)

104. (B) $T_2T_1(z) = T_2^{-1}\left(\dfrac{z+2}{z+3}\right)$

$$= \frac{\dfrac{z+2}{z+3}}{\dfrac{z+2}{z+3} - 1} = z + 2$$

105. (A) $T_2T_1(z) = T_1\left(\dfrac{z}{z+1}\right)$

$$= \frac{\dfrac{z}{z+1} + 2}{\dfrac{z}{z+1} + 3} = \frac{3z+2}{4z+3}$$

106. (B) We have

$$a_n = \frac{1}{n!}, \; a_{n+1} = \frac{1}{(n+1)!}$$

$$R = \lim_{n \to \infty} \frac{a_n}{a_{n+1}}$$

$$= \lim_{n \to \infty} \frac{(n+1)!}{n!}$$

$$= \lim_{n \to \infty} (n+1) = \infty$$

107. (A) We have

$$a_n = \frac{n!}{n^n} \; a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$R = \lim_{n \to \infty} \frac{a_n}{a_{n+1}}$$

$$= \lim_{n \to \infty} \frac{n!}{n^n} \cdot \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

108. (A) 109. (A) 110. (B) 111. (C) 112. (A)
113. (B) 114. (A) 115. (A) 116. (A) 117. (B)
118. (C) 119. (C) 120. (A) 121. (A) 122. (B)
123. (B)

Real Analysis

Sequences

Sequence is a function whose domain is the set of positive integers.

i.e., $\{a_n\}_{n=1}^{\infty}$

or $a_n = a(n)$,

$n = 1, 2, 3, \ldots\ldots$

e.g., (a) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\right\}$

(b) Let P_n be the nth Prime number

$\{P_n\}_{n=1}^{\infty} = \{2, 3, 5, \ldots\}$

Convergence of Sequence—A sequence $\{a_n\}_{n=1}^{\infty}$ converges to a real number A iff for each $\in\, = 0$, there is a positive integer N, such that for all $n \geq N$, we have $|a_n - A| < \in$.

Neighbourhood—A set N_x of real numbers is a neighbourhood of real number x iff, N_x contains an interval of positive length centered at x, *i.e.*, iff there is $\varepsilon > 0 : (x - \varepsilon, x + \varepsilon) \subset N_x$.

Convergent and divergent sequence— A sequence $\{a_n\}_{n=1}^{\infty}$ is convergent iff there is a real number. A such that $\{a_n\}_{n=1}^{\infty}$ converges to A. If $\{a_n\}_{n=1}^{\infty}$ is not convergent it is Divergent sequence.

Cauchy Sequence—A sequence $\{a_n\}_{n=1}^{\infty}$ is Cauchy iff for each $\varepsilon > 0$, there is a positive integer N : if $m, n \geq N$, then $|a_n - a_m| < \varepsilon$.

Limit of a Sequence—If a sequence is convergent the unique number to which it converges is the limit of the sequence.

Accumulation Point—For a set S of real numbers, a real number A is an accumulation point of S iff every neighbourhood of A contains infinitely many points of S.

Subsequence—Let $\{a_n\}_1^{\infty}$ be a sequence and $\{n_k\}_1^{\infty}$ be any sequence of positive integer such that $n_1 < n_2 < n_3 < \ldots$ the sequence $\{a_{nk}\}_{k=1}^{\infty}$ is called a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Increasing sequence—Sequence $\{a_n\}_{n=1}^{\infty}$ is increasing, iff $b_n \geq b_{n+1}$ for all n.

Monotone Sequence—Sequence that is either increasing or decreasing.

Bounded above sequence—Sequence $\{a_n\}_{n=1}^{\infty}$ is bounded above, iff there exist a real number N : $a_n \leq N$ for all n.

Bounded below sequence—Sequence $\{a_n\}_{n=1}^{\infty}$ is bounded below iff there exist a real number M : $a_n \geq M$ for all n.

Bounded Sequence—Sequence $\{a_n\}_{n=1}^{\infty}$ is bounded, if it is bounded both from above and below $\Leftrightarrow$ the exist a real number S : $|a_n| \leq S$ for all n.

Series

Infinite series—An infinite series is a pair $\left\{\{a_n\}_{n=1}^{\infty}, \{S_n\}_{n=1}^{\infty}\right\}$, where $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $S_n = \sum_{k=1}^{\infty} a_k$ for all n, a_n is the nth term of the series and S_n is the nth partial sum of the series.

Convergence of Series—If $\sum_{n=1}^{\infty} a_n$ converges, then $\{S_n\}_{n=1}^{\infty}$ converges.

Converges absolutely—An infinite series $\sum_{n=1}^{\infty} a_n$ converges absolutely iff $\sum_{n=1}^{\infty} |a_n|$ converges.

Converges Conditionally—If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Cauchy's Product—Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are two infinite series and for each n define $c_n =$

$\sum_{k=0}^{n} a_n b_{n-k}$. The infinite series $\sum_{n=0}^{\infty} c_n$ is called the Cauchy's product of two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$.

Rearrangement of Series—Let $\sum_{n=0}^{\infty} a_n$ be an infinite series. If T is one-one function from {0, 1, 2, ...} onto {0, 1, 2, ...}), then the infinite series $\sum_{n=0}^{\infty} a\mathrm{T}_{(n)}$ is called a rearrangement of $\sum_{n=0}^{\infty} a_n$.

Interval of Convergence—If $\sum_{n=0}^{\infty} a_n x^n$ is a power series, then the set of points at which series converges is either.

(1) R, a set of all real numbers $(-\infty, \infty)$, (Interval of infinite radius)

(2) {0}, (Interval of zero radius)

(3) An interval of positive finite length centered at zero which may contain all, none or one of its end points.

These intervals are called interval of convergence.

Radius of Convergence—If $\sum_{n=0}^{\infty} a_n x^n$ has an interval of convergence c which is different from R and {0}, then there is a unique real number r such that, $(-r, r) \subset C \subset [-r, r]$. This number r is called the radius of convergence of the power series.

Uniform Convergence—A sequence $\{f_n\}_{n=1}^{\infty}$ of functions is said to converge uniformly on E if there is a function $f : E \to R$ such that for each $\in s_0$, there is N such that for each positive integer n, $n \geq N$ implies that $|f_n(x) - f(x)| < \in$ for all $x \in E$.

Power Series—Let $\{a_n\}_{n=0}^{\infty}$ be a sequene of real number. For each real number, x, a series $\sum_{n=0}^{\infty} a_n x^n$ is power series.

Maxima and Minima

(1) **Maximum value**—A continuous function $f(x)$ is said to have a maximum value for $x = a$, if $f(a)$ is greater than any other value of $f(x)$ lying in small neighbourhood of $x = a$.

In other words, $f(a)$ is a maximum value of $f(x)$, if $f(x)$ is increasing in $(a - h, a)$ and decreasing in $(a, a + h)$ where h is a small quantity.

(2) **Minimum value**—A continuous function $f(x)$ is said to have a minimum value at $x = a$, if $f(a)$ is smallest of all $f(x)$ lying in small neighbourhood of $x = a$.

In other words, $f(a)$ is minimum value of $f(x)$, if $f(x)$ is decreasing in $(a - h, a)$ and increasing in $(a, a + h)$, where h is a small quantity.

(3) **Conditions for Maxima**—The function $f(x)$ has a maximum value $f(a)$ if $f'(a) = 0$ and $f'(x)$ changes sign from positive to negative as x passes through a from left to right.

In general, for any even number n.

$f'(a) = f''(a) = f'''(a) = \ldots = f^{n-1}(a) = 0$ and $f''(a) < 0$, then $f(a)$ is a maximum value of $f(x)$.

In particular, if $n = 2$ then $f'(a) = 0$, $f''(a) < 0$, then the function is maximum at $x = a$.

(4) **Conditions for Minima**—The function $f(x)$ has a minimum value $f(a)$. If $f'(a) = 0$ and $f'(x)$ change sign from negative to positive as x passes through a from left to right.

In general, for any even number n.

$f'(a) = f''(a) = f'''(a) = \ldots\ldots\ldots = f^{n-1}(a) = 0$ and $f^n(a) \leq 0$, then $f(a)$ is a minimum value of the function $f(x)$.

In particular, If $n = 2$ and $f'(a) = 0$, $f''(a)$ so the function is minimum at $x = a$.

Extreme value—Either a maximum value or minimum value $f(a)$ of the function $f(x)$ is said to be extreme value.

> **Note :** The tangent at maximum or minimum point of the curve is parallel to x-axis.

Stationary value—If $f'(a) = 0$, then $f(a)$ is said to be stationary value of the function $f(x)$ at $x = a$.

> **Note :** Every extreme value is stationary but every stationary value need not be an extreme value.

Greatest Value—The greatest value of a function in an interval (a, b) is either a maximum value of $f(x)$ at a point inside the interval or end value (*i.e.*, at $x = a$, or $x = b$) of $f(x)$ which ever is greater.

Least Value—The least value of $f(x)$ in an interval (a, b) is either a minimum value of $f(x)$ at a point insdie the interval or an end value (*i.e.* at $x = a$ or $x = b$) of $f(x)$ which ever is less.

Maxima and Minima for the function of two independent variable—

(i) **Maximum value**—Let $f(x, y)$ be continuous function. The value $f(a, b)$ is said to be maximum value of $f(x, y)$ if there exist some neighbourhood of the points (a, b) such that $(a+h, b+k)$ of this neighbourhood, other than (a, b).

$$f(a, b) > f(a+h, b+k)$$

(ii) **Minimum value**—Let $f(x, y)$ be a continuous function the value $f(a, b)$ is said to be maximum value of (x, y) if there exist some neighbourhood of the point (a, b) such that $(a+h, b+k)$ of this neighbourhood, other than (a, b)

$$f(a, b) < f(a+h, b+k)$$

Sufficient Condition for Maximum value— If $f_x(a, b) = 0, f_y(a, b) = 0$ and $f_x^2(a, b) = A$, $f_{xy}(a, b) = B$, $f_y^2(a, b) = C$, then $f(a, b)$ is maximum value if $AC - B^2 > 0$ and $A < 0$.

Sufficient Condition for Minimum value— If $f_x(a, b) = 0, f_y(a, b) = 0$ and $f_x^2(a, b) = A$, $f_{xy}(a, b) = B$, $f_y^2(a, b) = C$, then $f(a, b)$ is minimum value if $AC - B^2 > 0$ and $A > 0$.

Note : The value $f(a, b)$ is neither maximum nor minimum, if $AC - B^2 < 0$.

Extreme value—Either a maximum value or a minimum value $f(a, b)$ of the function $f(x, y)$ is said to be extreme value.

Sufficient Condition for Extreme value—If $f_x(a, b) = 0, f_y(a, b) = 0$ and $f_x^2(a, b) = A, f_{xy} = (a, b) = B, f_y^2(a, b) = C$, then $f(a, b)$ is an extreme value if $AC - B^2 > 0$.

Fourier Series

L(I)—Set of Lebesgue-integrable function on interval 1.

L^2(I)—Set of square integrable on I.

Orthogonal system and orthonormal on interval 1—Let $\delta = \{\phi_0, \phi_1, \phi_2, \ldots\}$ be a collection of function in $L^2(I)$.

If $(\phi_n, \phi_m) = 0$ whenever $m \neq n$, the collection S is orthogonal system on I.

If each ϕ_n has $\| \phi_n \| = 1$, then S is orthonormal on I.

Fourier Series and Coefficients—Let $S = \{\phi_1, \phi_2 \ldots\}$ be orthonormal on interval I and fE L^2 (I) then $f(x) \sim \sum_{n=0}^{\infty} c_n \phi_n(x)$ is called Fouries series of f relative to S and $c_n = (f_1\phi_n) = \int f(x) \phi_n (x)dx$ $(n = 0, 1, 2, \ldots)$ is called Fouries Coefficient of f relative to S.

Functions of Several Variables—If $D \subset R^n$ the function

$f(x) = f(x_1, x_2, \ldots x_n)$, where ($x \in \mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ is called function of several variables.

Limit—A function $f(x)$, $x \in D \subset R^n$ has a limit l, *i.e.*, $\lim_{x \to a} f(x) = l$, if for given $\in > 0$, there exist $\delta = 0$ such that $|f(x) - l| < \in$ for every $a \in D$, $\|x - a\| < \delta$.

Continuity—A function $f : D \to \mathbb{R}^n$ is continuous at $x = a$, if for each $\in > 0$ there exist $\delta > 0$ such that $|f(x) - f(a)| < \in$ whenever $\|x - a\| < \delta$ and $x \in D \subset \mathbb{R}^n$ or $f : D \to \mathbb{R}^n$ is continuous at a iff $\lim_{x \to a} f(x) = f(a)$.

Uniform Continuity—A function $f : D \to \mathbb{R}^n$ is uniformaly continuous on D if it is continuous at every $x \in D \subset \mathbb{R}^n$.

Some Important Results

1. $f : D \to \mathbb{R}^n, g : D \to \mathbb{R}^n$ then, $\lim (f \pm g)(x) = \lim f(x) \pm \lim g(x)$ $\lim (f \cdot g)(x) = \lim f(x) \cdot \lim g(x)$, $\lim (f/g)(x) = \lim f(x)/\lim g(x)$. If $\lim g(x) \neq 0$.
2. The range of a function continuous on a compact set is compact.
3. A real valued function continuous on a compact set is bounded and attains its bound.
4. A real valued function continuous on a closed rectangle $[a, b]$ is bounded and attains its bound.
5. A function continuous on a compact domain is uniformaly continuous.
6. Let f be a real valued function with domain $D \subset R^n$. Let D be such that

   $X, Y \in D \Rightarrow ex + (1 + \tau) Y \in D \ \forall \ \tau \in [0, 1]$,

   then f assumes every value between $f(x)$ and $f(y)$.
7. If $\lim_{x \to a} f(x) = b$

   and $b = (b_1, b_2, \ldots b_m)$

   $f = (f_1, f_2, \ldots f_m)$

   then $\lim_{x \to a} f_i(x) = b_i \ (1 \leq i \leq m)$ and conversely

Riemann (Stielties) Integration

Partition—A partition P of $[a, b]$ is a finite set $\{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \ldots < x_n = b$.

Refinement—If P and Q are partition of $[a, b]$ with $P \subset Q$, then Q is refinement of P.

Riemann Integration—Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function and P $\{x_1, x_2, \ldots, x_n\}$ is a partition of $[a, b]$. For each i, $(i = 1, 2, \ldots, n)$ define

$$M_i(f) = \sup \{f(x) : x \in [x_{i-1} - x_i]\}$$

$$m_i(f) = \inf \{f(x) : x \in [x_{i-1} - x_i]\}$$

$$U(P, f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1})$$

(Upper darboux sum of f)

$$L(P, f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1})$$

(lower darboux sum of f)

Then upper integration of f.

$$\int_a^{-b} f\,dx = \inf\{U(P, f) : P \text{ is a partition}\}$$

Lower integration of f.

$$\int_{-a}^{b} f\,dx = \text{Sup}\{L(P, f) : P \text{ is a Partition}\}$$

Riemann integrable on $[a, b]$—f is Riemann integrable if $\int_a^b f(x)\,dx$ exist.

i.e. $\int_a^{-b} f\,dx = \int_{-a}^{b} f\,dx = \int_a^b f\,dx$

Riemann—Stielties integration—Suppose $f : [a, b] \to R$ is bounded and $d : [a, b] \to R$ is an increasing function. For each partition

$$P = \{x_0, \ldots, x_n\} \text{ define}$$

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i(f)[x_i(\alpha) - x_{i-1}(\alpha)]$$

$$L(P, f, \alpha) = \sum_{i=1}^{n} m_i(f)[x_i(\alpha) - x_{i-1}(\alpha)]$$

$$\int_a^{-b} f\,d\alpha = \inf\{U(P, f, \alpha) : P \text{ is a partition}\}$$

$$\int_{-a}^{b} f\,d\alpha = \sup\{L(P, f, \alpha) : P \text{ is a partition}\}$$

f is Riemann-Stielties integrable with respect to α on $[a, b]$ if

$$\int_a^{-b} f\,d\alpha = \int_{-a}^{b} f\,d\alpha = \int_a^b f\,d\alpha$$

when $\alpha(x) = x$, the Riemann—Stielties integral with respect to α reduces to Riemann integration.

Some Important Theorems

1. Let $f : [a, b] \to \mathbb{R}$ be bounded and $\alpha : [a, b] \to R$ increasing function.
   Then if P and Q are any partions of $[a, b]$, we have
   (i) If $P \subset Q$, then $L(P, f, \alpha) \le L(Q, f, \alpha)$ and $U(Q, f, \alpha) \le U(P, f, \alpha)$
   (ii) $L(P, f, \alpha) \le U(Q, f, \alpha)$
   (iii) $\int_{-a}^{b} f\,d\alpha \le \int_a^b f\,d\alpha$
2. Let $f : [a, b] \to \mathbb{R}$ be bounded and $\alpha : [a, b] \to \mathbb{R}$ be increasing. Then f is Riemann integration on $[a, b]$ iff for each $\in$ so, there is a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) \le \in$.
3. If f'. $[a, b] \to R$ is monotone and $\alpha : [a, b] \to \mathbb{R}$ is increasing and continuous, then f is Riemann–Stielties integration on $[a, b]$.
4. If $f : [a, b] \to \mathbb{R}$ is continuous and $\alpha : [a, b] \to \mathbb{R}$ increasing, then f is Riemann–Stielties integration on $[a, b]$.
5. If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and f' is Riemann integration on $[a, b]$, then $\int_a^b f'\,dx = f(b) - f(a)$.
6. If $f_1, f_2 : [a, b] \to \mathbb{R}$ are bounded, $\alpha : [a, b] \to \mathbb{R}$ is increasing and f_1, f_2 are Riemann integration with respect to α on $[a, b]$, then—
   (a) For any real numbers $c_1, c_2, c_1f_1 + c_2f_2$ is also Riemann integration on $[a, b]$ and $\int_a^b (c_1f_1 + c_2f_2)\,d\alpha = c_1\int_a^b f_1 d\alpha + c_2\int_a^b f_2 d_\alpha$
   (b) If $f_1(x) \le f_2(x)$ for all $x \in [a, b]$, then $\int_a^b f_1(x)\,d\alpha \le \int_a^b f_2(x)\,d\alpha$
   (c) If $m \le f_1(x) \le M$ for all $x \in [a, b]$, then $m\{\alpha(b) - \alpha(a)\} \le \int_a^b f_1 d\alpha \le M\{a(b) - \alpha(a)\}$.
   (d) If $\beta : [a, b] \to R$ is increasing and f is Riemann integration with respect to β on $[a, b]$ and c_1 and c_2 are any non-negative real numbers, then f is Riemann integration with respect to $(c_1\alpha + c_2\beta)$ on $[a, b]$ and $\int_a^b fd\,(c_1\alpha + c_2\beta) = c_1\int_a^b fd\alpha + c_2\int_a^b fd\beta$.

7. Suppose $f : [a, b] \to R$ is bounded and $\alpha : [a, b] \to \mathbb{R}$ is increasing. If $a < c < b$, then f is Riemann–Stielties integration on $[a, b]$ iff f is Riemann–Stielties integration on $[a, c]$ and $[c, b]$ and $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$.
8. Suppose $f : [a, b] \to [c, d]$, $\alpha : [a, b] \to \mathbb{R}$ is increasing. f is Riemann Stielties integrable on $[a, b]$ and $\phi : [c, d] \to \mathbb{R}$ is continuous. Then ϕ of is Riemann–Stielties integrable on $[a, b]$.
9. If $f, g : [a, b] \to \mathbb{R}$, $\alpha : [a, b] \to \mathbb{R}$ is increasing, f, g is Riemann–Stielties integrable on $[a, b]$ then
   (a) fg is Riemann Stielties integrable on $[a, b]$
   (b) $|f|$ is Riemann Stielties integrable on $[a, b]$
   (c) $\left|\int_a^b f d\alpha\right| \le \int_a^b |f|\, d\alpha$
10. If $f : [a, b] \to \mathbb{R}$ is continuous and $\alpha : [a, b] \to \mathbb{R}$ is increasing, then there is $c \in [a, b]$ such that $\int_a^b f\,d\alpha = f(c)\,[\alpha(b) - \alpha[a)]$

Line and Surface Integrals

Line Integral

Curve—A curve Γ, in the xy plane is the set of points (x, y) such that $\Gamma = \{(x, y) : x = \phi(\tau), y = \psi(\tau); a \le \tau \le b\}$.

Closed Curve—If $\phi(a) = \phi(b)$ and $\psi(a) = \psi(b)$ the curve Γ is closed.

Double Points—For $a < \tau_1, \tau_2 < b$ and $\tau_1 \ne \tau_2 \Rightarrow \phi(\tau_1) = \phi(\tau_2)$ and $\psi(\tau_1) = \psi(\tau_2)$.

Jordan Curve—If curve closed and have no double points.

Regular Curve—If curve Γ is regular if it has no double points and if the interval (a, b) can be divided into a finite subintervals in each of which $[\psi'(\tau)]^2 + [\psi'(\tau)]^2 > 0$ where $\phi(\tau), \psi(\tau) \in c'$.

Regular region—A region is regular if it is bounded and closed and if its boundary consists of a finite number of regular Jordan curves which have no points in common with each other.

Plane region— $R_x = R\,[a, b\,\phi(x), \psi(x)]$

$R_y = R\,[a, b, \phi(y), \psi(y)]$

Line Integral—Let $f(x, y)$ is a function defined at every point $(x, y) \in \Gamma$ and Δ is a sub division of interval $[a, b]$ by $a = \tau_0, \tau_1, \ldots, \tau_n = b$. The line integral of $f(x, y)$ are

$$\int_\Gamma f(x, y)\,dx = \int_{x_0 y_0}^{x_1 y_1} f(x, y)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\phi(\tau_i'), \psi(\tau_i'))\,[\phi(\tau_i) - \phi(\tau_{i-1})]$$

$$\int_\Gamma f(x, y)\,dy = \int_{x_0 y_0}^{x_1 y_1} f(x, y)\,dx = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\phi(\tau_i'), \psi(\tau_i'))\,[\phi(\tau_i) - \phi(\tau_{i-1})]$$

where $x_0 = \phi(a)$, $y_0 = \psi(a)$

and $x_1 = \phi(b)$, $y_1 = \psi(b)$

and $\tau_{i-1} \le \tau_i' \le \tau_i$

Simply Connected domain—A domain D is simply connected if no Jordan curve in D contains in its interior a boundary point of D.

Some Important Theorems

1. If Γ is a regular curve and $f(x, y)$ on Γ exists, then $\int_\Gamma f(x, y)\,dx$ and $\int_\Gamma f(x, y)\,dy$ exists.
2. **Green theorem (Gauss's theorem) (First form)**—If R is region R_x and also R_y and Γ is bounded of R, where $P(x, y), Q(x, y) \in R$ then $\int_\Gamma P dx + Q dy = \int\int_R Q_x(x, y) - P_y(x, y]\,dx$ the line integral taken in positive sense.
3. **Green's theorem (Second form)**—If R is a region Rx and regular region S, Γ is bounded of R and $P(x, y), Q(x, y) \in R$, then $\int_\Gamma P dx + Q dy = \int\int_R [Q_x(x, y) - P_x(x, y)]\,ds$ the line integral taken in positive sense.
4. Area of the region R—
   $A = -\int_\Gamma y\,dx = \int_\Gamma x\,dy = \frac{1}{2}\int_\Gamma (-y)dx + x\,dy$
   the integral being in the positive sense.
5. If D is simply connected and $P(x, y)$, $Q(x, y) \in D$ and $Q_1(x, y) = P_2(x, y) \in D$ iff there exists $F(x, y) \in D$ such that $F_1 = P$, $F_2 = Q$.
6. If D is simply connected and $P(x, y), Q(x, y) \in D$, $Q_1(x, y) = P_2(x, y) \in D$ and Γ is a regular curve in D joining (a, b) with
   $x_0 = \phi(a)$ and $y_0 = \psi(a)$

Then line integral extended over Γ is independent of Γ.

7. If $P(x, y)$, $Q(x, y) \in D$, a simply connected domain, then $Q_x = P_y \in D$ iff $\int_\Gamma Pdx + Qdy = 0$ for every regular closed curve Γ in D.
8. If R (x, y, z) a point in $Vxy = V$ (R, $\phi(x, y)$, $\psi(x, y)$ and function ϕ, ψ exists in R and y is the angle between the positive z-axis and the exterior normal to $\in$, the boundary of V_{xy} then

$$\iiint_{V_{xy}} R_3 dy = \iint_{\in} R \cos y d\in$$

Line Integrals in Space—If $\Gamma = (x, y, z) : x = \phi(\tau)$, $y = \psi(\tau)$, $z = w\ (\tau)$, $a \leq \tau \leq 1$. The line integral is defined as

$$\int_\Gamma f(x, y, z)dx = \sum_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\psi(\tau_i'),$$
$$\psi(\tau_i'), \psi(\tau_i')) [\phi(\tau_i) - \phi(\tau_{i-1})]\tau_{i-1} \leq \tau_i' \leq \tau_i$$

Stokes' theorem—If $\in$ is the surface $z = f(x, y)$ bounded by the regular closed curve Γ and P, Q, y belongs to $\in$ and α, β, γ are direction angles to a directed normal to $\in$. Then

$$\int_\Gamma Pdx + Qdy + Rdz = \iint_{\in} [(R_2 - Q_3) \cos\alpha$$
$$+ (P_3 - R_1) \cos\beta + (Q_1 - P_2) \cos\gamma] d\in$$

where the direction of integration is clockwise to an observer facing in the direction of the directed normal.

Surface Integral

Subdivision Δ of surface $\in$—It is a set of closed curves $\{e_k\}_1^n$ lying on surface $\in$.

Diameter of a region on $\in$—The diameter of a region on $\in$ is the length of the largest straight line segment whose ends lie in the region.

Norm of $\|\Delta\|$—It is the largest of the n-diameters of the subregions produced by the sub-division.

Surface integral over $\in$—Let $P(x, y, z)$ be a function defined at every point of $\in$ and let (ξ_K, n_K, ξ_K) be a point on $\in$ inside or on the boundary of the sub-region bounded by e_k. Then the surface integral of $P(x, y, z)$ over $\in$ is

$$\iint_E P(x, y, z)\, d\in = \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} P(\xi_K, n_K, \xi_K)\Delta\in_K$$

where $(\Delta\in_K)$ is the area of sub-region $\in_K$ and the limit exists.

Some Important Theorems

1. If $\in$ is the surface $z = f(x, y)$ over the region R and $\in$ lines in V, then
   (a) $\iint_{\in} P(x, y, z)\, d\in$ exists
   (b) $\iint_E P(x, y, z)\, d\in = \iint_R P(x, y, f(x, y))$
   $$\sqrt{1 + f_1^2(x, y) + f_2^2(x, y)}\, ds$$
   This reduces a surface integral to an ordinary double integral.
2. **Green's (Causs's) theorem**—If $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ are the points on V and V is bounded by bounded region $\in^*$ and α, β, γ are the direction angle of the exterior normal to $\in^*$
   $$\iiint_V [P_1(x, y, z) + Q_2(x, y, z) + R_3(x, y, z)]\, dv$$
   $$= \iint_{\in} [P(x, y, z)\cos\alpha + Q(x, y, z)\cos\beta + R(x, y, z)\cos\gamma)]\, d\in$$

Metric Space

Metric Space—A metric space $<x, P>$ is a non-empty set X of elements (points) and $P : X \times X \to \mathbb{R}$ such that for $x, y, z \in X$.

1. $P(x, y) \geq 0$
2. $P(x, y) = 0$ iff $x = y$
3. $P(x, y) = P(y, x)$
4. $P(x, y) \leq P(x, y) + P(x, y)$

The function P is called metric.

Cartesian Product—If $<X, P_1>$ and $<Y, P_2>$ are two metric spaces, then $<X \times Y, \tau>$ is the cartesian product of $<X, P_1>$ and $<Y, P_2>$ defined as $X \times Y = \{(x, y) : x \in X, y \in Y\}$ and $\tau(<x_1, y_1>, <x_2, y_2>) = \sqrt{P_1(x_1, x_2)^2 + P_2(y_1, y_2)^2}$

Diameter—If (X, P) is a metric space, $E \neq \phi \subset X$, then diameter $E = \sup\{P(x, y); x, y \in E\}$.

Pseudometric—If $P(x, y) = 0$ for some $x \neq y$ then (X, P) is called Pseudometric space and P is a Pseudometric.

Extended Pseudometric—If $P(x, y) = \infty$ for some x, $y \in X$, then P is extended Pseudometric and (X, P) is extended Pseudometric space.

Ball—A set $\delta_{x,\delta}$ $\{Y : P(x, y) < \delta\}$ is called ball centered at $x \in <X, P>$.

Convergence and Completeness

Convergence—A sequence $<x_n>$ from metric space $<X, P>$, converges to the point $x \in X$ (x is a limit), if given $\in > 0$, there is N such that $P(x, x_n) < \in, \forall_n \in N$.

Cluster Point—x is a cluster point of $<x_n>$ if given $\in > 0$ and given N there is $n \geq N : P(x, x_n) < \in$.

Cauchy's Sequence—A sequence $<x_n>$ from a metric space $<X, P>$ is called Cauchy sequence, if given $\in > 0$. There is $N : n, m > N$, we have $P(x_n, x_m) < \in$.

Complete metric space—A metric space $<X, P>$ is complete if every Cauchy sequence converges (to some point of X).

Some Important Theorems

1. If x is limit of $<x_n>$, then x is cluster point of $<x_n>$ (converge is not true).
2. If $<x_n>$ a Cauchy sequence converges to some $x \in <X, P>$, then sequence converges to $x \in <X, P>$ (converges is not true).
3. If $<X, P>$ is an incomplete metric space, it is possible to find a complete metric space X^* in which X is isometrically embeded as a dense subset. If X is contained in an arbitrary complete space Y, then X^* is isometric with the closure of X in Y.

Lebesgue Measure

Measure

Length of an interval—The length of an interval I is the difference of end points of the interval.

Measure of a set—Let M be a collection of sets of real numbers and $E \in M$. Then non-negative extended real number mE is called the measure of E. If m satisfies.

(a) mE is defined for each set E of real numbers *i.e.* $M = P(R)$, the power set of sets of real number.

(b) For an interval I, $mI = I(I)$

(c) If $<E_n>$ is a sequence of disjoint sets (for which m is defined) in M. $m\ (UE_n) = \in_m.E_n$.

(d) m is translation invariant, *i.e.* if E is the set on which m is defined, and $E + Y = \{x + y : x \in E)$, then $m\ (E + y) = mE$.

Countable Additive Measure—Let M be a σ-algebra of sets of real numbers and $E \in M$. Then non-negative extended real number mE is countable additive measure, if $m\ (UE_n) = \in mE_n$ for each sequence $<E_n>$ of disjoint sets in M.

Countable sub additive measure—Let M be a σ-algebra of sets of real numbers and $E \in M$. Then non-negative extended real number mE is countable subadditive measure, if $m(UE_n) \leq \in mE_n$, for each sequence $<E_n>$ of sets in M.

Counting measure—If

$$nE = \begin{cases} \infty, \text{ for infinite set E} \\ \text{The number of elements in E,} \\ \text{ for finite set E} \end{cases}$$

Some Important Results

Let m be a countable additive measure defined for all sets in a σ-algebra. M then

(a) Monotonicity : $A, B \in M, A \subset B \rightarrow mA \leq mB$

(b) If for some set $A \in M$, $mA < \infty$, then $m\phi = 0$

Lebesgue outer Measure

Lebesque outer Measure—Let A be a set of real number $\{I_n\}$ be the countable collection of open intervals that covers A, *i.e.* $A \subset U\ I_n$. Then Lebesgue outer measure m^*A of A is

$$m^*A = \underset{A \subset UI_n}{Inf} \in l(I_n)$$

Some Important Results

1. $m * \phi = 0$
2. $A \subset B \Rightarrow m^* A \leq m^* B$
3. For singleton set $\{x\}$, $m^* \{x\} = 0$
4. The Lebesgue outer measure of an interval is its length.
5. If $\{A_n\}$ is a countable collection of sets of real number. Then $m^*\ (UA_n) \leq m^*\ A_n$.
6. If A is countable, $m^*\ A = 0$
7. The set [0, 1] is not countable.
8. Given any set A and any $\in > 0$, there is an open set $O : A \subset O$ and $m * 0 \leq m^* A + \in$. There is a $G_\delta : A \subset G$ and $m^* A = m^* G$.

Lebesgue Measurable sets and Lebesgue Measure

Lebesgue Measurable sets—A set E is Lebesgue measurable its for each set A we have $m^*A = m^*\ (A \cap E) + m^*(A \cap E^C)$

Lebesgue Measure—If E is a Lebesgue measurable set, the Lebesgue measure mE is the Lebesgue outer measure of E.

Some Important Results

1. If $m^*E = 0$, then E is Lebesgue measurable.
2. If E_1 and E_2 are Lebesgue measurable, so $E_1 \cup E_2$.
3. The family M of Lebesgue measurable sets is in algebra of sets.
4. If A is any set and $E_1, E_2 \ldots, E_n$ a finite set sequence of disjoint Lebesgue measurable sets. Then
$$m^*\left(A \cap \left[\overset{n}{\underset{j=1}{\cup}} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$
5. The collection M of Lebesgue measurable sets is a σ-algebra.
6. Every set with Lebesgue outer measure zero is Lebesgue measurable.
7. The interval (a, ∞) is Lebesgue measurable.
8. Every Borel set is Lebesgue measurable.
9. Each open set and closed set is Lebesgue measurable.
10. If $<E_i>$ is a sequence of Lebesgue measurable set. Then for Lebesgue measure $m(UE_i) \leq \in mE_i$.

    If the set E_i are pairwise disjoint then for Lebesgue measure $m(UE_i) = \in mE_i$.
11. Let $<E_i>$ be an infinite decreasing sequence of Lebesgue measurable sets, *i.e.*, $E_{n+1} \subset E_n$ for each n. Let Lebesgue measure mE, is finite then $m\left(\overset{\infty}{\underset{n=1}{\cup}} E_i\right) = \lim_{n \to \infty} mE_n$.
12. For a given set following are equivalent :—
    (a) E is measurable
    (b) Given $\in > 0$, there is an open set $0 \supset E$ with $m^*(O \sim E) < \in$.
    (c) Given $\in > 0$, there is a closed set $F \subset E$ with $m^*(E \sim F) < \in$.
    (d) $G \in G_\delta$ with $E \subset G$, $m^*(G \sim E) = 0$
    (e) $F \in F_\delta$ with $F \subset E$, $m^*(E \sim F) = 0$ If $m^*E < \infty$, then these statements are equivalent to.
    (f) Given $\in > 0$, there is a finite union $\cup$ of open interval : $m^*(\cup \Delta E) < \in$.

Lebesgue Measurable Functions

Lebesgue Measurable Function—An extended real valued function f is Lebesgue measurable if its domain is measurable end if it satisfies one of the following five; for each real number α.

(a) $\{x : f(x) < \alpha\}$ is measurable
(b) $\{x : f(x) > \alpha\}$ is measurable
(c) $\{x : f(x) \leq \alpha\}$ is measurable
(d) $\{x : f(x) \geq \alpha\}$ is measurable
(e) $\{x : f(x) = \alpha]$ is measurable

Almost everywhere Property—If a set of points where it fails to hold is a set of measure zero.

If $f = g$, almost everywhere if f and g have the same domain and $m\{x : f(x) \neq g(x)\} = 0$.

Characteristic function—χ_A : If A is any set, the characteristic function of set A is defined as
$$\chi_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases}$$

Simple Function—A real valued function ϕ is simple function, if it is Lebesgue measurable and assume only a finite number of values.

Borel measurability—A function f is Borel measurable if for each α, the set $\{x : f(x) > \alpha\}$ is a Borel set.

Some Important Results

1. If f is an extended real valued function whose domain is measurable then the following statements are equivalent : For each real number α.
   (a) The set $\{x : f(x) > \alpha\}$ is Lebesgue measurable
   (b) The set $\{x : f(x) < \alpha\}$ is Lebesgue measurable
   (c) The set $\{x : f(x) \leq \alpha\}$ is Lebesgue measurable
   (d) The set $\{x : f(x) \geq \alpha\}$ is Lebesgue measurable
2. If c is a constant and f and g two Lebesgue measurable real valued functions defined on the same domain. Then the functions $f + c$, cf, $f + g$, $g - f$ and fg are also Lebesgue measurable.
3. If $<f_n>$ is a sequence of Lebesgue measurable functions (with the same domain of definition). Then the function syp $\{f_1, \ldots, f_n\}$, inf $\{f_1, \ldots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\lim f_n$ and $\lim f_n$ are all Lebesgue measurable.
4. If f is a measurable function and $f = g$ almost everywhere, then g is measurable.

5. If f is Lebesgue measurable function defined an interval $[a, b]$ and assume that f takes the values $\neq \infty$ only on a set of measure zero. Then given $\in > 0$, we can find a step function g and a continuous function h such that $|f - g| < \in$ and $|f - g| < \in$.

The Lebesgue Integration

The Riemann Integral—If f is a bounded real valued function defined on the interval $[a, b]$ and $a = x_0 < x_1 < \ldots < x_n = b$ is a subdivision of $[a, b]$.

$$S = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i$$

$$S = \sum_{i=1}^{n} (x_i - x_{i-1}) m_i$$

where $M_i = \sup_{x_{i-1}} < x < x_i f(x)$

and $m_i = \inf_{x_{i-1} < x < x_i} f(x)$

The upper Riemann integral of f is

$$R \int_a^{-b} f(x)\,dx = \inf S$$

and the lower Riemann integral

$$R \int_a^{-b} f = R \int_{-a}^{b} f = R \int_a^b f(x)\,dx$$

Step Function—For the given subdivision $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ of the interval $[a, b]$, a function ϕ is a step function if,

$$\phi(x) = c_i,$$
$$x_{i-1} < x < x_i$$

Some Important Results

1. $\int_a^b \phi(x)\,dx = \sum_{i=1}^{n} e_i (x_i - x_{i-1})$
2. $R \int_a^{-b} f(x)\,dx = \inf \int_a^b \phi(x)\,dx$ for all step function $\phi(x) \geq f(x)$
3. $R \int_{-a}^{b} f(x)dx = \sup \int_a^b \phi(x)\,dx$ for all step function $\phi(x) \leq f(x)$

The Lebesgue Integral of a Bounded Function over a set of Finite Measure

Characteristic Function—A real valued function of set E.

$$\chi_E = \begin{Bmatrix} 1, & x \in E \\ 0, & x \notin E \end{Bmatrix}$$

Simple Function—A linear combination

$$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

where sets E_i are measurable.

Canonical representation of simple function—If ϕ is a simple function and $\{a_1, \ldots a_n\}$ is the set of non-zero values of ϕ, then $\phi = \in a_i \chi_{A_i}$, where $A_i = \{x : \phi(x) = a_i\}$ is called canonical representation of simple function. Here A_i are disjoint and a_i are distinct and non-zero.

Integral of Simple function—If simple function ϕ vanishes outside a set of finite measure, the integral of ϕ is $\int \phi(x)\,dx = \sum_{i=1}^{n} a_i (mA_i)$ when ϕ has a canonical reprsentation $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$. and mA_i is the Lebesgue measure of A_i. If E is any measurable set, we have $\int_E \phi = \int^{-} \phi\chi_E$.

The Lebesgue Integral—If f is bounded measurable function defined on a measurable set E with mE finite, the Lebesgue integral of f over E is $\int_E f(x)\,dx = \inf \int_E \psi.(x)dx$ for all simple function $\psi \geq f$.

Some Important Theorems

1. Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$, with $E_i \cap E_j = \phi$ for $i \neq j$ suppose each set E_i is a measurable set of finite measure. Then $\int \phi = \sum_{i=1}^{n} a_i m_{E_i}$.
2. If ϕ and ψ is simple function which vanish out side a set of finite measure then $\int (a\phi + b\psi) = a \int \phi + b \int \psi$ and if $\phi \geq \psi$ almost everywhere then $\int \phi \geq \int \psi$.
3. If f is defined and bounded on a measurable set E with mE and $\inf_{f \leq \psi} \int_E \psi(x)dx = \sup_{f \geq \phi} \int_E \phi(x)\,dx$ for all simple functions ϕ and ψ. Then of is measurable function.
4. Let f be bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$, then it is measurable and

$$R\int_a^b f(x)dx = \int_a^b f(x)dx$$

5. If f and g are bounded measurable functions defined on set E of finite measure, then
   (a) $\int_E (af + bg) = a\int_E f + b\int_E g$
   (b) If $f = g$ almost everywhere, then $\int_E f = \int_E g$.
   (c) If $f \leq g$ almost everywhere, then $\int_E f \leq \int_E g$.
   (d) $|\int f| = \int |f|$
   (e) If $A \leq f(x) \leq B$, then $A(mE) \leq \int_E f \leq B(mE)$.
   (f) If A and B are dispoint measureable sets of finite measure, then
   $$\int_{A\cup B} f = \int_A f + \int_B f$$
6. Let $<f_n>$ is a sequence of measurable functions defined on a set E of finite measure and suppose that there is a real number M such that $|f_n(x)| \leq M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each $x \in E$, then $\int_E f \lim \int_E f_n$.
7. A bounded function f on $[a, b]$ is Riemann integrable iff the set of points at which f is discontinuous has measure zero.

Fatou's Lemma—If $[f_n Y$ is a sequence of non-negative measurable function and $f_n(x) \to f(x)$, almost everywhere on a set E, then

$$\int_E f \leq \lim \int_E f_n$$

Dominated Convergence

Convergence in measure—A sequence $<f_n>$ of measurable functions is said to converge to f in measure if, given $\in > 0$, there is an N such that for all $n \geq N$, we have

$$m\{x : |f(x) - f_n(x)| \geq \in\} < \in$$

Cauchy sequence in measure—A sequence $<f_n>$ of measurable functions is Cauchy sequence in measure if given $\in > 0$ there is an N such that for all $m, n \geq N$, we have

$$m\{x : |f_n(x) - f_m(x)| \geq \in\} < \in$$

Some Important Theorem

1. If $<f_n>$ is a sequence of measurable functions that converges in measure to f. Then there is a subsequence $<f_{nk}>$ that converges to of almost everywhere.
2. If $<f_n>$ is a sequence of measurable functions defined on a measurable set E of finite measure. Then $<f_n>$ converges of f in measure iff every subsequence of $<f_n>$ has in turn a subsequence that converges almost everywhere to f.
3. Fatou's lemma and the monotone and Lebesgue convergence theorem remain valid if 'convergence almost everywhere' is replaced by 'convergence in measure'.

Weierstrass approximation theorem—A space X has Weierstrass property if every infinite sequence in X has atleast one limit point. Nowhere dense—$A \subset X$ is nowhere dense, A the closure of A has no interior point.

First category (Merger)—A set $A \subset X$ is of first category in X, if it is the union of countabley many nowhere dense sets in X.

Second Category (Non-merger)—A set $A \subset X$ is of second category in X if it is not of first category.

Compactness

Compact metric space—(1) A metric space (X, P) is compact if every infinite subset of X has atleast one limit point.

(2) A metric space X is compact if every open covering μ of X a has finite subcovering.

Set $K \subset X$ is compact if (K, P) is compact.

Relatively Compact—If X is a metric space, $K \subset X$ and closure of K, $\bar{K}$ is compact, then K is relatively compact to X.

Total Boundedness—A metric space X is totally bounded if for every $\in > 0$, X contains a finite set, called an $\in$-net, such that the finite set of open spheres of radius $\in$ and centres in the $\in$-net covers X.

Some Solved Examples

Example 1. Test the convergence of the integral $\int_a^\infty \frac{dx}{x^n}$, where $a > 0$.

Solution :

Here $\int_a^\infty \frac{dx}{x^n} = \lim_{x\to\infty} \int_a^x \frac{dx}{x^n}$

$$= \lim_{x \to \infty} \frac{1}{1-n}[x^{1-n} - a^{1-n}]$$

where $n \neq 1$

Case I. When $1 - n < 0$

i.e., $n > 1$

$$\lim_{x \to \infty} \int_a^x \frac{dx}{x^n} = \frac{a^{1-n}}{n-1}$$

which is finite. The given integral is convergent when $n > 1$.

Case II. When $1 - n < 0$

i.e., $n < 1$

$$\lim_{x \to \infty} \int_a^x \frac{dx}{x^n} = \infty,$$

Therefore, the integral is divergent.

Case III. When $1 - n = 0$

i.e. $n = 1$

$$\lim_{x \to \infty} \int_a^x \frac{dx}{x} = \lim_{x \to \infty} [\log x - \log a]$$
$$= \infty$$

Therefore, divergent.

Hence, the given integral is divergent except when $n > 1$.

Example 2. If $f(x, y) = 2x^2 - xy + 2y^2$ find $\frac{\delta f}{\delta x}$, $\frac{\delta f}{\delta y}$ at (1, 2)

Solution :

$$\left(\frac{\delta f}{\delta x}\right)_{(1,2)} = \lim_{h \to 0} \frac{f(1+h, 2) - f(1, 2)}{h}$$

$$= \lim_{h \to 0} \frac{\{2(1+h)^2 - (1+h)^2 + 2 \cdot 2^2\} - \{2 \cdot 1^2 - 1 \cdot 2 + 2 \cdot 2^2\}}{h}$$

$$= \lim_{h \to 0} \frac{2h^2 + 2h}{h}$$

$$= \lim_{h \to 0} (2h + 2)$$

$$= 2$$

$$\left(\frac{\delta f}{\delta y}\right)_{(1,2)} = \lim_{k \to 0} \frac{f(1, 2+k) - f(1, 2)}{k}$$

$$= \lim_{k \to 0} \frac{\{2 - (2+k) + 2(2+k)^2\} - \{2 - 2 + 8\}}{k}$$

$$= \lim_{k \to 0} \frac{2k^2 + 7k}{k}$$

$$= \lim_{k \to 0} (2k + 7)$$

$$= 7$$

Example 3. Show that $\lim_{(x, y) \to (0, 0)} \frac{2xy}{x^2 + y^2}$ does not exist.

Solution : Let $f(x, y) = \frac{2xy}{x^2 + y^2}$

$(x, y) \neq (0, 0)$

Taking $y = \phi(x) = mx$

where m is a real number

We have $f(x, y) = \frac{2mx^2}{x^2 + m^2x^2}$

$$= \frac{2m}{1 + m^2}$$

$$\therefore \quad \lim_{x \to 0} f(x, \phi(x)) = \lim_{x \to 0} \frac{2m}{1 + m^2}$$

$$= \frac{2m}{1 + m^2}$$

For $\phi_1(x) = m_1 x$,

$\phi_2(x) = m_2 x$

and $m_1 \neq m_2$

$$\lim_{x \to 0} f(x, \phi_1(x)) = \frac{2m_1}{1 + m_1^2} \quad \ldots\text{(i)}$$

and $$\lim_{x \to 0} f(x, \phi_2(x)) = \frac{2m_2}{1 + m_2^2} \quad \ldots\text{(ii)}$$

By (i) and (ii) we have

$$\lim_{x \to 0} f(x, \phi_1(x)) \neq \lim_{x \to 0} f(x, \phi_2(x))$$

$\therefore \lim_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

Also, $$\lim_{y \to 0} \left(\lim_{x \to 0} \frac{2xy}{x^2 + y^2}\right) = \lim_{y \to 0} 0 = 0$$

$$\lim_{x \to 0} \left(\lim_{y \to 0} \frac{2xy}{x^2 + y^2}\right) = \lim_{x \to 0} 0 = 0$$

$\therefore$ The two repeated limits exist and are equal.

Example 4. If ϕ, ψ of $[1, 2] \to \mathbb{R}$ and $c = \{(x, y) : x = at^2, y = 2at\}$, then find the value of $\int_c \frac{dx}{x + y}$.

Solution : Here $f(x, y) = \frac{1}{x + y}$,

$\phi(t) = at^2$,

$\psi(t) = 2at$,

$\psi'(t) = 2at$

$\therefore$ The conditions are satisfied

$$\therefore \quad \int_c \frac{1}{x+y}dx = \int_1^2 \frac{1}{at^2+2at}.2at\,dt$$

$$= \int_1^2 \frac{2}{t+2}dt$$

$$= [2 \log (t+2)]^2$$

$$= 2 \log (4/3)$$

Example 5. The integral $\int_0^{\pi/2} \sin^{m-1} x.\cos^{n-1} x\,dx$ converges if and only if , $m > 0, n > 0$.

Solution : The points $x = 0$ and $x = \frac{1}{2}\pi$ are the points of infinite discontinuous, when $0 \subset m < 1$.

$$\sin^{m-1}x \cos^{n-1}x = \left(\frac{\sin x}{x}\right)^{m-1} \cos^{n-1}x\, x^{m-1}$$

$$\therefore \lim_{x\to 0} x^{\mu}\left(\frac{\sin x}{x}\right)^{m-1} \cos^{n-1}x\, x^{m-1}$$

$$= \lim_{x\to 0} x^{\mu}x^{m-1}$$

$$= 1$$

If $\mu = 1-m$

$\therefore$ The integral will be convergent if

$\mu = 1-m<1,$

i.e., $m > 0$

and $1-m > 0$

or, $m < 1$

Hence, the integral is convergent, if $0 < m < 1$ when $0 < n < 1$.

$$\sin^{m-1}\cos^{n-1}x = x$$

$$= \sin^{m-1}x\,.\frac{\cos^{n-1}x}{\left(\frac{1}{2}\pi - x\right)^{n-1}}\left(\frac{\pi}{2}-x\right)^{n-1}$$

$$\text{and} \quad \lim_{x\to\pi/2}\left(\frac{\cos x}{\frac{1}{2}\pi - x}\right)^{n-1} = 1$$

$$\therefore \lim_{x\to\pi/2}\left(\frac{\pi}{2}-x\right)^{\mu}\sin^{m-1}x\left(\frac{\cos x}{\frac{1}{2}\pi - x}\right)^{n-1}\left(\frac{\pi}{2}-x\right)^{n-1}$$

$$= \lim_{x\to\pi/2}\left(\frac{\pi}{2}-x\right)^{\mu}.\left(\frac{\pi}{2}-x\right)^{n-1}$$

$$= 1$$

If $\mu = 1-n$

$\therefore$ The integral will be convergent if $0 < n < 1$ when $m > 1$ and $n > 1$.

When $m > 1$ and $n > 1$, the integral becomes a proper integral and hence converges.

$\therefore$ The given integral is convergent if and only if $m > 0$, $n > 0$.

Example 6. Show that (i) the area bounded by a simple closed curve c is given by $\frac{1}{2}\int_c x\,dy - y\,dx$.

(ii) Find the area of the ellipse $x = a \cos\theta$, $y = b \sin\theta$, $0 \le \theta \le 2\pi$.

Solution : (i) By Green's theorem

$$\int_c Pdx + Qdy = \iint_S \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dx\,dy$$

Put $P = -y$

and $Q = -x$

$$\because \quad \frac{\delta P}{\delta y} = -1,$$

$$\frac{\delta Q}{\delta x} = +1$$

$$\therefore \quad \oint_c xdy - y\,dx = 2\int_S dx\,dy$$

$$= 2A$$

where A is the area of the surface S.

(ii) The area of surface S.

$$A = \frac{1}{2}\int_c xdy - ydx$$

Now for the ellipse

$$x = a\cos\theta,$$

$$y = b\sin\theta$$

$$\text{Area} = \frac{1}{2}\oint xdy - ydx$$

$$= \frac{1}{2}\int_0^{2\pi}(a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\,d\theta$$

$$= \frac{1}{2}ab\int_0^{2\pi}d\theta$$

$$= \pi ab$$

Example 7. Evaluate by Green's theorem, $\oint_c (y - \sin x\,dx + \cos x dy$ where c is the triangle enclosed by the lines $x = 0$, $x = \pi/2$, $xy = 2x$, $P = y - \sin x$, $Q = \cos x$.

Solution : Here $\frac{\delta P}{\delta y} = 1,$

$$\frac{\delta Q}{\delta x} = -\sin x$$

Hence by Green's theorem.

$$\oint_c (y - \sin x)dx + \cos x dy$$

$$= \iint_S (-1 - \sin x)\, dx\, dy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} (1 + \sin x)\, dx\, dy$$

$$= \int_0^{2/\pi} (1 + \sin x) \frac{2x}{\pi}\, dx$$

$$= -\frac{2}{\pi} \int_0^{2/\pi} (x + x \sin x)\, dx$$

$$= (\pi/4 + 2/\pi)$$

Example 8. If $S \subset R^n$, prove that S is compact iff set S is closed and bounded.

Solution : Suppose S is closed and bounded by Heine Borel theorem there is an open covering F of S such that a finite subcollection of F also covers S, so S is compact by definition of compactness. Suppose S is compact to prove S is bounded, choose $\bar{x} \in S$, then $\{B(\bar{x}, K), K = 1, 2, \ldots\}$ a collection of open covering of S. Since S is compact, a finite subcollection also cover S and hence S is bounded.

To prove S is closed. Suppose S is not closed, there is an accumulation point y of S such that $y \notin S$. If $x \in S$ and let $r_x = \|x - y\|/2$. Then the collection $\{B(x, r_x) : x \in S\}$ is an open covering of S. Since S is compact there is a finite sub collection which covers.

i.e. $$S \subseteq \bigcup_{k=1}^{P} B(x_k, r_k)$$

Choose $$r = \min\{r_1, r_2, \ldots, r_P\}$$

Then for open ball $B(y, r)$.

If $$x \in B(y, r)$$

$$\Rightarrow \|x - y\| < r \le r_k \quad (\text{for some } r_k)$$

Also we have

$$\|y - x_k\| \le \|y - x\| + \|x - x_k\|$$

$$\Rightarrow \|x - x_k\| \ge \|y - x_k\| - \|x - y\|$$

$$= 2r_k - \|x - y\|$$

Example 9. Verify Stokes' theorem for $F = -y^3 i + x^3 j$, where j is the circular disc

$$x^2 + y^2 \le 1,$$

$$z = 0$$

Solution : Given $F = y^3 i + x^3 j,$

The boundary C of S is a circle in xy. Plane,

$$x^3 + y^3 = 1,$$

$$z = 0$$

In parametric form

$$x = \cos\theta,$$

$$y = \sin\theta,$$

$$z = 0 \qquad \text{where } 0 \le \theta \le 2\pi$$

$$\therefore \int_C F.dr = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_C (-y^3 dx + x^3 dy)$$

$$= \int_C [-\sin^3\theta(-\sin\theta) + \cos^3\theta \cos\theta]\, d\theta$$

$$= \int_0^{2/\pi} (1 - 2\sin^2\theta \cos^2\theta)\, d\theta,$$

$$= \int_0^{2/\pi} d\theta - 2\int_0^{2/\pi} \sin^2\theta \cos^2\theta\, d\theta$$

$$= 2\pi - 2(u) \int_0^{\pi/2} \sin^2\theta \,.\, \cos^2\theta\, d\theta$$

$$= 2\pi - 8 \,.\, \frac{1}{4} \,.\, \frac{1}{2} \,.\, \frac{\pi}{2}$$

$$= \frac{3\pi}{2}$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$= K(3x^2 + 3y^2)$$

$$\therefore \int (\nabla \times F).N ds = 3\int_S (x^2 + y^2) K.N ds$$

$$= 3\iint_R (x^2 + y^2)\, dx.dy \quad \ldots(i)$$

Since $(K.N)\, ds = dx \,.\, dy$

and R is the region of xy-Plane.

Put $$x = r\cos\phi,$$

$$y = r\sin\phi$$

$$\therefore \quad dx \,.\, dy = r\, dr d\phi$$

and v varies from 0 to 1

and $$0 \le \phi \le 2\pi$$

$$\therefore \int (\nabla \times F) N.ds = 3\int_{\phi=0}^{2\pi} \int_{r=0}^{1} r^2$$

$$= r dr d\phi$$

$$= \frac{3\pi}{2}$$

Hence, the verification of the theorem.

Example 10. Find the nth term of the sequence $\{0, 1, 0, 1, \ldots\}$.

Solution : The first term of the sequence.

$$a_1 = \frac{1-1}{2} = 0$$

and second term of the sequence.

$$a_2 = \frac{1+(-1)^2}{2} = 1$$

Third term of the sequence

$$a_3 = \frac{1+(-1)^3}{2} = 0$$

Fourth term of the sequence

$$a_4 = \frac{1+(-1)^4}{2} = 1$$

$\therefore$ nth term of the sequence

$$a_n = \frac{1+(-1)^n}{2}$$

Example 11. The sequence $\left\{\dfrac{\sin\frac{n\pi}{2}}{n}\right\}_{n=1}^{\infty}$ is a convergent sequence.

Solution :

The sequence $\left\{\sin\frac{n\pi}{2}\right\}_{n=1}^{\infty}$ is bounded sequence

$$\because \quad \left|\sin\frac{n\pi}{2}\right| \le 1$$

The sequence $\frac{1}{n}$ is a convergent sequence and it converges to 0.

By the theorem : If $\{a_n\}_{n=1}^{\infty}$ converges to 0 and $\{b_n\}_{n=1}^{\infty}$ is bounded then $\{a_nb_n\}_{n=0}^{\infty}$ converges to 0, we have $\left\{\sin\frac{n\pi}{2}.\frac{1}{n}\right\}$ converges to 0.

Example 12. The series $\sum\limits_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-n}$ is convergent.

Solution : We have $a_n = \left(1+\frac{1}{\sqrt{n}}\right)^{-n}$

$$\therefore \quad \sqrt{a_n} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

By Root test

$$\therefore \quad \lim_{n\to\infty}\sqrt{(a_n)} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$= \frac{1}{e} < 1$$

Hence the series is convergent.

Example 13. Every Cauchy sequence is bounded.

Solution : Let sequence $<x_n>$ is a Cauchy sequence.

For $\quad \in\ >\ 0,$

$\exists N : n \ge N,$

$m \ge N$

$\Rightarrow \quad |x_n - x_m| < \in$

$\because \quad |x_n - x_m| \le |x_n| - |x_m|$

$\Rightarrow \quad |x_n| > |x_m| < \in$

$\Rightarrow \quad |x_n| < |x_m| + \in \qquad \ldots$(i)

Choose $\quad M = \min(|x_m|) + \in$

$\therefore$ By (i), we have, $\in\ >\ 0,$

$\exists N : n \ge N,$

$|x_n| < M$

The sequence $<x_n>$ is bounded.

Example 14. The open interval is an open set.

Solution : Let (a, b) bean open interval,

If $x\in (a, b)$, then $a < x < b$

Choose $\quad \delta = \min\{x-a, b-x\}$

Thus $\forall\ x\in (a, b)$, $\delta > 0$, whenever

$|x-y| < \delta,$

$y \in (a, b)$

i.e. (a, b) is open set.

Example 15. The closed interval is a closed set.

Solution : Let x be a point of closure of $[a, b]$, then for every $\delta > 0$ there is

$y \in [a, b],$

such that $\quad |x-y| < \delta$

A closed interval $[a, b]$ is closed set if it contains all its points of closure

i.e. $\quad x \in [a, b]$

On the contrary,

Let $\quad x \notin [a, b]$

Choose $\quad \delta = \min\{b-y, y-a\}$

For every $\quad y \in [a, b]$

Such that $|x - y| < \delta$,

$x \in [a, b]$,

which is the contradiction

Thus $x \in [a, b]$

$\Rightarrow [a, b]$ contains all its closure points

$\Rightarrow [a, b]$ is closed set.

Example 16. Prove that the continuous image of a compact set is compact.

Solution : Let f be a continuous function mapping the compact set K onto a space Y. If μ is an open covering for Y, then the collection of sets $f^{-1}[0]$ for all $0 \in \mu$ is an open covering of K. By the compactness of K, there are a finite number 0_1 ……0_n of sets of μ such that the sets $f^{-1}[0_1]$ cover K. Since f is onto, the sets $0_1 \ldots 0_n$ cover Y. Thus the continuous image of a compact set is compact.

Example 17. Prove that a metric space X has the Bolzano-Weierstrass property iff X is sequentially compact.

Solution : Since every limit of a subsequence of $<x_n>$, sequential compactness implies the Bolzano-Weierstrass property. Conversely, if $<x_n>$ has x for a cluster point, then for each K we can find an $n_k > n_{k-1}$ such that the ball of radius 1/K about x contains x_{nk}. Then $x_{nk} \to x$. Thus a metric space with the Bolzano-Weierstrass property is sequentially compact.

Example 18. Prove that if E is a labesgue measurable set, then each translate E + y is also measurable.

Solution : Let E + y is lebesgue measurable, then for some set A.

$$m^* A = m^* [A \cap (E + y)] + m^* [A \cap (E + y)^c] \quad \ldots(1)$$

Since E is measurable, we have

$$m^* [A \cap (E + y)]] = m^* (A \cap (E + y)^c \cap E] + m^* [A \cap (E + y) \cap E^c]$$

$$m^* [A \cap (E + y)^c] = m^* [A \cap (E + y)^c \cap E] + m^* [A \cap (E + y)^c \cap E^c]$$

Adding These, we have

$$m^* (A \cap (E + y)] + m^* [A \cap (E + y)^c]$$
$$= m^* [A \cap E \cap (E + y)E] + m^* [A \cap E \cap (E + y)^c] + m^* [A \cap E^c \cap (E + y)] + m^* [A \cap E^c \cap (E + y)]$$
$$= m^* (A \cap E) + m^* (A \cap E^c) = m^* A.$$

This proves (E + y) is lebesgue measurable

OBJECTIVE TYPE QUESTIONS

1. The nth term of the sequence $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots 0\right\}$ is—
   (A) $\frac{1}{n-1}$
   (B) $\frac{1}{2n}$
   (C) $\frac{1}{2(n+1)}$
   (D) None of these

2. The following sequence {2, 3, 5, 7, …} is a sequence of—
   (A) Real number
   (B) Prime number
   (C) Even number
   (D) Odd number

3. The nth term of the sequence {1, 1/2, 1, 1/3, 1, 1/4, …} is—
   (A) For n even $\frac{1}{n - \frac{n}{2}}$ for n odd 1
   (B) For n even $\frac{1}{n + \frac{n}{2} + 1}$ for n odd 1
   (C) For n even $\frac{1}{n - \frac{n}{2} + 1}$ for n odd 1
   (D) For n even $\frac{1}{n + \frac{n}{2}}$ for n odd 1

4. The nth term of the sequence $\left\{2, \frac{-3}{2}, \frac{4}{3}, \frac{-5}{4}, \ldots\right\}$ is—
   (A) $1 + 1/n$
   (B) $(-1)^{n-1}(1 - 1/n)$
   (C) $(-1)^{n-1}(1 + 1/n)$
   (D) None of these

5. The nth term of the sequence $\left\{1, \frac{5}{2}, \frac{5}{3}, \frac{9}{4}, \frac{9}{5}, \ldots\right\}$ is—

(A) $\frac{2n+(-1)^n}{2n}$ (B) $\frac{2n+(-1)^n}{n}$

(C) $\frac{2n+(1)^n}{n}$ (D) None of these

6. The sequence {0, 1, 0, 1/2, 0, 1/3, ...} has the nth term—

(A) $\frac{1+(-1)^n}{n}$ (B) $\frac{n-(1)^n}{n}$

(C) $\frac{n+(1)^n}{n}$ (D) None of these

7. A monotone sequence $\{a_n\}_{n=1}^{\infty}$ is convergent—

(A) It is bounded

(B) It is unbounded

(C) It is decreasing

(D) None of these

8. If sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are divergent sequence, then—

(A) $\{a_n+b_n\}_{n=1}^{\infty}$ is always divergent

(B) $\{a_n+b_n\}_{n=1}^{\infty}$ is always convergent

(C) $\{a_n+b_n\}_{n=1}^{\infty}$ sometimes convergent

(D) None of these

9. If sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergetn then—

(A) $\{a_n+b_n\}_{n=1}^{\infty}$ is always convergent

(B) $\{a_n+b_n\}_{n=1}^{\infty}$ is always divergent

(C) $\{a_n+b_n\}_{n=1}^{\infty}$ is sometimes divergent

(D) None of these

10. The sequence $\left\{\frac{\cos\frac{n\pi}{2}}{\text{n}}\right\}_{n=1}^{\infty}$ is—

(A) Convergent to 0 (B) Divergent

(C) Convergent to 1 (D) None of these

11. (a) Every bounded sequence is convergent.

(b) Every convergent sequence is bounded

(A) (a) and (b) are true

(B) (a) is true, (b) is false

(C) (b) is true, (a) is false

(D) (a) and (b) are false

12. The sequence {1, 0, 1, 0, 1, 0, ...} is—

(A) Increasing sequence

(B) Decreasing sequence

(C) Monotone sequence

(D) None of these

13. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequence such that $\{a_n\}_{n=0}^{\infty}$ and $\{a_n\, b_n\}_{n=0}^{\infty}$ converges respectively to A and AB, then $\{b_n\}_{n=0}^{\infty}$ converges iff—

(A) $\text{A} \neq 0$ (B) $\text{A} = 0$

(C) $\text{B} = 0$ (D) None of these

14. If $\{b_n\}_{n=1}^{\infty}$ is an increasing bounded sequence, then for the sequence $\{b_n\}_{n=1}^{\infty}$ is following statement is false.

(A) $\{b_n\}_{n=1}^{\infty}$ is a convergent sequence

(B) $\{b_n\}_{n=1}^{\infty}$ is a divergent sequence

(C) $\{b_n\}_{n=1}^{\infty}$ is a monotone sequence

(D) $\{b_n\}_{n=1}^{\infty}$ is a Cauchy sequence

15. If $\{a_n\}_{n=0}^{\infty}$ converges to a, for all $n, a \geq 0$, then $\left\{\sqrt{a_n}\right\}_{n=0}^{\infty}$ is—

(A) Converges to $\sqrt{a}$

(B) Diverges to $\sqrt{a}$

(C) Converges to a

(D) Diverges to a

16. If $\{a_n\}_{n=0}^{\infty}$ converges to A, then—

(A) $\{|a_n|\}_{n=0}^{\infty}$ converges to A

(B) $\{|a_n|\}_{n=0}^{\infty}$ converges to |A|

(C) $\{|a_n|\}_{n=0}^{\infty}$ divergent sequence

(D) None of these

17. A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded iff there is a real number S such that—

(A) $|a_n| \leq \text{S}$ for all n (B) $|a_n| \geq \text{S}$ for all n

(C) $|a_n| = \text{S}$ for all n (D) None of these

18. A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded from below if for real number R—

(A) $a_n \leq \text{R}$ for all n (B) $a_n \geq \text{R}$ for all n

(C) $a_n = \text{R}$ for all n (D) None of these

19. If $\{a_n\}_{n=1}^{\infty}$ converges to A and B both, then—

(A) $\text{A} > \text{B}$ (B) $\text{A} = \text{B}$

(C) $\text{A} \leq \text{B}$ (D) None of these

20. If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to a real number A, then $\{a_n\}_{n=1}^{\infty}$ is—
(A) Unbounded sequence
(B) Bounded sequence
(C) Divergent sequence
(D) None of these

21. A sequence $\{a_n\}_{n=1}^{\infty}$ is bounded from above if for real number R—
(A) $a_n \geq$ R for all n
(B) $a_n \leq$ R for all n
(C) $a_n =$ R for all n
(D) None of these

22. If a sequence is not a Cauchy sequence y, then it is a—
(A) Divergent sequence
(B) Convergent sequence
(C) Bounded sequence
(D) None of these

23. (a) Every convergent sequence is a Cauchy sequence
(b) Every Cauchy sequence is a convergent sequence
(A) (a) and (b) both are false
(B) (a) is true
(C) (b) is true
(D) (a) and (b) both are true

24. Every Cauchy sequence is—
(A) Unbounded sequence
(B) Bounded sequence
(C) Divergent sequence
(D) None of these

25. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence converges to 0 and $\{b_n\}_{n=1}^{\infty}$ be a sequence that is bounded, then $\{a_n\ b_n\}_{n=1}^{\infty}$ is a sequence that—
(A) Converges to one
(B) Converges to zero
(C) Is divergent sequence
(D) None of these

26. Let sequence $\{a_n\}_{n=1}^{\infty}$ converges A and sequence $\{b_n\}_{n=1}^{\infty}$ converges to B, with $a_n \leq b_n$ for all n, then—
(A) A $\leq$ B (B) A = B
(C) A $\geq$ B (D) None of these

27. Let sequence $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converges to A and B respectively, then $\{a_n/b_n\}_{n=1}^{\infty}$ converges to A/B if—
(A) $b_n \neq 0$ for all n and B = 0
(B) $b_n \neq$ for some n
(C) $b_n \neq 0$ for all n and B $\neq$ 0
(D) None of these

28. If $\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded, then $\{a_n\}_{n=1}^{\infty}$ is—
(A) Convergent sequence
(B) Divergent sequence
(C) Non-Cauchy sequence
(D) None of these

29. Every Cauchy sequence is—
(A) Monotone sequence
(B) Divergent sequence
(C) Unbounded sequence
(D) None of these

30. A series $\sum_{n=1}^{\infty} a_n$ converges, absolutely iff—
(A) $\sum_{n=1}^{\infty} |a_n|$ converges
(B) $\left|\sum_{n=1}^{\infty} a_n\right|$ converges
(C) $\sum_{n=1}^{\infty} |a_n|$ diverges
(D) None of these

31. A series $\sum_{n=1}^{\infty} a_n$ converges, then sequence $\{a_n\}_{n=1}^{\infty}$ is—
(A) Diverges
(B) Converges to zero
(C) Converges to any number
(D) None of these

32. For infinite series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, $b_n \geq 0$ for n, and there is a real number N, such that for $n \geq$ N $\Rightarrow |a_n| \leq b_n$. If $\sum_{n=1}^{\infty} b_n$ converges, then—

(A) $\sum_{n=1}^{\infty} b_n$ is absolutely convergent

(B) $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(C) $\sum_{n=1}^{\infty} a_n$ is absolutely divergent

(D) $\sum_{n=1}^{\infty} b_n$ is absolutely divergent

33. For real number α, if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges to A and B respectively, then $\sum_{n=1}^{\infty} (a_n + \alpha b_n)$—
(A) Converges to $A + \alpha B$
(B) Diverges
(C) Converges to $\alpha A + B$
(D) None of these

34. The series $\sum_{n=1}^{\infty} [(-1)^n/(2n-1)]$ is—
(A) Convergent
(B) Divergent
(C) Unbounded
(D) None of these

35. The series $2 + 4 + 6 + 8 + \ldots\ldots$ is—
(A) Divergent (B) Convergent
(C) Unbounded (D) None of these

36. The series $1 + \frac{1}{2^2} + \frac{1}{3^3} + \ldots$ is—
(A) Divergent (B) Convergent
(C) Bounded (D) None of these

37. The series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is—
(A) Convergent (B) Divergent
(C) Unbounded (D) None of these

38. The series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$ is—
(A) Divergent (B) Convergent
(C) Unbounded (D) None of these

39. The series $1 + 3 + 5 + 7 + \ldots$ is—
(A) Divergent (B) Convergent
(C) Unbounded (D) None of these

40. The series $\sum_{n=1}^{\infty} (-1)^n$ is—
(A) Divergent (B) Convergent
(C) Unbounded (D) None of these

41. The series $1^3 + 2^3 + 3^3 + \ldots$ is—
(A) Divergent (B) Convergent
(C) Bounded (D) None of these

42. The series $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \ldots$ is—
(A) Convergent (B) Divergent
(C) Decreasing (D) None of these

43. The sequence $\{(-1)^n\}_{n=1}^{\infty}$ is—
(A) Bounded and convergent
(B) Convergent and unbounded
(C) Bounded and divergent
(D) Divergent and unbounded

44. The sequence $\{1/n\}$ is—
(A) Unbounded and convergent
(B) Bounded and convergent
(C) Bounded and divergent
(D) Divergent and unbounded

45. The sequence $\{(-1)^n/n\}$ converges to—
(A) Zero (B) 1
(C) 2 (D) None of these

46. A monotone increasing is bounded, then—
(A) It is divergent (B) It is convergent
(C) It is constant (D) None of these

47. Every Cauchy sequence is—
(A) Bounded (B) Unbounded
(C) Divergent (D) None of these

48. Every Cauchy sequence is—
(A) Unbounded (B) Convergent
(C) Divergent (D) None of these

49. If sequence $\{a_n\}_{n=1}^{\infty}$ converges to A and $\{b_n\}$ converges to B, then for all $a_n \geq b_n$ we have—
(A) $A \geq B$ (B) $A \leq B$
(C) $A = B$ (D) None of these

50. The sequence $\{1, 2, 3, \ldots\}$ is—
(A) Bounded below
(B) Bounded above
(C) Bounded
(D) None of these

51. The sequence $\{-1, -2, -3, \ldots\}$ is—
(A) Bounded below (B) Bounded above
(C) Bounded (D) None of these

52. The sequence $\left\{\frac{n+1}{n}\right\}$ is—
(A) Bounded (B) Bounded above
(C) Unbounded (D) None of these

53. The sequence $\left\{\frac{n}{n+1}\right\}$ is—
(A) Decreasing sequence
(B) Bounded
(C) Unbounded
(D) None of these

54. The sequence $\left\{\frac{1}{3^n}\right\}$ is—
(A) Divergent sequence
(B) Bounded
(C) Unbounded
(D) None of these

55. The sequence $\left\{\frac{(-1)^n}{n}\right\}$ is—
(A) Divergent sequence
(B) Bounded
(C) Unbounded
(D) None of these

56. The sequence $\left\{\frac{n+1}{n}\right\}$ is—
(A) Increasing sequence
(B) Decreasing sequence
(C) Unbounded
(D) None of these

57. The sequence $\left\{\frac{n}{n+1}\right\}$ is—
(A) Increasing sequence
(B) Decreasing sequence
(C) Unbounded
(D) None of these

58. The sequence $\left\{\frac{(-1)^n}{n}\right\}$ is—
(A) Unbounded (B) Decreasing
(C) Increasing (D) None of these

59. The sequence $\{1, 0, 1, 0, 1, 0, \ldots\}$ is—
(A) Increasing (B) Decreasing
(C) Bounded (D) None of these

60. The sequence $\left\{a + \frac{(-1)^n b}{n}\right\}$—
(A) Bounded (B) Unbounded
(C) Divergent (D) None of these

61. The series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ is—
(A) Convergent (B) Divergent
(C) Oscillatory (D) None of these

62. The series $\sum_{n=1}^{\infty} \frac{1}{(1 + 1/n)^{n^2}}$ is—
(A) Convergent (B) Divergent
(C) Oscillatory (D) None of these

63. The series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots$—
(A) Convergent (B) Divergent
(C) Oscillatory (D) None of these

64. The series $1 + x + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!} + \ldots$—
(A) Convergent (B) Divergent
(C) Oscillatory (D) None of these

65. The series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$, for $|x| > 1$ is—
(A) Convergent (B) Divergent
(C) Oscillatory (D) None of these

66. A polynomial for real values of x is—
(A) Continuous (B) Discontinuous
(C) Convergent (D) None of these

67. A function $f(x) = \frac{x^2 - 1}{x - 1}$ is—
(A) Discontinuous at $x = 1$
(B) Discontinuous at $x = 2$
(C) Continuous at $x = 1$
(D) Continuous at $x = 2$

68. A function $f(x) = |x|$ is—
(A) Discontinuous
(B) Discontinuous at $x = 0$
(C) Continuous everywhere
(D) None of these

69. A function $f(x) = |x|$ is—
(A) Continuous and differentiable at $x = 0$
(B) Continuous but not differentiable at $x = 0$
(C) Discontinuous and differentiable at $x = 0$
(D) None of these

70. Let $f: E \to R$ and $x_0 \in E$ and x_0 an accumulation point of E if every sequence $\{x_n\}_{n=1}^{\infty}$ converses to x_0 with $x_n \in E$ for all n $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$, then—
(A) f is discontinuous at x_0
(B) f is discontinuous at x_0
(C) f is differentiable at x_0
(D) None of these

71. $f: E \to Q$ and $g: E \to R$ and both are continuous at x_0, then—
(A) f/g is continuous at x_0
(B) f/g is discontinuous at x_0
(C) f/g is continuous at x_0 if $g(x_0) \neq 0$
(D) None of these

72. Statement A : f is continuous
Statement B : f is differentiable
(A) If A is true, then B is true
(B) If A is false, then B is false
(C) If B is true, then A is true
(D) If B is true, then A is false

73. A set $E \subset R$ is compact—
(A) It is closed and bounded
(B) It is open and bounded
(C) It is open and unbounded
(D) It is closed and unbounded

74. The function $f(x)$ is continuous on closed interval $[a, b]$, then—
(A) f is bounded on $[a, b]$
(B) f is unbounded on $[a, b]$
(C) f is constant on $[a, b]$
(D) None of these

75. The function $f(x)$ is continuous on closed interval $[a, b]$, and $f(a) \cdot f(b) < 0$ if at some point c, $f(c) = 0$, then—
(A) $c \notin [a, b]$
(B) $c \in [a, b]$
(C) $c \in [a, b]$
(D) None of these

76. The function $f(x)$ is continuous on closed interval $[a, b]$ and $m = \min f(x)$ and $M = \max f(x)$. If for any A, $m \leq A \leq M$ there is $x_0 \in [a, b]$ for which—
(A) $f(x_0) = A$
(B) $f(x_0) = 0$
(C) $f(x_0) \neq A$
(D) None of these

77. A real number set E is compact, if—
(A) E is bounded and open
(B) E is unbounded and closed
(C) E is bounded and closed
(D) E is unbounded and open

78. If f_1 and f_2 are two real valued bounded functions defined on $[a, b]$, then for every partition P on $[a, b]$—
(A) $U(P, f_1 + f_2) = U(P, f_1) + U(P, f_2)$
(B) $U(P, f_1 + f_2) \leq U(P, f_1) + U(P, f_2)$
(C) $U(P, f_1 + f_2) \geq U(P, f_1) + U(P, f_2)$
(D) None of these

79. Let Lower Riemann Integral be $f(x)$ on $[a, b]$ and $L(P, f)$ is the Lower Riemann sum over all partitions on $[a, b]$ then—
(A) $\int_{-a}^{b} f(x)\,dx = i.u.b\ \{L(P, f)\}$
(B) $\int_{-a}^{b} f(x)\,dx = g.l.b\ \{L(P, f)\}$
(C) $\int_{-a}^{b} f(x)\,dx = L(P, f)$
(D) None of these

80. If f is a Riemann integrable function on $[a, b]$ and λ is any real number then following holds—
(A) $\lambda \int_{a}^{b} f(x)\,dx = \int_{a}^{b} \lambda f(x)\,dx$
(B) $\lambda \int_{a}^{b} f(x)\,dx \neq \int_{a}^{b} \lambda f(x)\,dx$
(C) $\lambda \int_{a}^{b} f(x)\,dx \geq \int_{a}^{b} \lambda f(x)\,dx$
(D) None of these

81. A real valued bounded function $f(x)$ is Riemann integrable on $[a, b]$, then—
(A) $\int_{-a}^{b} (x)\, dx$ and $\int_{a}^{-b} (x)\, dx$ exist
(B) $\int_{-a}^{b} (x)\, dx = \int_{a}^{-b} f(x)\, dx$
(C) $\int_{a}^{-b} f(x)\, dx \neq \int_{-a}^{b} f(x)\, dx$
(D) None of these

82. If $\int_{-a}^{b} f\, dx$ and $\int_{a}^{-b} dx$ are lower and upper Riemann integrable on $[a, b]$ then—
(A) $\int_{-a}^{b} f\, dx \geq \int_{a}^{-b} f\, dx$
(B) $\int_{-a}^{b} f dx = \int_{a}^{-b} f\, dx$
(C) $\int_{-a}^{b} f\, dx \leq \int_{a}^{-b} f dx$
(D) None of these

83. If f is real valued bounded function on $[a, b]$ and m, M are greatest lower bound and least upper bound respectively, then—
(A) $m(b-a) = \text{M}(b-a)$
(B) $m(b-a) \geq \text{M}(b-a)$
(C) $m(b-a) \leq \text{M}(b-a)$
(D) None of these

84. The radius of convergence of the series
$$1 - \frac{x}{1} + \frac{x^2}{2!} - \frac{x^3}{3!}$$
is—
(A) ∞ (B) Zero
(C) 1 (D) None of these

85. The radius of convergence of the series
$$\frac{x+0{\cdot}2}{1} + \frac{(x+0{\cdot}2)^2}{2} + \ldots\ldots + \frac{(x+0{\cdot}2)^n}{n} + \ldots$$
(A) 1 (B) ∞
(C) Zero (D) None of these

86. The series $1{\cdot}x + 1{\cdot}2x^2 + 1{\cdot}2{\cdot}3x^3 + \ldots + \underline{|n}\; x^n + \ldots$ is—
(A) Divergent everywhere except $x = 0$
(B) Convergent everywhere except $x = 0$
(C) Divergent for $x = 0$
(D) None of these

87. The radius of convergence of the series $1 - x^2 + x^4 - x^6 + \ldots\ldots$ is—
(A) 1 (B) Zero
(C) 2 (D) None of these

88. The series $1 + x + x^2 + \ldots + x^n + \ldots$
(A) Converges for $|x| < 1$
(B) Diverges for $|x| > 1$
(C) Converges for $(x) = 1$
(D) None of these

89. The series $1 + \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \ldots\ldots$
(A) Diverges for $|x| > 1$
(B) Converges for $|x| < 1$
(C) Diverges for $|x| = 1$
(D) None of these

90. The domain of convergence for $1 + x + x^2 + \ldots$ is—
(A) $[-1, +1]$ (B) $[-1, +1]$
(C) $[-2, 2]$ (D) None of these

91. The domain of convergence for $1 + \frac{x}{1^2} + \frac{x^2}{2^2} + \ldots$ is—
(A) $[-1, +1]$ (B) $[-1, +1]$
(C) $[0, 1]$ (D) None of these

92. The domain of convergence for
$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$
is—
(A) $(-1, 1)$ (B) $(-1, 1)$
(C) $(-1, 2)$ (D) $(-1, 1)$

93. If $f : [a, b] \to \text{R}$ is continuous and monotone function, then—
(A) f is Riemann integrable on $[a, b]$
(B) f is not Riemann integrable on $[a, b]$
(C) f is Riemann integrable on R
(D) None of these

94. If f_1 and f_2 are Riemann integral functions, then $\int_a^b f_1 dx + \int_a^b f_2 dx$ is equal to—
(A) $\int_b^a (f_1 + f_2)\, dx$
(B) Zero
(C) $\int_b^a (f_1 + f_2)\, dx$
(D) None of these

95. If f_1 is Riemann integrable, then

$$\int_b^a f_1\,dx + \int_b^c f_1\,dx$$

is—

(A) $\int_a^b f_1 dx + c$ (B) $\int_b^c f_1 dx + c$

(C) $\int_a^c f_1 dx$ (D) None of these

96. $f : [a, b] \to R$, f is Riemann integrable then—

(A) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

(B) $\int_a^b |f(x)|\,dx \le \left|\int_a^b f(x)dx\right|$

(C) $\int_a^b |f(x)|\,dx = \left|\int_a^b f(x)\,dx\right|$

(D) None of these

97. $f : [a, b] \to R$, P and Q are partitions of $[a, b]$ such $P \subset Q$, then—

(A) $L(P, f) \le (Q, f)$

(B) $L(P, f) \ge L\,(Q, f)$

(C) (A) is true (B) is false

(D) (B) is true (A) is false

98. A function f is Riemann integrable on $[a, b]$ if—

(A) Only $\int_a^{-b} f\,dx$ exist

(B) Only $\int_{-a}^{b} f\,dx$ exist

(C) $\int_a^{-b} f\,dx \ne \int_{-a}^{b} f\,dx$

(D) $\int_a^{-b} f\,dx = \int_{-a}^{b} f\,dx$

99. If f is Riemann integrable with respect to α on $[a, b]$, then—

(A) f is increasing and α is bounded function

(B) f is bounded and α is increasing function

(C) f and α are both bounded

(D) f and α are both increasing

100. $f : [0, 1] \to R$ such that $f(x) = \begin{cases} 0 & x \text{ is rational} \\ 1 & x \text{ is irrational} \end{cases}$

then—

(A) The upper and lower integral of f does not exist

(B) f is riemann integrable

(C) f is not riemann integrable

(D) None of these

101. $f : [a, b] \to R$ is continuous, then f is Riemann integrable on $[a, b]$—

(A) The statement is true

(B) The statement is false

(C) Neither true nor false

(D) Partially true

102. Given a function $f(x) = |x|$ for all $x \in R$. The function f is—

(A) Differentiable at zero

(B) Continuous at zero

(C) (A) is true (B) is false

(D) (A) and (B) are both true

103. A rational function (The quotient of two polynomial functions) is—

(A) Differentiable everywhere

(B) Not differentiable any where

(C) Differentiable at each point, where deno-minator is non zero

(D) None of these

104. Let $f : D \to R$. A point $x_0 \in D$ is a relative minimum of f, iff there is a neighbourhood Q of x_0 such that $x \in Q \cap D$, then—

(A) $f(x) \le f(x_0)$ (B) $f(x) \ge f(x_0)$

(C) $f(x) \not\le f(x_0)$ (D) $f(x) \not\ge f(x_0)$

105. Let $f : [a, b] \to R$ is continuous on $[a, b]$ and f is differentiable on (a, b), if $f(a) = f(b) = 0$ there exist c such that $f'(c) = 0$, then—

(A) $c \in [a, b]$ (B) $c \in (a, b)$

(C) $c \in [a, b)$ (D) $c \in (a, b]$

106. The set of positive integers is—

(A) Bounded above

(B) Unbounded above

(C) Unbounded below

(D) None of these

107. If $x \in R$, set of real numbers then—

(A) $\infty < x < -\infty$ (B) $-\infty \le x \le \infty$

(C) $-\infty < x < \infty$ (D) $-\infty \ge x \ge \infty$

108. The value of $(\infty - \infty)$ is—

(A) 0 (B) ∞

(C) $-\infty$ (D) Undefined

109. The set of real number is—
(A) Bounded above (B) Unbounded above
(C) Finite set (D) Countable set

110. If R* is an extended real number system, then the least upper bound of R* is—
(A) $+\infty$
(B) $-\infty$
(C) 0
(D) No least upper bound

111. If S is non-empty set of real numbers and S is unbounded below then—
(A) Inf S $= -\infty$ (B) Inf S $= +\infty$
(C) Sup S $= \infty$ (D) Sup S $= -\infty$

112. If R is a set of real numbers, $x, y, \in$ R, then—
(A) $x + y \neq y + x$ (B) $x + y = y + x$
(C) $xy \neq xy$ (D) $x + y \geq y + x$

113. If R is a set of real numbers, $x \in$ R, there exist $-x \in$ R—
(A) $x + (-x) = 0$ (B) $x + (-x) = 1$
(C) $x(-x) = 0$ (D) $x(-x) = 1$

114. If R is a set of real number $x \in$ R, then—
(A) $x + x^{-1} = 0$ (B) $x + x^{-1} = 1$
(C) $xx^{-1} = 1$ (D) $xx^{-1} = 0$

115. If R is a set of real numbers $x, y \in$ R, then—
(A) $x > 0$ and $y > 0 \Rightarrow xy > 0$
(B) $x > 0$ and $y > 0 \Rightarrow xy < 0$
(C) $x > 0$ and $y > 0 \Rightarrow xy = 0$
(D) $x > 0$ and $y > 0 \Rightarrow xy = 1$

116. If A and B are two non empty set of R if C = $\{x + y : x \in \text{A}, y \in \text{B}\}$, then—
(A) Inf C = inf A + inf B
(B) Inf C $\neq$ inf A + inf B
(C) Inf C $<$ inf A + inf B
(D) Inf C $>$ inf A + inf B

117. If $x, y, z \in$ R the set of real number, then following relation not holds—
(A) $x > 0$ and $y > 0$
(B) $x, y > 0 \Rightarrow x + y > 0$
(C) $x, y, z > 0 \Rightarrow x + y + z > 0$
(D) $xy > 0 \Rightarrow x - y > 0$

118. Given a real number a and b such that—
$a \leq b + \in, \ \forall \in > 0$
(A) $a = b$ (B) $a \leq b$
(C) $a \geq b$ (D) $ab = \in$

119. If $x, y, z \in$ R and $x < y$, then—
(A) $xz < yz, z > 0$
(B) $xz < yz, z < 0$
(C) $xz < yz, z = 0$
(D) $xz < yz$, z is positive, negative or zero

120. If $x, y, z \in$ R and $x < y$, then—
(A) $xz > yz, z < 0$
(B) $xz > yz, z > 0$
(C) $xz > yz, z = 0$
(D) $xz > yz$, z is positive, negative or zero

121. If $w, x, y, z \in$ R, the set of real numbers and $x > y, z > w$, then—
(A) $xz > yw$ if $y, w > 0$
(B) $xz > yw$ if $y > 0, w < 0$
(C) $xz > yw$ if $y < 0, w > 0$
(D) $xz > yw$ if $y, w < 0$

122. The set of integers is—
(A) Ordered set
(B) Non-ordered set
(C) Set of irrational numbers
(D) Does not satisfies principle of induction

123. Composite number n is—
(A) A prime number and $n > 1$
(B) Non prime number and $n < 1$
(C) Non prime number and $n > 1$
(D) A prime number and $n < 1$

124. An integer n is a prime number if—
(A) $n > 1$ and n is divisible by itself only
(B) $n < 1$ and n is divisible by itself only
(C) $n = 1$
(D) $n = 1$ and n/n

125. Every integer $n > 1$ is—
(A) Prime number
(B) Prime number or product of prime numbers
(C) Product of prime numbers
(D) Sum of prime numbers

126. For open interval (a, b), where $a < b$, $x \in (a, b)$ we have—
(A) $a \leq x \leq b$ (B) $a < x < b$
(C) $a \leq x < b$ (D) $a < x \leq b$

127. For closed interval $[a, b]$, where $a < b$, $x \in [a, b]$ we have—
(A) $a \leq x \leq b$ (B) $a < x \leq b$
(C) $a \leq x < b$ (D) $a < x < b$

128. If P is a prime number and P | ab, then—
(A) P|a of P|b (B) P + ab = 0
(C) P – ab = 0 (D) ab|P

129. If S is a set of real numbers, and c_1 and c_2 are two least upper bounds of S, then—
(A) $c_1 = c_2$ (B) $c_1 > c_2$
(C) $c_1 < c_2$ (D) $c_1 \neq c_2$

130. If S is a set of real numbers, c is a least upper bound and b is any upper bound of S, then—
(A) $c \leq b$ (B) $c \geq b$
(C) $c \neq b$ (D) $c > b$

131. The set $R^+ = (0, +\infty)$ has—
(A) No upper bound
(B) Upper bound
(C) Least upper bound
(D) None of these

132. The Set $R^+ = [0, 1]$ has—
(A) Upper bound only
(B) Lower bound only
(C) Upper and lower bound both
(D) No upper and lower bound

133. An element a is an minimal element of set S, then—
(A) $a \in$ S
(B) $a \notin s$
(C) a is not lower bound of S
(D) a is not upper bound of S

134. The set of negative integers have the least upper bound—
(A) 1 (B) 2
(C) 0 (D) – 1

135. The non-empty set of real numbers which is bounded below has—
(A) Supremum (B) Infimum
(C) Upper bound (D) No lower bound

136. For every real number x, there is a positive integer n—
(A) $n > x$ (B) $n < x$
(C) $n = x$ (D) None of these

137. If S is a set of real numbers, c_1 and c_2 are two greatest lower bound of S, then—
(A) $c_1 > c_2$ (B) $c_1 < c_2$
(C) $c_1 = c_2$ (D) $c_1 \neq c_2$

138. If S is a set of real number, c is the greatest lower bound and b is any lower bound, then—
(A) $c < b$ (B) $c > b$
(C) $c \geq b$ (D) $c \leq b$

139. The set of positive integer have the greatest lower bound—
(A) 1 (B) 0
(C) ∞ (D) None of these

140. The set of positive real numbers is—
(A) Bounded above
(B) Bounded below
(C) Unbounded from below
(D) None of these

141. The non empty set S of real numbers which is bounded above has—
(A) Supremum (B) Infimum
(C) Lower bound (D) No upper bound

142. The set $R^+ = (0, +\infty)$ is—
(A) Bounded above
(B) Unbounded above
(C) Unbounded below
(D) None of these

143. The closed interval (0, 1) have—
(A) Maximal element only
(B) Minimal element only
(C) Both maximal and minimal element
(D) None of these

144. If an element is maximal element of Set S, then—
(A) $a \in$ S
(B) $a \notin$ S
(C) a is not upper bound of S
(D) a is lower bound of S

145. The set of natural numbers has—
(A) Upper bound
(B) Lower bound
(C) Maximal element
(D) None of these

146. The set $R^+ = (0, \infty)$ is—
(A) Bounded above
(B) Bounded below
(C) Unbounded below
(D) None of these

147. The set $R^+ = (0, \infty)$ has—
(A) Minimal element
(B) No minimal element
(C) Least upper bound
(D) Maximal element

148. The closed interval $S = [0, 1]$ is—
(A) Bounded above
(B) Unbounded below
(C) Unbounded above
(D) No maximal element

149. The half interval [0, 1] is—
(A) Bounded above only
(B) Bounded below only
(C) Bonded above and below both
(D) None of these

150. The half interval [0, 1] have—
(A) Maximal element only
(B) Minimal element only
(C) Maximal and minimal both elements
(D) No maximal and no minimal element

151. Let $S = [0, 1]$, the maximal element S is—
(A) 0 (B) 1
(C) ϕ (D) 2

152. Let $S = [0, 1]$, the least upper bound for S is—
(A) 0 (B) 1
(C) ϕ (D) 2

153. Let $S = [0, 1]$, the maximal element of S is—
(A) 0
(B) 1
(C) No maximal element
(D) ϕ

154. If S, T $\subset$ R and $\forall s \in S$ and $t \in T$, $s \le t$, it S and T have supremum, then—
(A) Sup S $\le$ Sup T
(B) Sup S $\ge$ Sup T
(C) Sup S = Sup T
(D) None of these

155. If S, T $\subset$ R and $\forall s \in S, t \in T, s \le t$ if S and T have infimum, then—
(A) Inf S $\ge$ Inf T (B) Inf S $\le$ Inf T
(C) Inf S = Inf T (D) None of these

156. If S is a non empty set and S has no upper bound, then—
(A) Sup S = ∞ (B) Inf S = ∞
(C) Sup S = $-\infty$ (D) Inf S = $-\infty$

157. The set of real numbers is—
(A) Unbounded
(B) Bounded from below
(C) Bounded from above
(D) Bounded

158. The completeness aniam states—
(A) Every non-empty set S of real number which is bounded above has supremum
(B) Every non empty set S of real number which is bounded above has infimum
(C) Every non empty set S of real number which is bounded have no supremum
(D) Every non empty set S of real number which is bounded have no infimum

159. If $0 < x < y$ and $0 < z < w$, then—
(A) $xz < yw$ (B) $xz > yw$
(C) $xz < y$ (D) $xz < w$

160. If $x < y < 0$ and $0 < z < w$, then—
(A) $xz < yw$ (B) $xz > yw$
(C) $xz < y$ (D) $xz < w$

161. The value of $(0, \infty)$ is—
(A) ∞ (B) 0
(C) Undefined (D) None of these

162. If $\mathbb{R}^*$ is an extended real number system then the inf $\mathbb{R}^*$ is—
(A) 0 (B) $+\infty$
(C) $-\infty$ (D) No infimum

163. If S, T $\subset \mathbb{R}$ and $S \le t \ \forall s \in S$ and $t \in T$, then—
(A) Inf T $\le$ Inf S (B) Inf T $\ge$ Inf S
(C) Inf T = Inf S (D) None of these

164. If S, T $\subset \mathbb{R}$ and $A = \{x + y : x \in S$ and $y \in T\}$, then—
(A) Sup A = Sup S + Sup T
(B) Sup A > Sup S + Sup T
(C) Sup A < Sup S + Sup T
(D) Sup A $\neq$ Sup S + Sup T

165. "Every nonempty set S of real number which is bounded below, has infimum" is stated in—
(A) Field aniom

(B) Ordered axiom
(C) Completeness axiom
(D) None of these

166. If $x, y, z \in \mathbb{R}$, then—
(A) $x(y+z) = xy + yz$
(B) $x(y+z) = xy + yz$
(C) $x(y+z) = xyz$
(D) $xy = xz, y \neq z$

167. If the sequence, $\{x_n\}$ of real numbers have limits l_1 and l_2, then—
(A) $l_1 > l_2$ (B) $l_1 < l_2$
(C) $l_1 = l_2$ (D) $l_1 \neq l_2$

168. Every Cauchy sequence of real numbers is—
(A) Convergent
(B) Divergent
(C) Limit does not exist
(D) None of these

169. If a sequence of real number has a cluster points, then—
(A) It is convergent
(B) It s divergent
(C) Limit exist
(D) Existence of limit not definite

170. If a sequence of real numbers has a limit, then—
(A) Cluster point exist
(B) Cluster point does not exist
(C) Sequence is divergent
(D) None of these

171. (a)Sequence $<x_n>$ is convergent
(b)Sequence $<x_n>$ is bounded
(A) $A \Rightarrow B$ (B) $B \Rightarrow A$
(C) $A \Leftrightarrow B$ (D) None is true

172. (a)Cauchy sequence is convergent
(b)Cauchy sequence is bounded.
(A) (a) and (b) both true
(B) (a) is false
(C) (b) is false
(D) (a) and (b) both false

173. For the sequence $<(-1)^n>$, $\overline{\lim}\, x_n$ is equal to—
(A) -1 (B) $+1$
(C) $-\infty$ (D) $+\infty$

174. For the sequence $<(-1)^n n>$, $\overline{\lim}\, x_n$ is equal to—
(A) -1 (B) $+1$
(C) $-\infty$ (D) $+\infty$

175. For the sequence $<x_n>$ where $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$, $\overline{\lim}\, x_n$ is equal to—
(A) -1 (B) $+1$
(C) $-\infty$ (D) $+\infty$

176. The set of real numbers $\mathbb{R}$ is—
(A) Countable (B) Uncountable
(C) Infinite (D) Bounded

177. Following inequality is false—
(A) $\underline{\lim}\, x_n + \overline{\lim}\, y_n \le \overline{\lim}\, x_n + \overline{\lim}\, y_n$
(B) $\lim x_n + \overline{\lim}\, y_n \le \overline{\lim}\,(x_n + y_n)$
(C) $\underline{\lim}\, x_n + \overline{\lim}\, y_n \le \underline{\lim}\, x_n + \underline{\lim}\, y_n$
(D) $\underline{\lim}\, x_n + \overline{\lim}\, y_n \ge \lim x_n + \lim y_n$

178. Following statement is true—
(A) $\underline{\lim}\, x_n \le \overline{\lim}\, x_n$ (B) $\overline{\lim}\, x_n \le \underline{\lim}\, x_n$
(C) $\underline{\lim}\, x_n \ge \lim x_n$ (D) $\overline{\lim}\, x_n \le \lim x_n$

179. If limit $x_n = l$ exist, then—
(A) $\lim x_n = \overline{\lim}\, x_n = l$
(B) $\underline{\lim}\, x_n \neq \overline{\lim}\, x_n = l$
(C) $l = \underline{\lim}\, x_n \neq \lim x_n$
(D) $\underline{\lim}\, x_n \le \overline{\lim}\, x_n \le l$

180. For the sequence $<x_n>$, where $x_n = (-1)^n n$, limit x_n is—
(A) $-\infty$ (B) $+\infty$
(C) 0 (D) 1

181. For the sequence $<(-1)^n>$, $\lim x_n$ is equal to—
(A) $-\infty$ (B) $+\infty$
(C) $+1$ (D) -1

182. For the sequence $<x_n>$ where $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$, $\lim x_n$ is equal to—
(A) $-\infty$ (B) -1
(C) $+\infty$ (D) $+1$

183. The sequence diverges to $+\infty$ if—
(A) $\underline{\lim}\, x_n = \overline{\lim}\, x_n = +\infty$
(B) $\underline{\lim}\, x_n \leq +\infty$
(C) $\overline{\lim}\, x_n \leq +\infty$
(D) $\underline{\lim}\, x_n = \overline{\lim}\, x_n < +\infty$

184. If $\forall\, x_n, y_n \in \mathbb{R}, x_n < y_n$, then—
(A) $\underline{\lim}\, x_n \geq \underline{\lim}\, y_n$
(B) $\underline{\lim}\, x_n \leq \underline{\lim}\, y_n$
(C) $\overline{\lim}\, x_n \leq \overline{\lim}\, y_n$
(D) $\overline{\lim}\, x_n \leq \underline{\lim}\, y_n$

185. If every $x_n < y_n$, then—
(A) $\overline{\lim}\, x_n \leq \underline{\lim}\, y_n$ (B) $\overline{\lim}\, x_n \leq \overline{\lim}\, y_n$
(C) $\underline{\lim}\, x_n \leq \overline{\lim}\, y_n$ (D) $\lim x_n \leq \lim y_n$

186. The open interval is—
(A) Open set (B) Closed set
(C) Empty set (D) Closure

187. The closed interval is—
(A) Open set (B) Closed set
(C) Empty set (D) Unbounded set

188. The half closed interval is $[a, \infty]$ is—
(A) Open set (B) Closed set
(C) Empty set (D) Unbounded set

189. The intersection of finite collection of open sets is—
(A) Open set (B) Closed set
(C) Empty set (D) None of these

190. The intersection of any collection of open sets is—
(A) Open set
(B) Closed set
(C) Can not defined
(D) Semi open

191. If S is a set of real numbers which is bounded above then Sup S is—
(A) A point of closure to S
(B) Not a point of closure to S
(C) Prime number
(D) None of these

192. If S is a set of real numbers which is bounded below the inf S is—
(A) A point of closure to S
(B) Not a point of closure to S
(C) Prime number
(D) None of these

193. If F is an open covering of a closed and bounded set A, then—
(A) There exist a infinite sub collection of F which covers A
(B) There exist a uncountable sub collection of F which covers A
(C) Both (A) and (B)
(D) None of these

194. Let F be an open covering of A, then—
(A) There exist countable sub collection of F which covers A
(B) There exist uncountable collection of F which covers A
(C) (A) and (B) both true
(D) (A) and (B) both false

195. The complement of open set is—
(A) Closed set (B) Open set
(C) Countable set (D) None of these

196. A finite set is—
(A) Open set (B) Closed set
(C) Uncountable set (D) None of these

197. Singleton set $\{x_0\}$ is—
(A) Open (B) Closed
(C) Uncountable (D) None of these

198. If f' exist and is monotonic on (a, b), then—
(A) f' is continuous on (a, b)
(B) f' is discontinuous on (a, b)
(C) f' is constant
(D) None of these

199. If f is continuous on $[a, b]$ and f' exist at each $x \in (a, b)$, then if $f' > 0$ in (a, b), then—
(A) f is strictly increasing
(B) f is strictly decreasing
(C) f is constant
(D) None of these

200. If f is continuous on $[a, b]$ and f' exist at each $x \in (a, b)$, then it $f' = 0$—
(A) f is increasing

(B) f is decreasing
(C) f is constant
(D) None of these

201. A function f is increasing function on $E \subset \mathbb{R}$ if $x, y \in E$—
(A) $x < y \Rightarrow f(x) \leq f(y)$
(B) $x < y \Rightarrow f(y) \leq f(x)$
(C) $x < y \Rightarrow f(x) = f(y)$
(D) None of these

202. A function f is strictly decreasing on $E \subset R$, if $x, y \in E$—
(A) $x < y \Rightarrow f(x) < f(y)$
(B) $x < y \Rightarrow f(y) < f(x)$
(C) $x < y \Rightarrow f(y) \leq f(x)$
(D) $x < y \Rightarrow f(y) \geq f(x)$

203. A function is a monotonic function f if—
(A) f is only a decreasing function
(B) f is only an incresing function
(C) f is either increasing or decreasing function
(D) None of these

204. Singular monotonic function on $[a, b]$ have—
(A) $f(x) = 0, \forall x \in [a, b]$
(B) $f(x) = 0, \forall x \in [a, b]$
(C) $f(x) \neq 0, \forall x \in [a, b]$
(D) None of these

205. If f is an increasing function, then—
(A) $-f$ is decreasing function
(B) $-f$ is increasing function
(C) $-f$ is constant
(D) None of these

206. If f is an increasing function on closed interval $[a, b]$, then $c \in [a, b]$ we have—
(A) $f(c^-) \geq f(c) \geq f(c^+)$
(B) $f(c^-) \leq f(c) \leq f(c^+)$
(C) $f(c^-) = f(c) = f(c^+)$
(D) None of these

207. If f and g are of bounded variation, then following is false—
(A) $f + g$ is of bounded variation
(B) $f - g$ is of bounded variation
(C) fg is of bounded variation
(D) f/g is of bounded variation

208. If f is bounded variation, then f/g is of bounded variation if—
(A) f is of bounded above from zero
(B) g is of bounded above from zero
(C) g is of bounded variation
(D) None of these

209. If f is of bounded variation on $[a, b]$ and $c \in (a, b)$, then—
(A) $V_f(a, b) \leq V_f(a, c) + V_f(c, b)$
(B) $V_f(a, b) \geq V_f(a, c) + V_f(a, b)$
(C) $V_f(a, b) = V_f(a, c) + V_f(c, b)$
(D) None of these

210. If f is of bounded variation on $[a, b]$ and $c \in (a, b)$, then—
(A) f is of bounded variation on $[a, c]$ and on $[c, b]$
(B) f is not of bounded variation on $[a, c]$ and on $[c, b]$
(C) f is constant on $[a, c]$ and $[c, b]$
(D) None of these

211. If f is of bounded variation on $[a, b]$ if—
(A) f is of difference of two monotonic real valued function on $[a, b]$
(B) f is the product of two monotonic real valued function on $[a, b]$
(C) f is the quotient of two monotone real valued function on $[a, b]$
(D) None of these

212. If f is continuous on $[a, b]$ then f is of bounded variation on $[a, b]$ if—
(A) f is the difference of two monotone continuous function on $[a, b]$
(B) f is the product of two monotone continuous function on $[a, b]$
(C) f is the quotient of two monotone continuous functions on $[a, b]$
(D) None of these

213. If $f : [a, b] \to R$ is monotonic, then—
(A) If is of bounded variation
(B) f is unbounded
(C) The set of discontinuous of f are uncountable
(D) None of these

214. If f is of bounded variation on $[a, b]$, then total variation of f on $[a, b]$ is—

(A) Finite number (B) Infinite

(C) Zero (D) None of these

Answers with Hints

1. (B) The first term $a_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$

The second term $a_2 = \frac{1}{2\cdot 2} = \frac{1}{4}$

The third term $a_3 = \frac{1}{3\cdot 2} = \frac{1}{6}$

and the nth term $a_n = \frac{1}{n \,.\, 2}$

2. (B) Prime number

3. (C) For n = odd $a_n = 1$

For n = even $a_n = \frac{1}{n - \frac{n}{2} + 1}$

4. (C) $a_1 = 2 = 1 + \frac{1}{1}$

$$a_2 = \frac{-3}{2} = (-1)^{2-1}\left(1 + \frac{1}{2}\right)$$

$$a_3 = \frac{4}{3} = (-1)^{3-1}\left(1 + \frac{1}{3}\right)$$

$$a_4 = \frac{-5}{4} = (-1)^{4-1}\left(1 + \frac{1}{4}\right)$$

$$a_n = (-1)^{n-1}\left(1 + \frac{1}{n}\right)$$

5. (B) $a_1 = 1 = 2 - 1,$

$$a_2 = \frac{5}{2} = 4 + 1,$$

$$a_3 = \frac{5}{3} = 6 - 1$$

$$\ldots\ldots a_n = \frac{2n + (-1)^n}{n}$$

6. (A) Given $a_1 = 0 = \frac{1-1}{1}$

$$a_2 = 1 = \frac{1+1}{2}$$

$$a_3 = 0 = \frac{1-1}{3}$$

$$a_4 = \frac{1}{2} = \frac{1+1}{4}$$

$$a_n = \frac{1 + (-1)^n}{n}$$

7. (A)	8. (C)	9. (A)	10. (A)	11. (C)
12. (D)	13. (A)	14. (B)	15. (A)	16. (B)
17. (A)	18. (A)	19. (B)	20. (B)	21. (A)
22. (A)	23. (D)	24. (B)	25. (B)	26. (A)
27. (C)	28. (A)	29. (A)	30. (A)	31. (B)
32. (B)	33. (C)			

34. (A) The nth term $a_n = \frac{(-1)^n}{2n-1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left|\frac{(-1)^n}{2n-1}\right| = 0$$

$\Rightarrow (|a_n|)$ is a convergent sequence

$\Rightarrow \in |a_n|$ is convergent

$\Rightarrow \in a_n$ is convergent and converges to zero.

35. (A) The nth term $a_n = 2n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (2n)$$

$$= \infty \neq 0$$

$\because \{a_n\}$ is not converges to zero

$\therefore \in a_n$ is divergent.

36. (A) The nth term

$$a_n = 1/n^n$$

$$a_{n+1} = \frac{1}{(n+1)^{n+1}}$$

(By Ratio test)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{(n+1)^{n+1}} \cdot \frac{n^n}{1}\right|$$

$$= \left|\frac{n^n}{(n+1)^n} \cdot \frac{1}{n+1}\right|$$

$$= \left|\left(\frac{n^n}{n+1}\right)^n \left(\frac{1}{n+1}\right)\right|$$

$$\therefore \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{e} \cdot \infty\right|$$

$$= \infty > 1$$

$$\left(\because \lim\left(\frac{n}{n+1}\right) = 1/e\right)$$

$\therefore$ Series diverges.

37. (A) $\frac{a_{n+1}}{a_n} = \frac{\underline{|n+1}}{(n+1)^{n+1}} = \frac{n^n}{\underline{|n}}$

$$= \frac{n^n}{(1+n)^n} = \frac{1}{(1+1/n)^n}$$

$\therefore \quad \lim \left|\frac{a_{n+1}}{a_n}\right| = 1 \div \left\{\lim_{n\to\infty} (1 + 1/n)^n\right\}$

$= \frac{1}{e} < 1$

$\therefore$ The series is convergent.

38. (B) $\frac{a_{n+1}}{a_n} = \frac{1}{3}(1 + 1/n)^2$

$\Rightarrow \quad \lim \frac{a_{n+1}}{a_n} = \frac{1}{3} < 1$

$\Rightarrow$ Series is convergent.

39. (A) The partial sum

$$S_n = \frac{1}{2} n\,[2(1) + (n-1).1]$$

$$= \frac{1}{2} n\,(n+1)\ \forall\, n \in N$$

$\therefore \quad \forall n \in N,\ 1 \le S_n$ (bounded below)

and there is

$n\,\theta M : M \ge S_n$

$\therefore$ $\{S_n\}$ is unbounded sequence

Also $\quad S_{n+1} - S_n = \frac{1}{2}(n+1)\,(n+2)\frac{-1}{2}\,n\,(n+1)$

$$= \frac{1}{2}(n+1)\,(n+2-n)$$

$$= n+1$$

$\Rightarrow \quad S_{n+1} - S_n \ge 0$

$\Rightarrow$ $\{S_n\}$ is increasing sequence

$\because$ $\{S_n\}$ is increasing and unbounded

$\therefore$ $\{S_n\}$ is divergent

$\Rightarrow 1 + 3 + 5 + \ldots$ is divergent.

40. (A) $S_n = \sum_{n=0}^{n} (-1)^K = 1$

or 0 as n is odd or even

$\Rightarrow \quad \{S_n\} = \{1, 0, 1, 0, \ldots\}$

It is divergent sequence

$\Rightarrow \in (-1)^n$ is divergent

41. (A) The partial sum $S^n = \sum_{K=0}^{n} (n)^n$

$\{S_n\}$ is increasing sequence and unbounded from above

$\therefore$ It is divergent sequence

$\Rightarrow$ The series is divergent series.

42. (A) The nth term

$$a_n = \frac{1}{n\,(n+2)}$$

$$= \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$$

$$S_n = \sum_{K=1}^{n} a_K$$

$$= \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

$\because \forall n \in N \quad S_n \ge \frac{1}{6}$

and $\quad S_n \le \frac{3}{4}$

$\therefore$ $\{S_n\}$ is bounded and also S_n is monotone sequence

$\Rightarrow$ $\{S_n\}$ is convergent

$\therefore$ The series is convergent.

43. (C) $x_n = (-1)^n$,

$\forall n \in N, \forall n \in N, -1 \le x_n$

and $\quad \forall n \in N,\ 1 \ge x_n$

$\Rightarrow$ $\{(-1)^n\}$ is bounded

For even number $\lim_{n\to\infty} (-1)^n = -1$

For odd number $\lim_{n\to\infty} (-1)^n = -1$

$\therefore$ Series is divergent.

44. (B) $x_n = \frac{1}{n}\ \forall n \in N, \forall n \in N,$

$1 \ge x_n, \forall n \in N,$

$0 \le 0$

$\Rightarrow \left\{\frac{1}{n}\right\}$ is bounded

$$x_n = \frac{1}{n}$$

and $\quad x_{n+1} = \frac{1}{n+1}$

$\Rightarrow \quad x_{n+1} - x_n = \frac{1}{n+1} - \frac{1}{n}$

$$= \frac{n-n-1}{(n+1)n}$$

$$= \frac{-1}{(n+1)^n}$$

$\Rightarrow \quad x_{n+1} - x_n \le 0$

$\Rightarrow \quad x_{n+1} \le x_n,$

$\forall n \in N$ decreasing sequence

$\because \left\{\frac{1}{n}\right\}$ is bounded and monotone sequence

$\therefore \left\{\frac{1}{n}\right\}$ is convergent.

45. (A) $x_n = a_n b_n$

$= (-1)^n \cdot \left(\frac{1}{n}\right) \; \forall n \in N,$

$a_n = (-1)^n$

$\Rightarrow \quad \forall n \in N \; -1 \leq a_n$

and $\quad 1 \geq a_n$

$\Rightarrow \{a_n\}$ is a bounded sequence

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

$\Rightarrow \{a_n\}$ is bounded sequence and $\{b_n\}$ converges to zero

$\therefore \{a_n b_n\}$ converges to zero.

46. (B) 47. (A) 48. (B) 49. (B)

50. (A) Let $x_n = n,$

$n \in N, \forall n \in N, 1 \leq x_n$ (bounded below)

But there exist no real number

$M : x_n \leq M$

$\therefore \{n\}$ is unbounded.

51. (B) The sequence $\{-n\}$ is bounded above since there exist a real number

$-1 : -1 \geq -n, \forall n \in N.$

52. (A) $x_n = \frac{n+1}{n} = 1 + 1/n$

$\forall n \in N, 1 \leq x_n$ (bounded below)

$\forall n \in N, 2 \geq x_n$ (bounded above)

$\therefore \left\{\frac{n+1}{n}\right\}$ is bounded sequence.

53. (B) $x_n = \frac{n}{n+1} = \frac{1}{1+1/n}$

$\forall n \in N, 1 \geq x_n$ (bounded above)

and $\forall n \in N, \frac{1}{2} \leq x_n$ (bounded below)

$\therefore \{x_n\}$ is bounded

54. (B) $x_n = 1/3^n,$

$\forall n \in N, 0 \leq x_n$ (bounded below)

$\forall n \in N, 1 \geq x_n$ (bounded above)

$\therefore \{x_n\}$ is bounded.

55. (B) $x_n = \frac{(-1)^n}{n}$

$\forall n \in N, -1 \leq x_n$

$\forall n \in N, 0 \geq x_n$

$\therefore \{x_n\}$ is bounded

56. (B) $x_n = \frac{n+1}{n}$

$= 1 + 1/n$

$x_{n+1} = \frac{(n+1)+1}{(n+1)}$

$= 1 + \frac{1}{n+1}$

$x_{n+1} - x_n = \frac{1}{n+1} \cdot \frac{-1}{n}$

$= \frac{-1}{n(n+1)}$

$\Rightarrow \quad x_{n+1} - x_n < 0$

$\Rightarrow \quad x_{n+1} < x_n$

$\Rightarrow \{x_n\}$ is decreasing sequence.

57. (A) $x_n = \frac{n}{n+1},$

$x_{n+1} = \frac{n+1}{n+1+1}$

$= \frac{n+1}{n+2}$

$x_{n+1} - x_n = \frac{n+1}{n+2} - \frac{n}{n+1}$

$= \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)}$

$= \frac{1}{(n+2)(n+1)}$

$\Rightarrow \quad x_{n+1} - x_n > 0$

$\Rightarrow \quad x_{n+1} > x_n$

$\Rightarrow \left[\frac{n}{n+1}\right]$ is increasing sequence.

58. (A) $x_n = \frac{(-1)^n}{n},$

$x_{n+1} = \frac{(-1)^{n+1}}{n+1}$

$x_{n+1} - x_n = \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^n}{n}$

$= \frac{n(-1)^{n+1} - (n+1)(-1)^n}{(n+1)^n}$

$$= \frac{(-1)^n(-n-n+1)}{(n+1)\,n}$$

$$= \frac{(-1)^n}{n\,(n+1)}$$

Increasing n even, Decreasing n odd.

$\forall n \in \mathrm{N}, -1 \le x_n \le 0$, bounded,

59. (C) $\forall n \in \mathrm{N}, 0 \le x_n$,

$\forall n \in \mathrm{N}, 1 \ge x_n$

$\therefore$ {1, 0, 1, 0, ...} is bounded.

60. (A) $\forall n \in \mathrm{N}, a \le x_n$,

$\forall n \in \mathrm{N}, a - b \ge x_n$

$\therefore \{x_n\}$ is bounded.

61. (A) The series $\Sigma \frac{1}{n^{\mathrm{P}}}$ is convergent iff P > 1

62. (A) Here $a_n = \frac{1}{(1+1/n)^{n^2}}$

By Root test

$$\lim_{n\to\infty} \sqrt{a_n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n}$$

$$= \frac{1}{e} < 1$$

$\therefore$ The series is convergent.

63. (A) The absolute series

$$\frac{\sin x}{1^3} + \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \ldots \text{ is convergent.}$$

$\therefore$ The series is convergent.

64. (A) By Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^n}{\lfloor n} \times \frac{\lfloor n-1}{x^{n-1}}\right|$$

$$= \left|\frac{x}{n}\right|$$

$$\lim_{n\to\infty} \left|\frac{x}{n}\right| = 0 < 1,$$

The series is convergent.

65. (B) By Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|(-1)^{n+1} \cdot \frac{x^{n+1}}{n+1} \times \frac{n}{(-1)^n \times x^n}\right|$$

$$= \left|\frac{x \,.\, n}{n+1}\right|$$

$$= x \,.\, \left|\frac{1}{1+1/n}\right|$$

$$x \lim_{n\to\infty} \left|\frac{1}{1+1/n}\right| = x > 1$$

when $|x| > 1$

$\therefore$ The series diverges for $|x| > 1$

66. (A)	67. (A)	68. (C)	69. (B)	70. (B)
71. (C)	72. (C)	73. (A)	74. (A)	75. (B)
76. (A)	77. (C)	78. (C)	79. (A)	80. (A)
81. (B)	82. (C)	83. (C)		

84. (A) Here $a_n = \frac{(-1)^n}{\lfloor n}$

and $\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{\lfloor n+1} \times \frac{\lfloor n}{1}$

$$= \frac{1}{n}$$

$\therefore$ $\mathrm{L} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

$$= \lim_{n\to\infty} \frac{1}{n}$$

$$= 0$$

and the radius of convergence

$$\mathrm{R} = \frac{1}{\mathrm{L}} = \frac{1}{0}$$

$$= \infty$$

85. (A) Here $x_0 = 0{\cdot}2$

and $a_n = 1/n$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1} \cdot \frac{n}{1}$$

$$= \frac{n}{n+1}$$

$$= \frac{1}{1+1/n}$$

$\Rightarrow$ $\mathrm{L} = \lim_{n\to\infty} \frac{1}{1+1/n}$

$$= 1$$

and the radius of convergence.

$$\mathrm{R} = \frac{1}{\mathrm{L}} = 1$$

86. (A) Here $a_n = \lfloor n$

and $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\lfloor n+1}{\lfloor n}$

$$= n+1$$

$$\Rightarrow \qquad L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \infty$$

The radius of convergence

$$R = 0$$

∴ The series converges only at

$$x = 0$$

87. (A)	88. (A)	89. (A)	90. (A)	91. (A)
92. (B)	93. (A)	94. (C)	95. (C)	96. (B)
97. (A)	98. (D)	99. (B)	100. (B)	101. (A)
102. (C)	103. (C)	104. (B)	105. (B)	106. (B)
107. (C)	108. (D)	109. (B)	110. (A)	111. (A)
112. (B)	113. (A)	114. (C)	115. (A)	116. (A)
117. (D)	118. (B)	119. (A)	120. (A)	121. (A)
122. (A)	123. (C)	124. (A)	125. (B)	126. (B)
127. (A)	128. (A)	129. (A)	130. (A)	131. (A)
132. (C)	133. (A)	134. (D)	135. (B)	136. (A)
137. (C)	138. (C)	139. (A)	140. (B)	141. (A)
142. (B)	143. (D)	144. (A)	145. (B)	146. (B)
147. (A)	148. (A)	149. (C)	150. (B)	151. (B)
152. (B)	153. (C)	154. (A)	155. (A)	156. (A)
157. (A)	158. (A)	159. (A)	160. (A)	161. (B)
162. (C)	163. (A)	164. (A)	165. (C)	166. (B)
167. (C)	168. (A)	169. (D)	170. (A)	171. (C)
172. (A)	173. (B)	174. (D)	175. (B)	176. (A)
177. (C)	178. (A)	179. (A)	180. (A)	181. (D)
182. (B)	183. (A)	184. (B)	185. (B)	186. (A)
187. (B)	188. (B)	189. (A)	190. (C)	191. (A)
192. (A)	193. (C)	194. (A)	195. (A)	196. (B)
197. (B)	198. (A)	199. (A)	200. (C)	201. (A)
202. (B)	203. (C)	204. (B)	205. (A)	206. (B)
207. (D)	208. (B)	209. (C)	210. (A)	211. (A)
212. (A)	213. (A)	214. (A)		

●●

Ordinary Differential Equations

Ordinary Differential Equation—Ordinary differential equation is a differential equation in which unknown functions depends on single independent variables.

Order (of differential equation) is the order of the highest derivative in the equation.

Degree (of differential equation) is the degree of highest of derivative occurring in differential equation, after the equation has been expressed in a form free from radicals and fractions as far as the derivatives are concerned.

Solution (of differential equation)/ Integral—

(a) A relation between variables, which satisfies the given differential equation.

(b) Any function which reduces the differential equation to an identity when substituted for dependent variable.

e.g., $y = \sin x$ is a solution (integral) of $y'' + y = 0$ because substituting $y = \sin x$ in $y'' + y = 0$ $\Rightarrow (\sin x)'' + \sin x = 0$, which is an identity.

General (complete) solution is a solution in which the number of arbitrary constants is equal to the order of the differential equation.

Particular solution is a solution obtained from general solution, by giving particular value to arbitrary constant.

Singular solution is an additional solution, other than general solution. It is a solution that can not be obtained by assigning particular values to arbitrary constants of general solution.

e.g. $$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} - y = 0$$

General solution

$$y = cx - c^2$$

Particular solution—Giving arbitrary values to c in general solution.

Singular solution

$$y = \frac{x^2}{4}$$

which can not be obtained from general solution by substituting value to c.

First Order Differential Equation

The general form

$$F(x, y, y') = 0, \; y' = \frac{dy}{dx} \quad \text{(Implicit from)}$$

$$y' = f(x, y) \quad \text{(Explicit form)}$$

Initial Value Problem

$$y' = f(x, y), \; y(x_0) = y_0$$

where x_0 and y_0 are given values

Solution—A function $y = f(x)$ such that $y' = f(x, y)$ becomes identity if we replace y by h and y' by h'.

General Solution—A solution involving an arbitrary constant.

Particular Solution—A solution obtained by given particular value to an arbitrary constant of general solution.

(A) Solution of different differential equations—

$$y' = f(x)$$

$$\frac{dy}{dx} = f(x)$$

$$\Rightarrow \quad dy = f(x)\, dx$$

Integration gives

$$y = \int f(x)\, dx + c$$

(B) Separable differential equation—

$$y' = \frac{f(x)}{g(y)}$$

$$\Rightarrow \quad g(y)\, dy = f(x)\, dx$$

Integration gives

$$\int g(y)\, dy = \int f(x)\, dx + c$$

Equation reducible to separable form,

(a) $\quad y' = g(y/x)$

let $\quad y = ux$

$$\Rightarrow \quad y' = u'x + u \qquad \ldots(a)$$

Substitution gives

$$u'x + u = g(u) \Rightarrow x\frac{du}{dx} + u = g(u)$$

$$\Rightarrow \quad \frac{du}{g(u) - u} = \frac{dx}{x} \quad \ldots (b)$$

(b) $y' = f(ax + by)$ substitution $z = ax + by$, $z' = a + by'$ gives equation (a).

(c) $y' = f\left(\frac{ax + by + c}{px + gx + r}\right)$

(i) $c = r$ equation (a)

(ii) $\frac{a}{p} = \frac{b}{q}$ equation (b)

(iii) Substitute $x = u + \alpha$, $y = v + \beta$

$\therefore \frac{dy}{dx} = \frac{dv}{du}$, where α, β solution of

$$\begin{cases} a\alpha + b\beta + c = 0 \\ p\alpha + q\beta + c = 0 \end{cases}$$

gives an equation (a) in u and v.

Existence and Uniqueness Theorem for $\frac{dy}{dx} = f(x, y)$

Lipschitz Condition—If $f(x, y)$ is a function defined for (x, y) in a domain D, then the function $f(x, y)$ is said to satisfy the Lipschitz condition on D. If there exists a positive constant K such that $|f(x, y) - f(x_1, y_1)| \le K\,|(y_2 - y_1)|$ for every pair of points (x, y_2) and (x, y_1) in D, the constant K is called Lipschitz constant.

Existence Theorem—The initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ has atleast one solution $y(x)$ provided the function $f(x, y)$ is continuous and bounded for all values of x in a domain D and there exists positive constants M and K such that $|f(x_1, y)| \le M$ and satisfies the Lipschitz condition $|f(x, y_2) - f(x, y_1) \le K\,|y_2 - y_1|$ for all points in D.

Uniqueness Theorem—The initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ has a unique solution provided the function $f(x, y)$ is continuous and bounded for all values of x in domain D and there exists positive constants M and K such that $|f(x, y)| \le M$ and satisfies the Lipschitz condition $|f(x, y_2) - f(x, y_1)| \le K|y_2 - y_1)$ for all points in domain D.

Existence and Uniqueness Theorem—The initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, has a unique solution for all values of x in the range $|x - x_0| \le a$ provided the function $f(x, y)$ is continuous and satisfy the conditions.

(i) $|f(x, y)| \le M$

(ii) $|f(x, y_2) - f(x, y_1)| \le K\,|y_2 - y_1|$ for all values of x and y. M, K are positive constants in the rectangle R defined by $|x - x_0| \le a$ and $|y - y_0| \le Ma$.

Theorem—If δ is either a rectangle $|x - x_0| \le a$, $|y - y_0| \le b$ $(a, b > 0)$ or a strip $|x - x_0| \le a$, $|y| \le \infty$, $(a > 0)$ and if f is real valued function defined on δ such that $\delta f/\delta y$ exists, is continuous on S and

$$\left|\frac{\delta}{\delta y} f(x, y)\right| \le K,\ (x, y) \sum\delta, \text{ for } K \ge 0, \text{ then } f$$

satisfies a Lipschitz condition on δ with Lipschitz constant K.

Systems of Linear First Order Differential Equations

An equation of the form $\frac{dy}{dx} + Py = Q$.

where P and Q are functions of x (or constants) is called a linear equation of the first order.

Suppose R is an integrating factor, then the left hand side of $R\frac{dy}{dx} + RPy = RQ$ is the differential coefficient of some product. Since the term $R\frac{dy}{dx}$ can only be derived by differentiating Ry, we put.

$$R\frac{dy}{dx} + R\cdot Py = \frac{d}{dx}(Ry) = R\frac{dy}{dx} + y\frac{dR}{dx}$$

i.e., $$RP = \frac{dR}{dx}$$

or $$\log R = \int P\,dx, \text{ when } R = e^{\int P dx}$$

Hence, we have determined the integrating factor, R and so we have the rule.

To solve $\frac{dy}{dx} + Py = Q$, multiply both sides by $e^{\int P dx}$ which is an integrating factor and then we get the integral as

$$y\cdot e^{\int P dx} = \int Q\cdot e^{\int P dx}\cdot dx$$

Linear Ordinary Differential Equations of Higher Order with Coefficients

An ordinary linear differential equation of n^{th} order has the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_{n-1}\frac{dy}{dx} + P_n y = 0 \qquad \ldots(1)$$

where $P_1, P_2, \ldots P_n$ are functions of x or contains and do not contain y or derivatives of y.

If $P_1, P_2, \ldots P_n$ are constants and Q is any function of x, the equation of the form given by (1) are called ordinary differential equation with constant coefficients.

Theorem—If $y = f(x)$ is the general solution of

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_n y = 0 \qquad \ldots(2)$$

and $y = \phi(x)$ is the solution of equation (1), then

$$y = f(x) + \phi(x)$$

is the general solution of

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_{n-1}\frac{dy}{dx} + P_n y = Q \qquad \ldots(3)$$

Substituting the values of y from (3) in (1), we get the left hand side of (1),

$$= \left(\frac{d^n f}{dx^n} + P_1 \frac{d^{n-1}f}{dx^{n-1}} + \ldots + P_n f\right) + \left(\frac{d^n \phi}{dx^n} + P_1 \frac{d^{n-1}\phi}{dx^{n-1}} + \ldots + P_n \phi\right)$$

If the coefficient of $\frac{d^n y}{dx^n}$ is not unity, is can always be made unity by dividing the equation by its, coefficient. Thus (1) is the most general form.

Now, since $y = f(x)$ is a solution of (2), the first group of terms within the brackets is zero. Similarly the second group of terms is equal to Q, since $y = \phi(x)$ is a solution of (1).

Hence (3) is a solution of (1).

This theorem enables us to divide the method of solving a linear equation into two parts.

First we find the general solution of the equation (2) which we will denote, say, by

$$y = f_1(c_1, c_2, \ldots\ldots c_n, x)$$

Next, we find a solution of (1) which does not contain an arbitrary constant. Let us denote this solution, say, by

$$y = f_2(x)$$

$$\therefore \qquad y = f_1(c_1, c_2, \ldots\ldots cn_x) + f_2(x)$$

gives evidently, the general solution of equation (1)

Linear Second Order Ordinary Differential Equations with Variable Coefficients—Linear equations of second order of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \ldots(1)$$

where P, Q and R are functions of x.

If an integral included in the complementary function of a second order linear differential equation be known its order can be depressed and the complete solution can be found.

Let $y = u$ be an integral in the complementary function of (1). Put $y = uv$, so that $y_1 = u_1 v + v_1 u$ and $y_2 = u_2 v + 2u_1 v_1 + uv_1$.

Substituting the values of y, y_1 and y_2 in (1), we get

$$uv_2 + (2u_1 + Pu)\, v_1 + (u_2 + Pu_1 + Qu)\, v = R \qquad \ldots(2)$$

The coefficient of v varnishes since $y = u$ is a solution of

$$y_2 + Py_1 + Qy = 0$$

Thus, (2) becomes

$$v_2 + \left(2\frac{u_1}{u} + P\right) v_1 = \frac{R}{u}$$

This is a linear equation in v_1. Hence v_1 can be determined; and then

$y = u\int v_1 dx + c_1 u$ is the integral of equation (1), the other constant of integration will occur in expression for v_1. Thus we get the complete primitive.

Change of Independent Variable—Let the linear equation of second order be

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P, Q and R are functions of x and let the independent variable be changed from x to z, z being a given function of x.

$$\text{Since} \quad \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$\text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)\frac{dz}{dx}$$

$$= \frac{d^2y}{dx^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\cdot\frac{d^2z}{dx^2}$$

the original equation becomes.

$$\left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dx^2} + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = R$$

or $$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad \ldots(1)$$

where $$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2}$$

and $$R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2}$$

P_1, Q_1, R_1 are functions of x as shown above but can be readily expressed as functions of z by the given relation between z and x.

If by equating $\dfrac{Q}{\left(\dfrac{dz}{dx}\right)^2}$ to a constant quantity we find that P, also becomes constant then equation (2) is at once integrable. Since z is quite arbitrary. It may therefore be chosen to satisfy an assignable condition. Thus, we may choose z to make the coefficient of $\dfrac{dy}{dz}$ vanish hence if we put

$P_1 = 0$ or $\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx} = 0$ *i.e.*, $z = \int e^{-\int P dx}\, dx$.

Above is the required relation between z and x which will make $P_1 = 0$ and the equation reduce to the form

$$\frac{d^2y}{dz^2} + Q_1 y = R_1$$

If the value of Q_1, comes out to be a constant or a constant divided by z^2, then equation (1) becomes integrable.

If we choose z such that $Q_1 = a^2$, then

$$a\frac{dz}{dx} = \sqrt{Q}$$

$$\therefore \qquad az = \int\sqrt{Q}dx$$

Now the equation will be reduced to

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + a^2 y = R$$

If P_1 comes out to be a constant, then the above equation can be easily integrated.

Differential Equation with Variable Coefficients

Consider the differential equation of the type

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \ldots(1)$$

where R, S, T are continuous function of x and y possessing continuous partial derivatives and

$$r = \frac{\delta^2 z}{\delta x^2},\ s = \frac{\delta^2 z}{\delta y \delta x},\ t = \frac{\delta^2 z}{\delta y^2}$$

Sometimes r may be written as u_{zz}, s as u_{yx}, t as u_{yy}.

Classification

If $S^2 - 4RT > 0$, the differential equation of this type is hyperbolic.

If $S^2 - 4RT = 0$, the differential equation of this types is parabolic .

If $S^2 - 4RT < 0$, the differential equation of this type is elliptic.

Power Series Method—The power series method is the method for solving linear differential equations with variable coefficients.

Power Series— $\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots$

where a_0, a_1 ... are constants (coefficients) and x_0 a constant called centre of the series.

Power Series in Power of x— $\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots\ldots$

Real Analytic Function—A real analytic function $f(x)$ is called analytic at a point $x = x_0$ if it can be represented by a power series of $x - x_0$ with radius of convergence.

$$R = \frac{1}{\lim\limits_{m \to \infty} \sqrt[m]{|a_m|}} > 0$$

Existence of Power Series Solution—If p, q, r in differential equation $y'' + p(x)\, y' + q(x)\, y = r(x)$ are analytic at $x = x_0$, then every solution of differential equation is analytic at $x = x_0$ and can be represented by power series in power of $x - x_0$ with radius of convergence $R > 0$.

Legendre's Equation and Polynomials

Legendreis equation—

$$(1-x^2)\,y'' - 2xy' + n\,(n+1)\,y = 0$$

where n is a parameter and a real number.

The solution of Legendre's equation is called Legendre function.

Legendre's polynomial (of degree n)—

$$\mathrm{P}_n(x) = \sum_{m=0}^{\mathrm{M}} (-1)^m \frac{\underline{|2n-2m}}{2^n\,\underline{|m}\,\underline{|n-m}\,\underline{|m-2m}}\, x^{n-2m}$$

$$= \frac{\underline{|2n}}{2^n\left(\underline{|n}\right)^2} - \frac{\underline{|2n-2}}{2^n\,\underline{|1}\,\underline{|n-1}\,\underline{|n-2}}\, x^{n-2} + \ldots$$

where $\mathrm{M} = \dfrac{n}{2}$ (n is even) or $\mathrm{M} = \dfrac{n+1}{2}$ (n is odd)

Rodriques formula—

$$\mathrm{P}_n(x) = \frac{1}{2^n\,\underline{|n}}\,\frac{dx}{dx^n}\,[(x^2-1)^n]$$

Recurrence relations for Legendre polynomials—

$$(2n+1)\,x\,\mathrm{P}_n(x) = (n+1)\,\mathrm{P}_{n+1}(x) + n\mathrm{P}_{n-1}(x)$$

$$n\mathrm{P}_n = x\mathrm{P}'_n - \mathrm{P}'_{n-1}$$

$$(2n+1)\,\mathrm{P}_n = \mathrm{P}'_{n+1} - \mathrm{P}'_{n-1}$$

$$(n+1)\,\mathrm{P}_n = \mathrm{P}'_{n+1} - x\mathrm{P}'_n$$

$$(1-x^2)\,\mathrm{P}'_n = n\,(\mathrm{P}_{n-1} - x\mathrm{P}_n)$$

$$(1-x^2)\,\mathrm{P}'_n = (x+1)\,(x\mathrm{P}_n - \mathrm{P}_{n+1})$$

Laplace first integral for $\mathrm{P}_n(x)$—

$$\mathrm{P}_n(x) = \frac{1}{\pi}\int_0^{\pi} \{x \pm \sqrt{(x^2-1)}\cos\phi\}^n\, d\phi$$

Laplace second integral for $\mathrm{P}_n(x)$—

$$\mathrm{P}_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\{x \pm \sqrt{(x^2-1)}\cos\phi\}^{n+1}}$$

Generating function for $\mathrm{P}_n(x)$—

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n\,\mathrm{P}_n(x).$$

Bessel's Equations and Bessel's Functions

Bessel's differential equation—

$$x^2y'' + xy' + (x^2-n^2)\,y = 0$$

where n is a parameter and real number.

Bessel functions of first kind (of order n)—

$$\mathrm{J}_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m\, x^{2m}}{2^{2m+n}\,\underline{|m}\,\underline{|m+n}}$$

General Solution of Bessel's Equation—If n is not an integer, the general solution of Bessel's equation for all $x \neq 0$ is $y(x)$ $= c_1\,\mathrm{J}_n(x) + c_2\,\mathrm{J}_{-n}(x)$

Linear Dependence of Bessel Function—For integer n, the Bessel function $\mathrm{J}_n(x)$ and $\mathrm{J}_{-n}(x)$ are linearly dependent because.

$$\mathrm{J}_{-n}(x) = (-1)^n\,\mathrm{J}_n(x).$$

Recurrence relations for $\mathrm{J}_n(x)$—

$$\frac{d}{dx}\{x^n\,\mathrm{J}_n(x)\} = x^n\,\mathrm{J}_{n-1}(x)$$

$$\frac{d}{dx}\{x^{-n}\,\mathrm{J}_n(x)\} = -x^{-n}\,\mathrm{J}_{n+1}(x)$$

$$\mathrm{J}_{n-1}(x) + \mathrm{J}_{n+1}(x) = \frac{2n}{x}\,\mathrm{J}_n(x)$$

$$\mathrm{J}_{n-1}(x) - \mathrm{J}_{n+1}(x) = 2\mathrm{J}'_n(x)$$

$\mathrm{J}_n(x)$ for $n = \pm\frac{1}{2}$—

$$\mathrm{J}_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x$$

$$\mathrm{J}_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\cos x$$

Elementary Bessel's Functions—Bessel functions $\mathrm{J}_n(x)$, $n = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$ are elementary functions and can be expressed by finitely many sine and cosines and power of x.

Generating Functions of $\mathrm{J}_n(x)$—

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} t^n\,\mathrm{J}_n(x)$$

Some Solved Examples

Example 1. Show that $x\mathrm{L}_n''(x) + (1-x)\,\mathrm{L}_n'(x) + n\mathrm{L}_n(x) = 0$.

Solution : Since $\mathrm{L}_n(x)$ satisfies Laquerre's equation

$$xy'' + (1-x)\,y' + ny = 0$$

$$\Rightarrow x\mathrm{L}_n''(x) + (1-x)\,\mathrm{L}_n'(x) + n\mathrm{L}_n(x) = 0$$

Example 2. Show that $L_n'(0) = -n\lfloor n$

Solution : Since

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{\lfloor n} t^n = \frac{1}{1-t} e^{-\frac{xt}{1-t}}$$

Differentiating with respect to x, we have

$$\sum_{n=0}^{\infty} \frac{L_n(x)}{\lfloor n} t^n = -\frac{1}{(1-t)^2} e^{-\frac{xt}{1-t}}$$

For $x = 0$,

$$\sum_{n=0}^{\infty} \frac{L_n'(0)}{\lfloor n} t^n = -\frac{1}{(1-t)^2} = -t(1-t)^{-2}$$

$$= -t(1 + 2t + 3t^2 + \dots nt^{n-1} + \dots)$$

$$= -\sum_{n=1}^{\infty} nt^n$$

Equating the coefficient of t^n,

$$\frac{L_n'(0)}{\lfloor n} = -n$$

$$\Rightarrow \quad Ln'(0) = -n\lfloor n$$

Example 3. Classify and solve the following :

$u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0.$

Solution : Comparing with the general second order differential equation, we have

$A = 1$, $B = -2\sin x$, $C = -\cos^2 x$, $D = 0$, $E = -\cos x$, $F = 0$, $G = 0$. The discriminant $B^2 - 4AC = 4(\sin^2 x + \cos^2 x) = 4 > 0$

Hence, the given partial differential equation is hyperbolic.

The characteristic equation are

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A}$$

$$= -\sin x - 1$$

and

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A}$$

$$= 1 - \sin x$$

$$\Rightarrow \quad y = \cos x - x + c_1$$

and

$$\mu = \cos x + x + c_2$$

$$\Rightarrow \quad \zeta = x + y - \cos x = c_1$$

$$\eta = -x + y - \cos x = c_2$$

Integrating with respect to ζ, we obtain $\mu_\eta = f(\eta)$

Where f is arbitrary. Integrating once again with respect to η, we have

$$\mu = \int f(\eta)\, d\eta + g(\zeta)$$

$$\Rightarrow \quad \mu = \psi(\eta) + g(\zeta)$$

where $g(\zeta)$ is and arbitrary function. Returning to the old variables x, y, the solution of the given partial differential equation is

$$\mu(x, y) = \psi(y - x - \cos x) + g(y + x - \cos x)$$

Example 4. Show that

$$\int_0^x x^{-n} J_{n+1}(x)\, dx = \frac{1}{2^n \lceil (x+1)} - x^n J_n(x), \; n > 1$$

Solution : Since

$$\frac{d}{dx}\{x^{-n} J_n(x)\} = -x^{-n} J_{n+1}(x)$$

Integrating, we get

$$[x^{-n} J_n(x)]_0^x = \int_0^x x^{-n} J_{n+1}(x)\, dx$$

$$\Rightarrow \int_0^x x^{-n} J_{n+1}(x)\, dx$$

$$= -x^{-n} J_n(x) - \lim_{x \to 0} \left[\frac{J_n(x)}{x^n}\right]$$

$$= -x^n J_n(x) - \lim_{x \to 0} \frac{1}{x^n} \left[\frac{x^n}{2^n \lceil n+1} \left\{1 - \frac{x^2}{2 \cdot 2(n+1)} + \dots\right\}\right]$$

$$= -x^{-n} J_n(x) - \frac{1}{2^n \lceil (n+1)}$$

Example 5. Prove that : $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$.

Solution : Since

$$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$

$$\Rightarrow \quad J_{n+1}(x) = \frac{1}{x}[2n J_n(x) - x J_{n-1}(x)] \quad \dots(1)$$

For $n = 1, 2, 3$, we have

$$J_2(x) = \frac{1}{x}[2J_1(x) - xJ_0(x)] \quad \dots(2)$$

$$J_3(x) = \frac{1}{x}[4J_2(x) - xJ_1(x)] \quad \dots(3)$$

$$J_4(x) = \frac{1}{x}[6J_3(x) - x J_2(x)] \quad \dots(4)$$

By (2) and (3)

$$J_3(x) = \frac{8}{x^2} J_1(x) - \frac{4}{x} J_0(x) - J_1(x)$$

$$= \frac{8-x^2}{x^2} J_1(x) - \frac{4}{x} J_0(x) \quad \ldots(5)$$

By (4) and (5), we have $J_4(x) = \frac{48-6x^2}{x^3} J_1(x)$

$$-\frac{24}{x^2} J_0(x) - \frac{2}{x} J_1(x) + J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

Example 6. Show that

(i) $(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$

(ii) $P_n(x) = x P_n'(x) - P'_{n-1}(x)$

(iii) $(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n+1}(x)$.

Solution :

(i) Since $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n$.

Differentiating partially with respect to t, we get,

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum n P_n(x) t^{n-1}$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2) \sum n P_n(x) t^{n-1}$$

Equating coefficients of t^n from both sides we get,

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2nx P_n(x) + (n-1) P_{n-1}(x)$$

$$\Rightarrow (n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$$

(ii) Since $(1-2xt+t^2)^{-1/2}$

$$= \sum_{n=0}^{\infty} P_n(x) t^n \quad \ldots(1)$$

Differentiating partially with respect to x

$$\Rightarrow -\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2t) = \sum P_n'(x) t^n$$

$$\Rightarrow t(1-2xt+t^2)^{-3/2} = \sum P_n'(x) t^n \quad \ldots(2)$$

Again differentiating (1) partially with respect to t, we have

$$\Rightarrow (x-t)(1-2xt+t^2)^{-3/2} = \sum n P_n(x) t^{n-1} \quad \ldots(3)$$

Dividing (3) by (2), we get

$$\frac{x-t}{t} = \frac{\sum n P_n(x) t^{n-1}}{\sum n P_n'(x) t^n}$$

$$\Rightarrow \sum n P_n(x) t^n = (x-t) \sum P_n'(x) t^n$$

Equating coefficients of t^n from both sides, we get

$$n P_n(x) = x P_n'(x) - P'_{n-1}(x)$$

(iii) Since $(1-2xt+t^2)^{-1/2}$

$$= \sum_{n=0}^{\infty} P_n(x) t^n \quad \ldots(1)$$

Differentiating with respect to x, we get

$$(n+1) P'_{n+1}(x) = (2n+1) P_n(x) + (2n+1) x P'_n(x) - n P'_{n-1}(x) \quad \ldots(2)$$

From (1) and (2), we get

$$(n+1) P'_{n+1}(x) = (2n+1) P_n(x) + (2n+1)[n P'_n(x) + P'_{n-1}(x)] - n P'_{n-1}(x)$$

or $(2n+1) P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

Example 7. Prove that $\int x J_0^2(x)\, dx = \frac{1}{2} x^2 [J_0^2(x) + J_1^2(x)]$.

Solution : Here

$\int x J_0^2(x)\, dx = \int J_0^2(x)\, dx \cdot x$ [integrate by parts]

$$= J_0^2(x) \cdot \frac{1}{2} x^2 - \int 2 J_0(x) J_0'(x) \cdot \frac{1}{2} x^2\, dx$$

$$= \frac{1}{2} x^2 J_0^2(x) + \int x^2 J_0(x) J_1(x)\, dx$$

$$= \frac{1}{2} x^2 J_0^2(x) + \int x J_1(x)\, d/dx\, [x J_1(x)]\, dx$$

$$\left[\because \frac{d}{dx}[x J_1(x)] = x J_0(x)\right]$$

$$= \frac{1}{2} x^2 J_0^2(x) + \frac{1}{2}[x J_1(x)]^2$$

$$= \frac{1}{2} x^2 [J_0^2(x) + J_1^2(x)]$$

Example 8. Show that $\int J_3(x)\, dx = c - J_2(x) - \frac{2}{x} J_1(x)$.

Solution : Since $\frac{d}{dx}[x^{-n} J_n(x)]$

$$= -x^{-n} J_{n+1}(x) \quad \ldots(1)$$

$$\Rightarrow \int x^{-n} J_{n+1}(x)\, dx = -x^{-n} J_n(x) \quad \ldots(2)$$

$$\therefore \int J_3(x)\, dx = \int x^2 \cdot x^{-2} J_3(x)\, dx + c$$

$$= x^2 \int x^{-2} J_3(x)\, dx - \int 2x \left[\int x^{-2} J_3(x)\, dx\right] dx + c$$

$$= x^2 [-x^{-2} J_2(x)] - \int 2x [-x^{-2} J_2(x)]\, dx + c$$

[by (2) when $n = 2$]

$$= c - J_2(x) + \int \frac{2}{x} J_2(x)\,dx$$

$$= c - J_2(x) - \frac{2}{x} J_1(x) \text{ [by (2) when } n = 1]$$

Example 9. Are the following linear independent $\sin x$, $\cos x$, $\sin 2x$?

Solution : The Wronskion of the given variable

$$w(x) = \begin{vmatrix} \sin x & \cos x & \sin 2x \\ \cos x & -\sin x & 2\cos 2x \\ -\sin x & -\cos x & -4\sin 2x \end{vmatrix}$$

$$= \sin x \begin{vmatrix} -\sin x & 2\cos 2x \\ -\cos x & -4\sin 2x \end{vmatrix}$$

$$-\cos x \begin{vmatrix} \cos x & \sin 2x \\ -\cos x & -4\sin 2x \end{vmatrix}$$

$$-\sin x \begin{vmatrix} \cos x & \sin 2x \\ -\sin x & -2\cos 2x \end{vmatrix}$$

$$= \sin x\,(4\sin 2x \sin x + 2\cos 2x \cos x)$$
$$- \cos x\,(-4\sin 2x \cos x + \sin 2x \cos x)$$
$$- \sin x\,(2\cos x \cos 2x + \sin x \sin 2x)$$
$$= 3\sin^2 x \sin 2x + 3\cos^2 x \sin 2x$$
$$+ 2\sin x \cos x \cos 2x - 2\sin 2x \cos x \cos 2x$$
$$= 3(\sin^2 x + \cos^2 x)\sin 2x$$
$$+ \sin 2x \cos 2x - \sin 2x \cos 2x$$
$$= 3\sin 2x \neq 0$$

Since the Wronskion is not zero; they are linear independent.

Example 10. Show that $(2n+1)(x^2-1)$ $P_n'(x)$

$$= n(n+1)\{P_{n+1}(x) - P_{n-1}(x)].$$

Solution : Since

$$(x^2-1)P_n'(x) = n(x P_n(x) - P_{n-1}(x)) \quad \ldots(1)$$

$$\text{and } (x^2-1)P_n'(x) = (n+1)\{P_{n+1}(x) - x P_n(x)\} \quad \ldots(2)$$

Eliminating $xP_n(x)$ from (1) and (2), we get

$$(x^2-1)P'_n(x) = n\left\{P_{n+1}(x) - \frac{x^2-1}{n+1}P'_n(x) - P_{n-1}(x)\right\}$$

$$\text{or } \left\{x^2 - 1 + n\frac{x^2-1}{n+1}\right\} P_n'(x)$$
$$= n\{P_{n+1}(x) - P_{n-1}(x)\}$$

$$\text{or } (2n+1)(x^2-1)P_n'(x)$$
$$= n(n+1)\{P_{n+1}(x) - P_{n-1}(x)\}$$

Example 11. Show that $\int_{-1}^{1}(x^2-1)P_{n+1}(x)$ $P_n'(x)\,dx = \frac{2n(n+1)}{(2n+1)(2n+3)}$.

Solution : Since

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \quad \ldots(1)$$

$$\text{and } (x^2-1)P_n'(x) = (n+1)[P_{n+1}(x) - xP_n] \quad \ldots(2)$$

Multiplying (2) by $P_{n+1}(x)$ and integrating with respect to x, between the limit -1 and to $+1$, we get

$$\int_{-1}^{1}(x^2-1)P_{n+1}(x)P'_n(x)\,dx$$

$$= (n+1)\int_{-1}^{1}P_{n+1}(x)[P_{n+1}(x) - xP_n(x)]dx$$

$$= (n+1)\int_{-1}^{1}P^2_{n+1}(x)\,dx - (n+1)\int_{-1}^{1}P_{n+1}(x)$$

$$\left\{\frac{(n+1)P_{n+1}(x) - nP_{n-1}(x)}{2n+1}\right\}dx \text{ (from (1))}$$

$$= (n+1)\frac{2}{2n+3} - \frac{(n+1)^2}{2n+1}\cdot\frac{2}{2n+3}$$

$$= \frac{2n(n+1)}{(2n+1)(2n+3)}$$

Example 12. Show that $f(x, y) = xy^2$ satisfies the Lipschitz condition on the rectangle $|x| \le 1$, $|y| \le 1$ but does not satisfy a Lipschitz condition on the strip $|x| \le 1$, $|y| < \infty$.

Solution : $f(x, y) = xy^2$ and $\left|\frac{\delta}{\delta y}f(x,y)\right| = |2xy| \le 2$ for (x, y) on rectangle $|x| \le 1$, $|y| \le 1$.

$\therefore$ $\frac{\delta f}{\delta y}$ exists and is continuous and bounded in the rectangle $|x| \le 1$, $|y| \le 1$.

Thus Lipschitz condition is satisfied on $|x| \le 1$, $|y| \le 1$ on strip $|x| \le 1$, $|y| \le \infty$, there does not exists any positive constant $K > 0$, such that

$$\left|\frac{\delta}{\delta y}f(x,y)\right| = |2xy| \le K$$

Hence, Lipschitz condition not satisfied.

Examples 13. Prove that continuous function $f(x, y) = y^{2/3}$ on rectangle $|x| \le 1$, $|y| \le 1$ does not satisfies Lipschitz condition on a rectangle.

Solution :

Here $f(x, y) = x^{2/3}$ is continuous but $\left|\frac{\delta}{\delta y} f(x, y)\right|$

$= \left|\frac{2}{3y^{1/3}}\right| \neq K > 0$

for $y = 0$ which is a point of the rectangle.

Hence, Lipschitz condition is not satisfied on the rectangle.

Example 14. Show that

$$\int_0^a rJ_0(xr)\,dr = \frac{a}{x}J_1(ax)$$

Solution : $\int_0^a rJ_0(xr)\,dr$

$$= \int_0^a r\left[1 - \frac{(xr)^2}{2^2} + \frac{(xr)^4}{2^2 4^2} - \frac{(xr)^6}{2^2.4^2.6^2} + \ldots\right] dr$$

$$= \int_0^a \left[r + \frac{x^2}{2^2}r^3 + \frac{x^4}{2^2.4^2}r^5 + \frac{x^6}{2^2.4^2.6^2}r^7 + \ldots\right] dr$$

$$= \left[\frac{x^2}{2} - \frac{x^2r^4}{2^2.4} + \frac{x^4r^6}{2^2.4^2.6} - \frac{x^6r^8}{2^2.4^2.6^2.8} + \ldots\right]_0^a$$

$$= \frac{a}{x}\left[\frac{ax}{2} - \frac{(ax)^3}{2^2.4} + \frac{(ax)^5}{2^2.4^2.6} + \ldots\right]$$

$$= \frac{a}{x}J_1(ax)$$

Example 15. Prove that $4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$.

Solution : Since

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Differentiating it with respect to x, we get

$$2J_n''(x) = J'_{n-1}(x) - J'_{n+1}(x)$$

or
$$4J''_n(x) = 2J'_{n-1}(x) - 2J'_{n+1}(x)$$
$$= [J_{n-2}(x) - J_n(x)] - [J_n(x) - J_{n+2}(x)]$$
$$= J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$$

Example 16. Show that if δ is either a rectangle $|x - x_0| \leq a$, $|y - y_0| \leq b$ $(a, b > 0)$ or strip $|x - x_0| \leq a$, $|y| < \infty$ $(a > 0)$ and if f is real valued continonus function defined on δ and $\frac{\delta f}{\delta y}$ exists and also.

Also,

$\left|\frac{\delta}{\delta y} f(x, y)\right| \leq K$, $(x, y) \sum \delta$ for a positive constant K, then f satisfies Lipschitz condition on δ with Lipschitz constant K.

Solution : By fundamental theorem of calculus

$$f(x, y_2) - f(x, y) = \int_{y_1}^{y_2} \frac{\delta}{\delta y} f(x, y)\,dy$$

$$\Rightarrow \quad |f(x, y_2) - f(x, y) =$$

$$\left|\int_{y_1}^{y_2} \frac{\delta}{\delta y} f(x, y)\,dy\right| \leq \int_{y_1}^{y_2} \left|\frac{\delta}{\delta y} f(x, y)\right| |dy|$$

$$\leq K \int_{y_1}^{y_2} |dy|$$

$\Rightarrow \quad |f(x, y_2) - f(x, y_1) \leq K\,|y_2 - y_1)$

for all $(x, y_1), (x, y_2) \sum \delta$

$\Rightarrow f(x, y)$ satisfy Lipschitz condition on δ, where K is Lipschitz constant.

Example 17. Show that : $\int_{-1}^{1}(1 - x^2)[P'_n(x)]^2\,dx = \frac{2n(n+1)}{2n+1}$.

Solution : $\int_{-1}^{1}(1 - x^2)[P'_n(x)]^2\,dx$

$$= \int_{-1}^{1}[(1 - x^2)P'n(x)]P'_n(x)\,dx$$

$$= [(1 - x^2)P'_n(x)P_n(x)]_1^1 - \int_{-1}^{1}\frac{d}{dx}[(1 - x^2)P'_n(x)]P_n(x)\,dx$$

$$= -\int_{-1}^{1}P_n(x)\frac{d}{dx}[(1 - x^2)P'_n(x)]\,dx$$

$$= -\int_{-1}^{1}P_n(x)[-n(n + 1)P_n(x)]\,dx$$

Since $\frac{d}{dx}[(1 - x^2)P'_n(x)] + n(n + 1)P_n(x) = 0$

$$= n(n + 1)\int_{-1}^{1}P^2_n(x)\,dx$$

$$= n(n + 1)\cdot\frac{2}{2n + 1}$$

$$= \frac{2n(n + 1)}{2n + 1}$$

Example 18. Prove that

$$(1 - x^2)P'_n(x) = \frac{n(n+1)}{2n+1}[P_{n-1}(x) - P_{n+1}(x)].$$

Solution : Since

$$(1 - x^2)P'n(x) = n[P_{n-1}(x) - x\cdot P_n(x)] \quad \ldots(a)$$

$$x\,P_n(x) = \frac{1}{2n+1}\left[(n+1)\,P_{n+1}(x) + nP_{n-1}(x)\right] \quad \text{...(b)}$$

By (a) and (b), eliminating $P_n(x)$, we get

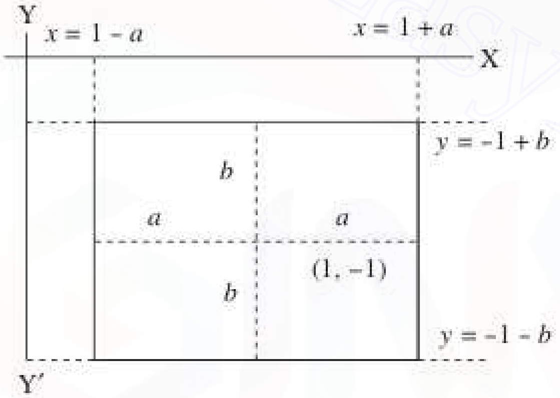
$$(1-x^2)\cdot P'_n(x) = n\cdot \left(P_{n-1}\left\{x\right) - \frac{1}{2n+1}\left[(n+1)\,P_{n+1}(x) + n\,P_{n-1}(x)\right]\right\}$$
$$= \frac{n\,(n+1)}{2n+1}\left[P_{n-1}(x) - P_{n+1}(x)\right\}$$

Example 19. Show the existence and a uniqueness of solution for the initial value problem $\frac{dy}{dx} = y^2$, $y(1) = -1$.

Solution : The existence—Given,

$$f(x, y) = \frac{dy}{dx} = y^2 \text{ and } y(1) = 1.$$

Here $\frac{\delta f}{\delta y} = 2y$, so f, and $\frac{\delta f}{\delta y}$ are both continuous for all (x, y) consider the rectangle R : $|x-1| < a$, $|y+1| \le b$, about $(1, -1)$



Then,

$|f(x, y_2) - f(x, y_1)| = |y_2^2 - y_1^2| = |y_2 + y_1|\,|y_2 - y_1| \le (2+2b)\,|y_2 - y_1|$ and Lipschitz condition is satisfied in the rectangle R.

Thus by existence theorem, there exist atleast one solution.

To uniqueness—Now let

M = $max\,|f(x, y)|$ for $x, y \in$ R and $m = \min(a, b/M)$

M = $max\,|f(x, y)| = \max|y^2| = |(-1-b^2)| = (1+b^2)$

and $m = \min(a, b/M) = \min\left(a, \frac{b}{(1+b)^2}\right)$

For maximum or minimum of

$$\frac{b}{(1+b)^2} = \phi(b)$$

$$\phi'(b) = \frac{1-b}{(1+b)^3}$$

and $$\phi''(b) = \frac{2b-4}{(1+b)^4}$$

$\Rightarrow$ when $b = 1$, $\phi'(1) = 0$ and $\phi''(1) = \frac{-2}{2^4} < 0$

$\therefore$ $\phi(b)$ is max. at $a + b = 1$ and $\phi(1) = \frac{1}{4}$

$\therefore$ If $a < \frac{1}{4}$, $m < \frac{1}{4}$ if $a \ge \frac{1}{4}$, $m = \frac{1}{4}$

and thus $m = \min\left\{a, \frac{b}{M}\right\} = \min\left\{a, \frac{1}{4}\right\}$
$= \frac{1}{4}$ for $b = 1$

Hence, the given problem possesses a unique solution when $|x - 1| < \frac{1}{4}$.

Example 20. Show that :

$$J_n(x) = (-2)^n x^n \left[\frac{d}{d(x)^2}\right]^n J_0(x).$$

Solution : Since

$$\frac{d}{dx}\left[x^{-n}J_n(x)\right] = x^{-n}J_{n+1}(x) \quad \text{...(1)}$$

When $n = 0$

$$\frac{d}{dx}\left[J_0(x)\right. = -J_1(x)$$

or $$\frac{1}{x}\frac{d}{dx}\left[J_0(x)\right] = -x^{-1}J_1(x) \quad \text{...(2)}$$

Differentiating with respect to x, we have

$$\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}J_0(x)\right) = -\frac{d}{dx}\left[x^{-1}J_1(x)\right]$$

or $$\frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}J_0(x)\right) = (-1)^2 x^{-2}J_2(x)$$

or $$\left(\frac{1}{x}\frac{d}{dx}\right)^2 J_0(x) = (-1)^2 x^{-2}J_2(x)$$

and so $$\left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x) = (-1)^n x^{-n}J_n(x)$$

$$\Rightarrow \quad J_n(x) = (-1)^n\cdot x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)$$
$$= (-1)^n x^n\left(x\cdot\frac{d}{d(x^2)}\right)^n J_0(x)$$
$$= (-2)^2 x^n\left(\frac{d}{d(x^2)}\right)^n J_0(x)$$

i.e., $$J_n(x) = (-2)^n x^n\left(\frac{d}{d(x^2)}\right)^n J_0(x)$$

Example 21. Find the solution of the one dimensional diffusion satisfying the following boundary conditions.

(i) T is bounded at $t \to \infty$

(ii) $\left.\dfrac{\delta T}{\delta x}\right|_{x=0} = 0$ for all t

(iii) $\left.\dfrac{\delta T}{\delta x}\right|_{x=0} = 0$ for all t

(iv) $T(x, 0) = x(a - x), 0 < x < a$

Solution : The possible solution is $T(x, t) = \exp(-a\lambda^2 t)(A \cos \lambda x + B \sin \lambda x)$ and $\dfrac{\delta T}{\delta x}$ $= \exp(-a\lambda^2 t)(-A\lambda \sin \lambda x + B\lambda \cos \lambda x)$ ... (1)

Using boundary conditions (ii), equation (1) gives B = 0 and boundary conditions (iii) into equation (1) gives.

$\sin \lambda a = 0$ implying $\lambda a = n\pi$, $\pi = 0\ 1, 2, \ldots$

By the principle of superposition, we have

$$T(x, t) = \in A_n \exp(-a\lambda^2 t) \cos \lambda x$$

$$= \sum_{n=0}^{\infty} A_n \exp\left[-a\left(\frac{n\pi}{a}\right)^2 t\right] \cos\left(\frac{n\pi}{a}\right) x$$

The boundary conditions (iv) gives

$$T(x, 0) = x(a - x) = A_0 + \sum_{n=1}^{\infty} A_n \exp.\left[-a\left(\frac{n\pi}{a}\right)^2 t\right] \cos\left(\frac{n\pi}{a}\right) x$$

where

$$A_0 = \frac{2}{a}\int_0^a (ax - x^2)\,dx = \frac{a^2}{6}$$

$$A_n = \frac{2}{a}\int_0^a (ax - x^2)\cos\left(\frac{n\pi}{a}x\right) dx$$

$$= \frac{2a^2}{n^2\pi^2}(1 + \cos n\pi)$$

$$= \frac{2a^2}{n^2\pi^2}(1 + (-1)^n]$$

$$\therefore \quad A_n = \begin{cases} \dfrac{-4a^2}{n^2\pi^2}, & \text{for } n \text{ even} \\ 0, & \text{for } n \text{ odd} \end{cases}$$

The required solutions is

$$T(x, t) = \frac{a^2}{6} - \frac{4a^2}{\pi^2}\sum_{n=2,4,\ldots\text{even}}^{\infty} \frac{1}{n^2} \cos\left(\frac{n\pi}{a}\right) x \exp.\left[-\alpha\left(\frac{n\pi}{a}\right)^2 t\right]$$

Example 22. Show that :

(i) $P'_n(1) = \dfrac{1}{2} n(n + 1)$

(ii) $P'_n(-1) = (-1)^{n-1}\dfrac{1}{2} n(n + 1)$

Solution : We know that $P_n(x)$ is Legendre's function of first kind of the Legender's equation.

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0$$

$\therefore (1 - x^2)P''_n(x) - 2xP'_n(x) + n(n+1)P_n(x) = 0$ ... (1)

(i) Putting $x = 1$ in (1), we get, $0 - 2P'_n(1) + n(n + 1)P_n(1) = 0$

or $P'n(1) = \dfrac{1}{2} n(n + 1)P_n(1) = \dfrac{1}{2} n(n + 1)$

$[\because P_n(1) = 1]$

(ii) Putting $x = -1$ in (1), we get

$0 + 2P'_n(-1) + n(n + 1)P_n(-1) = 0$

or
$$P'_n(-1) = -\frac{1}{2} n(n+1)P_n(-1)$$
$$= -\frac{1}{2} n(n + 1)(-1)^n P_n(1)$$
$$= (-1)^{n-1}\frac{1}{2} n(n + 1)$$

$[\because P_n(-1) = (-1)^2 P_n$ and $P_n(1) = 1]$

Example 23. Find the solution of the wave equation $\mu_{tt} = c^2\mu_{xx}$

Given : (i) $\mu(0, t) = \mu(2, t) = 0$

(ii) $\mu(x, 0) = \sin^3 x\pi/2$

(iii) $\mu_t(x, 0) = 0$

Solution : The possible solution is $\mu(x, t) = c_1 \cos \lambda x + c_2 \sin \lambda x)[c_3 \cos(c\lambda t) + c_4 \sin(c\lambda t)]$

Using the condition $\mu(0, t) = 0$, we obtain $c_1 = 0$. Also, condition (iii) $\Rightarrow c_4 = 0$.

The condition $\mu(2, t) = 0$ gives, $\sin 2\lambda = 0$, implying that $\lambda = \dfrac{n\pi}{2}$, $n = 1, 2, \ldots$

Thus, the solution is, $\mu(x, t) = \sum_{n=1}^{\infty} A_n \sin\dfrac{n\pi x}{2} \cos\dfrac{n\pi ct}{2}$.

Using condition (ii), we have,

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{2} = \sin^3\frac{\pi x}{2} = \frac{3}{4}\sin\frac{\pi x}{2} - \frac{1}{4}\sin\frac{3\pi x}{2}$$

which gives $A_1 = 3/4$, $A_3 = -1/4$, while all other $A'_n s$ are zero. Hence the required solution is

$$\mu(x, t) = \frac{3}{4}\sin\frac{\pi x}{2}\cos\frac{\pi ct}{2} - \frac{1}{4}\sin\frac{3\pi x}{2}\cos\frac{3\pi ct}{2}$$

Example 24. Prove that

(i) $P_n(1) = 1$ (ii) $P_n(-x) = (-1)^n P_n(x)$.

Solution : (i) Since

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2}$$

Substituting $x = 1$, we have,

$$\begin{aligned}\sum_{n=0}^{\infty} h^n P_n(1) &= (1 - 2h + h^2)^{-1/2} \\ &= (1-h)^{-1} \\ &= 1 + h + h^2 + \ldots \\ &= \sum_{n=0}^{\infty} h^n\end{aligned}$$

Equating the coefficients of h^n, we have $P_n(1) = 1$

(ii) Since $\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2}$

Substituting $x = -x$, we have, $\sum_{n=0}^{\infty} h^n P_n(-x)$

$$= (1 + 2xh + h^2)^{-1/2} \quad \ldots(a)$$

Also replacing h by $-h$, we have

$$\sum_{n=0}^{\infty} (-h)^n P_n(x) = (1 - 2xh + h^2)^{-1/2} \quad \ldots(b)$$

From (a) and (b), we have

$$\sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n h^n P_n(x)$$

Equating the coefficients of h^n, we get

$$P_n(-x) = (-1)^n P_n(x)$$

Example 25. Show that :

$$\frac{d}{dx}[x J_n(x) J_{n+1}(x)] = x[J^2_n(x) - J^2_{n+1}(x)]$$

Solution : Since $\frac{d}{dx}[x J_n(x) J_{n+1}(x)]$

$$= J_n(x) J_{n+1}(x) + x\,[J_n(x) J'_{n+1}(x) + J'_n(x) J_{n+1}(x)] \quad \ldots(1)$$

and since $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

$$\Rightarrow x^{-n} J'_n(x) - nx^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$$

$$\Rightarrow \quad J'_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \quad \ldots(2)$$

Also since

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\Rightarrow x^n J'_n(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$$

Replace n by $n + 1$, we have

$$x^{n+1} J'_{n+1}(x) + (n+1)x^n J_{n+1}(x) = x^{n+1} J'_n(x)$$

$$\Rightarrow \quad J'_{n+1}(x) = J_n(x) - \frac{n+1}{x} J_{n+1}(x) \quad \ldots(3)$$

By (2), (3) and (1) we have

$$\begin{aligned}&\frac{d}{dx}[x J_n(x) J_{n+1}(x)] \\ &= J_n(x) J_{n+1}(x) \\ &\quad + x\left\{J_n(x)\left[J_n(x) - \frac{n+1}{x} J_{n+1}(x)\right] + J_{n+1}\left[\frac{n}{x} J_n(x) - J_{n+1}(x)\right]\right\} \\ &= x \cdot [J^2_n(x) - J^2_{n+1}(x)]\end{aligned}$$

Example 26. Prove that :

(a) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

(b) $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

Solution : (a) Since,

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\underline{|r}\ \overline{|n+r+1}}$$

Multiplying by x^n, we have

$$\begin{aligned}x^n J_n(x) &= x^n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{r!\ \overline{|(n+r+1)}} \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r x^{2(n+r)}}{2^{n+2r} r!\ \overline{|(n+r+1)}}\end{aligned}$$

$$\begin{aligned}\therefore \frac{d}{dx}[x^n J_n(x)] &= \sum_{r=0}^{\infty} \frac{(-1)^r 2(n+r) x^{2(n+r)-1}}{2^{n+2r} r!\ \overline{|(n+r+1)}} \\ &= x^n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n-1+2r}}{r+\ \overline{|(n-1+r+1)}} \\ &= x^n J_{n-1}(x)\end{aligned}$$

(b) Since, $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\underline{|r}\ \overline{|n+r+1}}$

Multiplying by x^{-n}, we have

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{2^{n+2r} r!\ \overline{|(n+r+1)}}$$

$\therefore \dfrac{d}{dx}[x^{-n}J_n(x)]$

$= \sum_{r=0}^{\infty} \dfrac{(-1)^r\, 2rx^{2r-1}}{2^{n+2r}\, r!\, \overline{|(n+r+1)}}$

$= x^{-n} \sum_{r=0}^{\infty} \dfrac{(-1)^{r-1}\, x^{n+1+2(r-1)}}{2^{n+1+2(r-1)}\,(r-1)!\, \overline{|(n+r+1)}}$

$= -x^{-n} \sum_{k=0}^{\infty} \dfrac{(-1)^k (x/2)^{n+1+2k}}{k!\, \overline{|(n+1+k+1)}}$

$= -x^{-n} J_{n+1}(x)$, where $k = r-1$

$\Rightarrow \dfrac{d}{dx}[x^{-n}J_n(x)] = -x^n J_{n+1}(x)$

Example 27. Show that $J_n(x) = \dfrac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$.

Solution : Since, $\dfrac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

$\Rightarrow \quad x^n J_n'(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$

or dividing by x^n, $J_n'(x) + (n/x) J_n(x) = J_{n-1}(x)$ ... (1)

Since, $\dfrac{d}{dx}[x^{-n} J_n(x)] = -x^{-n}J_{n+1}(x)$

$\Rightarrow x^{-n} J_n'(x) - nx^{-n+1} J_n(x)$

$= x^{-n} J_{n+1}(x) - J_n'(x) + \dfrac{n}{x} J_n(x)$

$= J_{n+1}(x)$ ...(2)

Adding (1) and (2), we obtain

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$\Rightarrow \quad J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$$

Example 28. $\int_{-1}^{1} (x^2-1) P_{n+1}(x) P'_n(x)\, dx$

$$= \frac{2n(n+1)}{(2n+1)(2n+3)}$$

Solution : Since, $(x^2-1) P'_n(x) = (n+1)[P_{n+1}(x) - x \cdot P_n(x)]$

Multiplying it by $P_{n+1}(x)$ and integrating, we have,

$\int_{-1}^{1} (x^2-1) P_{n+1}(x) P'_n(x)\, dx$

$= (n+1) \int_{-1}^{1} [P_{n+1}(x) - x P_n(x)] P_{n+1}(x)\, dx$

$= (n+1) \int_{-1}^{1} P^2_{n+1}(x) - (n+1) \int_{-1}^{1} P_{n+1}(x) \left\{ \dfrac{(n+1) P_{n+1}(x) - nP_{n-1}(x)}{(2n+1)} \right\} dx$

$[\because (2n+1)\, x\, P_n(x) = (n+1) P_{n+1}(x) + nP_{n-1}(x)]$

$= (n+1) \dfrac{2}{2n+3} - \dfrac{(n+1)^2}{2n+1} \cdot \dfrac{2}{2n+3}$

$= \dfrac{2n(n+1)}{(2n+1)(2n+3)}$

Example 29. Prove that :

$$\int P_n(x)\, dx + c = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$$

Solution : Since

$$(2n+1) P_n = P'_{n+1}(x) - P'_{n+1}(x)$$

or $\quad P_n(x) = \dfrac{P'_{n+1}(x) - P'_{n-1}(x)}{2n+1}$

Integrating, we have

$$\int P_n(x)\, dx + c = \frac{P_{n+1}(x) - P_{n-1}(x)}{2n+1}$$

Example 30. Given the equation $x = A \cos(Pt - \alpha)$, eliminate A, P and α.

Solution : $\dfrac{dx}{dt} = -PA \sin(Pt - \alpha)$

$\dfrac{d^2x}{dt^2} = -P^2 A \cos(Pt - \alpha)$

$= -P^2 x$

$\dfrac{d^3x}{dt^2} = -P^2 \dfrac{dx}{dt}$

$\Rightarrow \quad \dfrac{\dfrac{d^3x}{dt^3}}{\dfrac{dx}{dt}} = -P^2 = d^2x/dt^2/x$

$\Rightarrow \quad x \cdot \dfrac{d^3x}{dt^3} = \dfrac{dx}{dt} \cdot \dfrac{d^2x}{dt^2}$

Example 31. Solve $\dfrac{dx}{x} = \tan y dy$.

Solution : $\log x = -\log \cos y + c$

$\Rightarrow \quad \log(x \cos y) = c$

$\Rightarrow \quad x \cos y = e^c = a$ (say)

Example 32. Solve $\dfrac{dy}{dx} = 2xy$.

Solution : $\dfrac{dy}{y} = 2x \cdot dx$

$\Rightarrow \quad \log y = x^2 + \log a$

$\Rightarrow \quad y = ae^{x^2}$

Example 33. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$.

Solution : Auxiliary equation

$$m^2 - 6m + 13 = 0$$

$$\Rightarrow \quad m = 3 \pm 2i$$

$\therefore$ General solution is

$$y = Ae^{(3+2i)x} + Be^{(3-2i)x}$$

Example 34. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 50\,e^{2x}$.

Solution : $(D^2 - 2D + 4)\,y = 50e^{2x}$

$$\Rightarrow \quad (D-2)^2\,y = 50e^{2x}$$

$$\text{C.F. } (m-2)^2 = 0$$

$$\Rightarrow \quad m = +2, +2$$

$$\Rightarrow \quad y = (A + Bx)\,e^{2x}$$

$$\text{P.I.} \frac{1}{(D-2)^2}\cdot 50e^{2x} = 50e^{2x}\cdot\frac{1}{D^2}$$

$$= 50e^{2x}\cdot\frac{1}{2}x^2$$

$$= 25x^2e^{2x}$$

$\therefore$ General solution $y = 25x^2e^{2x} + (A + Bx)\,e^{2x}$

Example 35. Find the complete integral of the following equation

$$2xz - \left(\frac{\delta z}{\delta x}\right)x^2 - 2\left(\frac{\delta z}{\delta y}\right)xy + \left(\frac{\delta z}{\delta x}\right)\left(\frac{\delta z}{\delta y}\right) = 0$$

Solution : By Charpit's method the subsidiary equation of F $(x, y, z, p, q) = 0$

$$\frac{dx}{-\frac{\delta F}{\delta P}} = \frac{dy}{-\frac{\delta F}{\delta q}} = \frac{dz}{-\frac{\delta F}{P\,\delta P} - \frac{\delta F}{q\delta q}}$$

$$= \frac{dP}{\frac{\delta F}{\delta x} + P\frac{\delta F}{\delta z}}$$

$$= \frac{dz}{\frac{\delta F}{\delta y} + q\frac{\delta F}{\delta x}} = \frac{\delta f}{0}$$

$$\Rightarrow \frac{dx}{x^2 - q} = \frac{dy}{2xy - P} = \frac{dz}{Px^2 + 2xyq - 2Pq}$$

$$= \frac{dP}{2z - 2qy} = \frac{dq}{0} = \frac{df}{0}$$

Integrating gives $q = a$ ...(2)

and (1) and (2) gives

$$P = \frac{2x\,(z - ay)}{x^2 - a}$$

Hence, $dz = Pdx + 2dy$

$$= \frac{2x\,(z-ay)\,dx}{x^2 - a} + ady$$

i.e., $\frac{dz - ady}{z - ay} = \frac{2xdx}{x^2 - a^2}$

$\Rightarrow$ $z = ay + b\,(x^2 - a^2)$

Example 36. Solve $\frac{dy}{dx} + 3y = e^{2x}$.

Solution : $\int Pdx = 3x$

$\therefore$ Integrating factor e^{3x}

$$\therefore \quad e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{5x}$$

$$\Rightarrow \quad \frac{d}{dx}(ye^{3x}) = e^{5x}$$

$$\Rightarrow \quad ye^{3x} = \frac{1}{5}e^{5x} + c$$

$$\Rightarrow \quad y = \frac{1}{5}e^{2x} + ce^{-3x}$$

Example 37. Show that $V = \frac{A}{r} + B$ is a solution of $\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0$

Solution : Differentiating $V = \frac{A}{r} + B$ twice with respect to r,

$$\frac{dV}{dr} = -\frac{A}{r^2} \text{ and } \frac{d^2V}{dr^2} = \frac{2A}{r^3}$$

$$\therefore \quad \left(\frac{d^2V}{dr^2}\right) / \left(\frac{dV}{dr}\right) = \frac{2A}{r^3}\times\frac{r^2}{-A}$$

$$= \frac{-2}{r}$$

$$\Rightarrow \quad \frac{d^2V}{dr^2} = \frac{-2}{r}\frac{dV}{dr}$$

$$\Rightarrow \quad \frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0$$

Example 38. The differential equation satisfied by the system of parabolas,

$$y^2 = 4a\,(x + y)$$

Solution : Differentiating

$$y^2 = 4a\,(x + y) \qquad \text{...(1)}$$

with respect to x,

$$2y\frac{dy}{dx} = 4a \qquad \text{...(2)}$$

Dividing (1) by (2), we get

$$\frac{y^2}{2\left(\frac{dy}{dx}\right)} = x + a$$

$$\Rightarrow \qquad a = \frac{y^2}{2\left(\frac{dy}{dx}\right)} - x$$

Substituting this value of 'a' in (2) a is eliminated and we get

$$y\left(\frac{dy}{dx}\right) = 2\left(\frac{y}{2\left(\frac{dy}{dx}\right)} - x\right)$$

or $$y\left(\frac{dy}{dx}\right)^2 - y + 2x\frac{dy}{dx} = 0$$

or $$y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} - y = 0$$

Example 39. Eliminate the arbitrary constants a, b and c from $z = a(x + y) + b(x - y) + abt + c$

Solution : $$\frac{\delta z}{\delta x} = a + b$$

$$\frac{\delta z}{\delta y} = a - b$$

$$\frac{\delta z}{\delta t} = ab$$

$$\because \quad (a + b)^2 - (a - b)^2 = 4ab$$

$$\Rightarrow \quad \left(\frac{\delta z}{\delta x}\right)^2 - \left(\frac{\delta z}{\delta y}\right)^2 = 4\frac{\delta z}{\delta t}$$

Example 40. Given $cy = c^2x + 1$ find the c–discriminant.

Solution : $f(x, y, c) = 0$

$$\Rightarrow \qquad cy = c^2x + 1 = 0$$

$$\Rightarrow \qquad y - cx - \frac{1}{c} = 0 \qquad \ldots(1)$$

and $$\frac{\delta f}{\delta c} = 0$$

$$\Rightarrow \qquad -x + \frac{1}{c^2} = 0$$

$$\Rightarrow \qquad c = \pm\frac{1}{\sqrt{x}} \qquad \ldots(2)$$

Substituting c in (1) given

$$y = \pm 2\sqrt{x}$$

or $$y^2 = 4x$$

Example 41. Find the P-discriminant of $(x + c)^2 + y^2 = r^2$.

Solution : $(x + c)^2 + y^2 = r^2$

$$\Rightarrow \qquad x + c = \pm\sqrt{r^2 - y^2}$$

or $$1 = \pm yp/\sqrt{r^2 - p^2}$$

$$\Rightarrow \qquad y^2p^2 + y^2 - r^2 = 0$$

The P-discriminant of $y^2p^2 + y^2 - r^2 = 0$ is $\frac{\delta f}{\delta p} = 0$

$$\Rightarrow \qquad y^2(y^2 - r^2) = 0$$

Example 42. Given $y = px \pm \sqrt{(p^2 + m^2)}$, find its singular solution.

Solution : The Chairaut's form,

$$y = px + f(p)$$

$$\Rightarrow \qquad y = cx \pm \sqrt{(c^2 + m^2)}$$

or $$(y^2 - cx)^2 = (c^2 + m^2) = 0$$

or $$c^2(x^2 - 1) - 2xyc + (y^2 - m^2) = 0$$

The x-discriminant is

$$4x^2y^2 - 4(x^2 - 1)(y^2 - m^2) = 0$$

or $$y^2 - m^2x^2 = m^2$$

which is the singular solution.

OBJECTIVE TYPE QUESTIONS

1. The equation $e^x dx + e^y dy = 0$ is of order—
   (A) Zero (B) One
   (C) Two (D) None of these

2. $y = x\frac{dy}{dx} + x/dy/dx$ is of degree—
   (A) Zero (B) Two
   (C) Three (D) One

3. The order of the equation $d^2y/dx^2 + n^2x = 0$ is—
   (A) Zero (B) One
   (C) Two (D) Three

4. The degree of the equation $d^2y/dx^2 + n^2x = 0$ is—
   (A) Zero (B) One
   (C) Two (D) Three

5. The degree of the equation $y\, dy./dx = x\left(\frac{dy}{dx}\right)^2 + x$ is—
(A) Zero (B) One
(C) Two (D) Three

6. The number of arbitrary constant. A general solution of first order equation contains—
(A) Zero (B) One
(C) Two (D) Three

7. The general solution of first order equation $\frac{dy}{dx} = \cos x$ is given by—
(A) $y = \cos x$ (B) $\sin x$
(C) $y = \cos x + c$ (D) $y = \sin x + c$

8. The general solution of the equation $\frac{dy}{dx} = -\frac{x}{y}$ is—
(A) $x^2 + y^2 = a^2$ (B) $x^2 - y^2 = a^2$
(C) $x = -y$ (D) $x + y = a$

9. The particulars solution of the initial value problem $xdx + ydy = 0, x_0 = 4, y_0 = -3$—
(A) $y = \pm\sqrt{25 - x^2}$ (B) $y = \pm\sqrt{x^2}$
(C) $y = -4x$ (D) $y^2 = 3x$

10. The general solution of the equation $\sin x\, dx + \frac{dy}{\sqrt{y}} = 0$ is—
(A) $2\sqrt{y} + \cos x = c$ (B) $2\sqrt{y} - \cos x = c$
(C) $2\sqrt{y} = \sin x$ (D) $\sin x + \cos x = \sqrt{y}$

11. The general solution of the equation of $y\, dx - xdy = 0$ is—
(A) $\frac{x}{y} = c$ (B) $x + y = c$
(C) $xy = c$ (D) $x - y = c$

12. Particular solution is a solution, that can be obtained from general solution by giving particular values to arbitrary constants—
(A) True
(B) False
(C) Neither true nor false
(D) All are true

13. The complete solution of differential equation contains arbitrary constants—
(A) More than the order of equation
(B) Can't say
(C) Equal to the order of equation
(D) Less than the order of equation

14. Singular solution of differential equation contains—
(1) Arbitrary constant
(2) Can be obtained from general
(3) Do not contains arbitrary constant
(4) Can not be obtained from general solution
(A) 1, 2 are true (B) 3, 4 are true
(C) 1, 4 are true (D) 2, 4 are true

15. A first order differential equation $M(x, y)\, dx + N(x, y)\, dy = 0$ is exact if—
(A) $\frac{\delta M}{\delta x} = \frac{\delta N}{\delta y}$ (B) $\frac{\delta M}{\delta x} \neq \frac{\delta N}{\delta y}$
(C) $\frac{\delta M}{\delta y} \neq \frac{\delta M}{\delta x}$ (D) $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$

16. The integrating factor for the differential equation $(x + 1)\frac{dy}{dx} - y = e^{3x}(x + 1)^2$ is—
(A) $\frac{1}{x + 1}$ (B) $x + 1$
(C) $\frac{1}{x^2 + 1}$ (D) $x^2 + 1$

17. The integrating factor for the differential equation $2ydx + xdy = 0$ is—
(A) x (B) y
(C) xy (D) y^2

18. The integration factor for the Leibnitz linear equation $\frac{dy}{dx} + Py = Q$ is—
(A) $\int Pdx$ (B) $\int Qdx$
(C) $\exp(\int Qdx)$ (D) $\exp(\int Pdx)$

19. In linear ordinary differential equation, the dependent variable and its differential coefficients are not multiplied together and occurs only in—
(A) First degree (B) Second degree
(C) Third degree (D) Fourth degree

20. If $M(x, y)\, dx + N(x, y)\, dy = 0$ and $\frac{\delta M}{\delta y} = \frac{\delta M}{\delta x}$, then—
(A) The equation is exact
(B) Not exact
(C) Linear
(D) Non-linear

21. If an equation Mdx + Ndy = 0 is not exact. Let F (x, y) be such that FMdx + FNdy = 0 is exact, then the function F is called—
(A) Differential function
(B) Arbitrary function
(C) Integrating factor
(D) None of these

22. For the equation $Y' + P(x)\,y = r(x)$ to be homogeneous—
(A) $r(x) \neq 0$ (B) $r(x) = 0$
(C) $r(x) = P(x)$ (D) $P(x) = 0$

23. For non homogeneous equation $y' + P(x)\,y = r(x)$, if y_1 and y_2 are its solutions, then the solution of homogeneous equation $y' + P(x)\,y = 0$ is—
(A) $y = y_1 - y_2$ (B) $y = \frac{y_1}{y_2}$
(C) $y = \frac{y_2}{y_1}$ (D) None of these

24. The homogeneous differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ can be reduced to a differential equation, in which the variable are separated, by the substitution—
(A) $y = vx$ (B) $xy = v$
(C) $x + y = v$ (D) $x - y = v$

25. Which of the following is not a solution of $y'' + y = 0$?
(A) $y = \sin x$ (B) $y = \cos x$
(C) $y = 3\cos x$ (D) $y = \sin x + \frac{1}{2}$

26. Which of the following is not a solution of $xy' + y = 0$?
(A) $xy = \sqrt{3}$ (B) $xy = -2$
(C) $x = \sqrt{3}y$ (D) $xy = \frac{\pi}{2}$

27. What is the order and degree of a differential equation

$$\frac{d^2y}{dx^2} + \sqrt{1 + \left(\frac{dy}{dx}\right)^3} = 0\ ?$$

(A) First order, second degree
(B) First order, first degree
(C) Second degree, first order
(D) Second order, second degree

28. What is the order and degree of a differential equation $[1 + (y')^2]^{3/2} - Py'' = 0$—
(A) First order, second degree
(B) Second order, first degree
(C) Second order, second degree
(D) Third order, second degree

29. The solution of $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$ is—
(A) $y\sin x + (\sin y + y)\,x = c$
(B) $y\sin x + (\sin x + x) = c$
(C) $y = \sin x + y\sin y + c$
(D) None of these

30. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$, has the solution—
(A) $y = c_1e^{-2x} + c_2e^{-x}$
(B) $y = ce^{-2x}$
(C) $y = c_1e^{-2x} + c_2e^{-x} + c_3$
(D) None of these

31. $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4 = 0$, has the solution—
(A) $y = c_1e^{-x} + xc_2e^{-x} + x^2c_3e^{-x}$
(B) $y = c_1\cos 2x + c_2\sin 2x$
(C) $y = c_1e^{-x} + c_2\cos x + c_3\sin 2x$
(D) None of these

32. The particular solution of $\frac{d^3y}{dx^3} + y = \cos(2x - 1)$ is—
(A) $\frac{1}{65}[\cos(2x - 1) - 8]$
(B) $\frac{1}{65}[\cos(2x - 1) - 8\sin(2x - 1)]$
(C) $\frac{1}{65}[\cos(2x + 1) + 8]$
(D) None of these

33. The nth order ordinary linear homogenous differential equation have—
(A) n-singular solution
(B) No singular solution
(C) One singular solution
(D) None of these

34. General solution of nth order ordinary linear differential equation contains—
(A) Every solution (B) Some solutions
(C) No solution (D) None of these

35. The n-order differential linear equation with constant coefficient, $y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$, $a_1 \ldots, a_n$ are constant and characteristic equation are—
(A) All distinct
(B) All same
(C) Partially distinct
(D) None of these

36. The difference of two solutions of non homogeneous n-order ordinary differential equation is a solution of corresponding homogeneous equation. The statement is—
(A) Always true (B) Always false
(C) Partially true (D) Partially false

37. If y_1 is the solution of non homogeneous nth order ordinary differential equation and y_2 is the solution of corresponding homogeneous equation. Then the solution $y_1 + y_2$ is also a solution of—
(A) Homogeneous equation
(B) Non-homogeneous equation
(C) Both (A) and (B)
(D) None of these

38. The linearity principle for ordinary differential equation holds for—
(A) Non-homogeneous equation
(B) Non-linear equation
(C) Linear differential equation
(D) None of these

39. For homogeneous linear ordinary differential equation, the linear combination of two solutions is again a solution of the equation. The statement is—
(A) True
(B) False
(C) Neither true nor false
(D) Can't say

40. If y_1 and y_2 are two solutions of $y'' + p(x) y' + g(x) y = 0$, then for general solution of this given equation. y_1 and y_2 are—
(A) Linearly dependent
(B) Linearly independent
(C) Proportional
(D) None of these

41. If y_1 and y_2 are two solutions of initial value problem $y'' + p(x) y' + g(x) y = 0$, $y(x_2) = y_0$, $y'(x_0) = y'_0$ and the wronskion $w(y, y_2) = 0$, then y_1 and y_2 are—
(A) Linearly dependent
(B) Linearly independent
(C) Proportional
(D) None of these

42. The nth order homogeneous linear differential equation $y^{(n)} + P_0^{(n-1)} + \ldots + P_n y = 0$ has general solution if the coefficievts $P_0(x), \ldots, P_n(x)$ on some internal I are—
(A) Continuous
(B) Discontinuous
(C) Discontinuous and differentiable
(D) None of these

43. The n-order ordinary linear differential equation have no other solution other than general solution—
(A) Always true (B) Always false
(C) Partially true (D) None of these

44. For $\frac{d^2y}{dx^2} + 4y = \tan 2x$ solving by variation of parameters. The value of wronskion w is—
(A) 1 (B) 2
(C) 3 (D) 4

45. Solving by variation of parameter $y'' - 2y' + y = e^x \log x$, the value of wronskion w is—
(A) e^{2x} (B) 2
(C) e^{-2x} (D) None of these

46. The value of wronskion $w(x, x^2, x^3)$ is—
(A) $2x^4$ (B) $2x^2$
(C) $2x^3$ (D) None of these

47. Solving by variation of parameter for the equation $y'' + y = \sec x$, the value of wronskion is—
(A) 1 (B) 2
(C) 3 (D) 4

48. The solution of $\frac{\delta^2 z}{\delta x^2} - 5\frac{\delta^2 z}{\delta x \delta y} + 4\frac{\delta^2 z}{\delta y^2} = \sin(4x + y)$ is—
(A) $z = \frac{1}{3} x \cos(4x + y)$
(B) $z = f_1(y + x) + f_2(y + 4x)$

(C) $z = f(x+y) - \frac{1}{3} x \cos(4x+y)$

(D) $z = f_1(y+x) + f_2(y+4x) - \frac{1}{3} x \cos(4x+y)$

49. $\frac{\delta^3 z}{\delta x^3} - 4\frac{\delta^3 z}{\delta x^2 \delta y} + 4\frac{\delta^3 z}{\delta x \delta y^2} = 0$ has the solution—

(A) $z = \phi(y) + f_1(y+2x) + f_2(y-2x)$

(B) $z = \phi(y) + 2f_1(y+2x)$

(C) $z = \phi(y) + f_1(y+2x) + xf_2(y+2x)$

(D) None of these

50. $\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} = 12(x+y)$ has the solution—

(A) $z = f_1(y+ix) + f_2(y-ix)$

(B) $z = f_1(y+ix) + f_2(y-ix) + (x+y)^3$

(C) $z = (x+y)^3$

(D) None of these

51. $\frac{\delta^2 z}{\delta x^2} - \frac{\delta^2 z}{\delta y^2} + \frac{\delta z}{\delta x} - \frac{\delta z}{\delta y} = 0$ has the solution—

(A) $z = f_1(y-x) + e^x f_2(y-x)$

(B) $z = f_1(y+x) + f_2(y-x)$

(C) $z = e^{-x} f(y-x)$

(D) $z = f_1(y+x) + e^{-x} f_2(y-x)$

52. The solution of the equation $\frac{\delta^2 z}{\delta x \delta y}\left(\frac{\delta z}{\delta x} - 2\frac{\delta z}{\delta y} - 3\right) = 0$ is—

(A) $z = f_1(y) + f_2(x)$

(B) $z = e^{2x} f_1(y+3x)$

(C) $z = f_1(y) + e^{2x} f_2(y+3x)$

(D) $z = f_1(y) + f_2(x) + e^{3x} f_3(y+3x)$

53. $\frac{\delta^2 z}{\delta x^2} - \frac{\delta^2 z}{\delta x^2} + \frac{\delta z}{\delta x} = 1$ has the solution—

(A) $z = f_1(y) + e^{-x} f_2(y+x) + x$

(B) $z = f_1(y) + e^{-x} f_2(y+x)$

(C) $z = x$

(D) None of these

54. $\frac{\delta^2 z}{\delta x^2} + 2\frac{\delta^2 z}{\delta x \delta y} + \frac{\delta^2 z}{\delta y^2} + 2\left(\frac{\delta z}{\delta x} + \frac{\delta z}{\delta y}\right) + z = 0$—

(A) $z = 2e^{-x} f_1(y+x)$

(B) $z = e^{-x} f_1(y-x) + xe^{-x} f_2(y-x)$

(C) $z = f_1(y-x) + xf_2(y-x)$

(D) None of these

55. The solution of $\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} = \frac{z}{a}$ is—

(A) $z = e^{y/a} f(x-y)$ (B) $z = e^{y/a} f(x)$

(C) $z = e^y f(y)$ (D) $z = e^a f(x+y)$

56. The general integral of $yzp + zxp = xy$ is—

(A) $f(x+y, y+z) = 0$

(B) $f(x^2+y^2, x^2+z^2) = 0$

(C) $f(x^2-y^2, x^2-z^2) = 0$

(D) None of these

57. The general integral of $\frac{\delta z}{\delta y} = 3\left(\frac{\delta z}{\delta x}\right)^2$ is—

(A) $z = ax + 3ay + c$

(B) $z = ax + 3a^2 y + c$

(C) $z = ax^2 + 3ay + c$

(D) None of these

58. The solution of $\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} = x + y$ is—

(A) $z = \frac{2}{3}[(x+a)^{3/2} + (y-a)^{3/2}] + b$

(B) $z = \frac{2}{3}[(x+a) + (y-a)] + b$

(C) $z = \frac{2}{3}[(x+a)^2 + (y-a)^2] + b$

(D) $z = \frac{2}{3}[(x+a)^3 + (y-a)^3 + b$

59. The equation $\frac{\delta z}{\delta x} e^y = \frac{\delta z}{\delta y} e^x$ gives the general solution—

(A) $z = ae^x + be^x$

(B) $z = e^x + e^y$

(C) $z = a(e^x + e^y) + b$

(D) None of these

60. The complete solution of $z = px + qy + p^2 + q^2$ is—

(A) $z = ax + by + a^2 + b^2$

(B) $z = ax + by$

(C) $z = a^2x^2 + z^2 + by^2$

(D) $z = ax^2 + by^2 + z^2$

61. The solution of the differential equation $\left(x \sin\left(\frac{y}{x}\right)\right) dx - \left(y \sin\left(\frac{y}{x}\right) - x\right) dx = 0$

(A) $\cos(y/x) = 0$

(B) $\sin(y/x) = 0$

(C) $\cos(y/x) - \log x =$ constant

(D) $\sin(y/x) - \log x =$ constant

62. The solution of the differential equation

$$\frac{dy}{dx} = \frac{3x^2y^4 + 2xy}{x^2 - 2x^3y^3} \text{ —}$$

(A) $x^3y^2 + \frac{x^2}{y} = \frac{x}{y}$ (B) $x^3y^2 + \frac{x^2}{y^2} = x$

(C) $x^3y^2 + \frac{x^2}{y} = c$ (D) None of these

63. The equation $y = Ae^{3x} + Be^{5x}$ can be represented as—

(A) $y'' - 8y' + 15y = 0$

(B) $y'' + 8y' = 0$

(C) $y' + 8y' = 0$

(D) $y'' + 8y' + 15y = 0$

64. The equation of the curve where sub-normal is equal to a constant a is—

(A) $y = ax + b$ (B) $y^2 = 2ax + 2b$

(C) $ay^2 - x^3 = a$ (D) None of these

65. The differential $\frac{dy}{dx}$ + Py = Q, P and Q are functions of x only, have the integrating factor—

(A) $e^{\int P dx}$ (B) $e^{\int Q dy}$

(C) $e^{\int P dy}$ (D) $e^{\int Q dx}$

66. A solution of a differential equation which contains no arbitrary constants is—

(A) Particular solution

(B) General solution

(C) Primitive

(D) None of these

67. A general solution of linear differential equation with constant coefficients is—

(A) Sum of particular solution and complementary function

(B) Product of particular solution and complementary function

(C) Quotient of particular solution and complementary function

(D) None of these

68. (a) The solution of ordinary differential equation of order n have n arbitrary constants.

(b) The solution of partial differential equation of order n have n arbitrary functions.

(A) (a) is true, (b) is false

(B) (a) is false (b) is true

(C) (a) and (b) are both true

(D) (a) and (b) both are false

69. The solution of ordinary differential equation of n order contains—

(A) n arbitrary constants

(B) More than n-arbitrary constants

(C) No arbitrary constants

(D) None of these

70. The solution of equation of $dx + xdy = 0$ is—

(A) $xy = c$ (B) $x = y + c$

(C) $x - y = c$ (D) None of these

71. Given a differential equation of order n, then its completes primitive contains—

(A) n-arbitrary constants

(B) More than n-arbitrary

(C) Less than n-arbitrary constants

(D) No arbitrary constant

72. The necessary condition for the equation M $(x, y)\, dx$ + N $(x, y)\, dy = 0$ to be exact is—

(A) $\frac{\delta N}{\delta y} = \frac{\delta M}{\delta x}$ (B) $\frac{\delta N}{\delta y} = -\frac{\delta M}{\delta x}$

(C) $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$ (D) $\frac{\delta M}{\delta y} = -\frac{\delta N}{\delta x}$

73. The equation $ydx + xdy = 0$ is—

(A) Exact differential equation

(B) Not exact differential equation

(C) Partial differential equation

(D) None of these

74. The solution of differential equation contains as may as arbitrary constants as the order of a differential equation then the solution is said—

(A) Particulars solution

(B) Complete primitive

(C) Singular solution

(D) None of these

75. The solution derived from complete primitive by giving particular values to arbitrary constants is—

(A) Singular solution

(B) Particular integral

(C) Complete solution

(D) None of these

76. The number of arbitrary constants complete primitive of differential equation

$$\frac{d^5y}{dx^5} + 2\frac{d^4y}{dx^4} = 0 \text{ contains—}$$

(A) 5 (B) 4
(C) 1 (D) None of these

77. Any equation contains n-arbitrary constants, then the order of differential equation derived from it, is—
(A) n (B) $n - 1$
(C) 2 (D) $n + 1$

78. The differential equation, derived from $y = Ae^{2x} + Be^{-2x}$ have the order, where A, B are constants—
(A) 3 (B) 2
(C) 1 (D) None of these

79. A differential equation of first order and first degree is homogeneous is—
(A) $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ (B) $\frac{dy}{dx}$ = constant
(C) $\frac{dy}{dx} = f(x)$ (D) None of these

80. The differential equation $\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}$ is—
(A) Homogeneous equation
(B) Non homogenous equation
(C) Non-exact equation
(D) None of these

81. The differential equation $\frac{dy}{dx} = \frac{y^3}{x^2}$ is—
(A) Homogeneous (B) Non-homogenous
(C) Exact equation (D) None of these

82. The equation $\frac{dy}{dx} + Py = Q$ is linear differential equation of first order if—
(A) P, Q are function of x only
(B) P, Q are function of y only
(C) P, Q are function of x and y
(D) None of these

83. The equation $x = A \cos (pt - \alpha)$ can be expressed as—
(A) $\frac{d^2x}{dt^2} = x$ (B) $\frac{d^2x}{dt^2} = -p^2x$
(C) $\frac{d^2x}{dt^2} = 0$ (D) None of these

84. Given equation $\frac{d^3y}{dx^3} = \frac{dy}{dx}$ and a solution of it is $y = a_0 + a_1 \sinh/x + a_2 \cosh/x$, where a_0, a_1, a_2 are arbitrary constants, then this solution is—
(A) Particular solution
(B) Complete primitive
(C) Singular solution
(D) None of these

85. General solution of $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$ is—
(A) $y = Ae^x + Be^{2x} + Ce^{3x}$
(B) $y = 3e^x$
(C) $y = A + Be^{2x}$
(D) None of these

86. The differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 50e^{2x}$ have particular integral—
(A) $\frac{2e^{2x}}{3}$ (B) $2e^{2x}$
(C) e^{2x} (D) None of these

87. The general solution of $\frac{dy}{dx} + \frac{1}{x}y = x^2$ is—
(A) $xy = \frac{1}{4}x^4 + c$ (B) $xy = c$
(C) $\frac{x}{y} = c$ (D) None of these

88. Solution of $\frac{dy}{dx} + \frac{1}{x}y = x^3$ is—
(A) $e^{\log x} = x$ (B) $e = x$
(C) $\log x = y$ (D) None of these

89. General solution of $\frac{dy}{dx} + 2xy = 2e^{-x^2}$ is—
(A) $y = (2x + c)\, e^{-x^2}$ (B) $y = 2xe^{-x}$
(C) $y = e^{-x}$ (D) None of these

90. General solution of $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 0$ is—
(A) $y = Ae^{-2x}$
(B) $y = Be^{1/2x}$
(C) $y = Ae^{-2x} + Be^{-1/2x}$
(D) None of these

91. For a given differential equation—
    (a) An envelope gives a singular solution
    (b) Node locus gives a solution
    (c) Cusp-locus gives a solution
    (A) (a) is true, (b) and (c) are false
    (B) (a), (b), (c) are true
    (C) (a) is false, (b) and (c) are true
    (D) (a), (b), (c) are false

92. The p-discriminant does not contains one of the following—
    (A) The envelop
    (B) The tac-locus
    (C) The cusp-locus
    (D) The node-locus

93. The c-discriminant when equated to zero includes nodal locus—
    (A) Once (B) Twice
    (C) Thrice (D) None of these

94. The c-discriminant and p-discriminant both contains—
    (A) Envelope (B) Tac-locus
    (C) Node-locus (D) None of these

95. The c-discriminant of the equation $(y-c)^2 = x(x-a)^2$ is—
    (A) $x(x-a^2)=0$ (B) $x-a^2=0$
    (C) $x=a$ (D) None of these

96. The p-discriminant of the equation $y = px + \frac{1}{p}$ is—
    (A) $y^2 = x$ (B) $y = x$
    (C) $y^2 = 4x$ (D) None of these

97. The p-discriminant of equation $4xp^2 = (3x-a)^2$ is—
    (A) $x = 0$ (B) $3x = b^2$
    (C) $(3x-a)^2 = 0$ (D) None of these

98. The p-discriminant of equation $y^2p^2 + y^2 - r^2 = 0$ is—
    (A) $y^2 = 0$ (B) $y^2 = r^2$
    (C) $y^2(y^2 - r^2) = 0$ (D) $y^4 = 0$

99. Following is the solution of differential equation—
    (A) Envelope (B) Cusp locus
    (C) Node-locus (D) Tac-locus

100. If $\phi(x, y) = 0$ is a singular solution, then $\phi(x, y)$ is a factor of—
    (A) p-discriminant only
    (B) c-discriminant only
    (C) p-and c-discriminant both
    (D) None of these

101. The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is—
    (A) $b^2 - 4ac$ (B) b^2
    (C) $4ac$ (D) None of these

102. The nodes have a double point—
    (A) With distinct tangent
    (B) With coincident tangent
    (C) Same tangent
    (D) None of these

103. (a) Singular solution contains no arbitrary constants.
    (b) Singular solution can be obtained from complete primitive.
    (A) (a) is true, (b) is false
    (B) (a) and (b) both true
    (C) (a) and (b) both false
    (D) None of these

104. Which of the following is an ordinary differential equation ?
    (A) $y + 2x = \frac{d}{dx}(\sin x)$
    (B) $\frac{d}{dx}(y + x) = x^2 + \sin x$
    (C) $\frac{d}{dx}(\cos x + \sin x) = x + y$
    (D) $y^2 + x^2 = \frac{d}{dx}(x\alpha + 1)$

105. The degree of the differential equation $\left[y + x\left(\frac{d^2y}{dx^2}\right)^2\right]^{1/4} = \frac{d^3y}{dx^3}$ is given by—
    (A) 2 (B) 3
    (C) 4 (D) 1

106. The order of the differential equation $\left[1 + \left(\frac{d^3y}{dx^3}\right)^2\right]^{4/3} = \frac{d^2y}{dx^2}$ is given by—
    (A) 1 (B) 2
    (C) 3 (D) 4

107. Which of the following is false ?
(A) The degree of the differential equation is the highest power of the highest order differential coefficient occurring it
(B) The order of the differential equation is the highest order derivative occurring in the equation
(C) The order of a differential equation is unique
(D) The order of a differential equation is any integer

108. If m and n are the order and degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + 4\left(\frac{dy}{dx}\right)^2 + 3y^2 + y = 0$, then—
(A) $m = n$ (B) $m < n$
(C) $m > n$ (D) $m \geq n$

109. The general solution of the differential equation $ydx - xdy = xydx$ is—
(A) $y = \text{A}xe^x$ (B) $\text{A}x = ye^x$
(C) $y = \text{A}e^x$ (D) $y = \text{A}e^{-x}$

110. The solution of the equation $\frac{dy}{dx} + \frac{1+y}{1+x} = 0$, is—
(A) $1 + x = c(1 + y)$
(B) $y + \frac{y^2}{2} = x + \frac{x^2}{2} + c$
(C) $1 + x + y + xy = c$
(D) None of these

111. The differential equation which is linear is—
(A) $\frac{dy}{dx} + x^2y = \sin y$
(B) $\frac{dy}{dx} - x^2y = \sin x$
(C) $(1 + y)\frac{dy}{dx} + \sin x = 0$
(D) $\frac{dy}{dx} + y(y + x) = x^2$

112. Solution of the equation $\frac{dy}{dx} = 1 + x + y + xy$ is —
(A) $1 + y = x + \frac{x^2}{2} + c$
(B) $\log(1 + x) = y + \frac{y^2}{2} + c$
(C) $\log(1 + y) = x + \frac{x^2}{2} + c$
(D) $\log(x + y) = y + \frac{y^2}{2} + c$

113. If $(x + y)\,dx + (x - y)\,dy = 0$, then—
(A) $x^2 + 2xy - y^2 = c$
(B) $x^2 - 2xy - y^2 = c$
(C) $x^2 + 2xy + y^2 = c$
(D) $x^2 - 2xy + y^2 = c$

114. The solution of the equation $(a^2 - 2xy - y^2)\,dx - (x + y)^2\,dy = 0$ is—
(A) $a^2x - x^2y + \frac{x^3}{3} - \frac{y^3}{3} = c$
(B) $a^2x - xy^2 + \frac{x^3}{3} - \frac{y^3}{3} = 0$
(C) $a^2x - x^2y - y^2x - \frac{y^3}{3} = c$
(D) $a^2x + x^2y + y^2 + \frac{y^3}{3} = c$

115. An integrating factor of the differential equation $(1 - x^2)\frac{dy}{dx} - xy = 1$ is—
(A) $-x$ (B) $\frac{x}{1 - x^2}$
(C) $\sqrt{1 - x^2}$ (D) $\frac{1}{2}\log_e(1 - x^2)$

116. The solution of the differential equation
$$\frac{dy}{dx} = e^{x-y} + x^2e^{-y} \text{ is—}$$
(A) $e^y = e^x + \frac{1}{3}x^3 + c$
(B) $e^y = e^{-x} + \frac{1}{3}x^3 + c$
(C) $e^y = e^x + x^3 + c$
(D) $e^{-y} = \frac{1}{3}x^3 + e^x + c$

117. If $\frac{1}{\text{M}}\left(\frac{\delta \text{N}}{\delta x} - \frac{\delta \text{M}}{\delta y}\right) = f(y)$, a function of y alone, then the integrating factor of $\text{M}dx + \text{N}dy = 0$ is—
(A) $e^{-\int f(y)\,dy}$
(B) $e^{\int f(y)\,dy}$
(C) $f(y)\int e^{f(y)\,dy}$
(D) $\int e^{f(y)} f(y)\,dy$

118. The solution of the differential equation $(x + y)^2 \frac{dy}{dx} = a^2$ is given by—

(A) $(y + x) = a \tan\left(\frac{y-c}{a}\right)$

(B) $(y - x) = a \tan (y - c)$

(C) $(y - x) = \tan\left(\frac{y-c}{a}\right)$

(D) $a\,(y - x) = \tan\left(\frac{y-c}{a}\right)$

119. The solution of $\frac{dy}{dx} = \frac{1}{x+y+1}$ is—

(A) $y = \log (x + y + 2) + 1 + c$

(B) $y + 1 = \log (x + y + 2) + c$

(C) $y + \log (x + y + 2) = c$

(D) $y + 1 + \log (x + y + 2) = c$

120. The differential equation $ydx - 2xdy = 0$ represents—

(A) A family of straight lines

(B) A family of parabolas

(C) A family of circles

(D) A family of catenaries

121. The integrating factor of the differential equation $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$ is—

(A) $e \tan^{-1} y$ (B) $e \cot^{-1} y$

(C) $\log (1 + y^2)$ (D) None of these

122. If P, Q are function of x, then solution of differential equation $\frac{dy}{dx} + \text{P}y = \text{Q}$ is—

(A) $y \cdot e^{\int Pdx} = \int Q \cdot e^{\int Pdx}\, dx + c$

(B) $y = e^{\int Pdx} \int Q \cdot e^{\int Pdx} + c$

(C) $y^P = \int Q \cdot e^{\int Pdx}\, dx + c$

(D) None of these

123. The differential equation of the form $\frac{dy}{dx}$ + $P(x)\, y = Q(x)\, y^n$ is called—

(A) Auxiliary equation

(B) Clairaut's equation

(C) Bessel's equation

(D) Bernoulli's equation

124. The general solution of the equation $\frac{dy}{dx} = \frac{x+y}{x-y}$ is—

(A) $\tan^{-1} \frac{y}{x} = \log (x^2 + y^2) + c$

(B) $2\tan \frac{y}{x} = \log (x^2 + y^2) + c$

(C) $2\tan^{-1} \frac{y}{x} = \log (x^2 + y^2) + c$

(D) $\log \frac{y}{x} = 2\log (x^2 + y^2) + c$

125. $Pdx + x \sin y\, dy = 0$ is exact, then P can be—

(A) $\sin y + \cos y$ (B) $-\sin y$

(C) $x^2 - \cos y$ (D) $\cos y$

126. The integrating factor for the differential equation $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy$ is given by—

(A) $\frac{1}{xy}$ (B) xy

(C) x^2y^2 (D) $\frac{1}{x^2y^2}$

127. To solve the equation $(x - y - 2)\,dx - (2x - 2y - 3)\,dy = 0$ we shall put—

(A) $y = vx$ (B) $x - y = v$

(C) $y = c \sin x$ (D) $y^2 = x^2 + v$

128. Solution of the differential equation

$\frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2 + y^2}}$ is—

(A) $\sqrt{x^2 + y^2} = a \sin (\tan^{-1} y + cx)$

(B) $\sqrt{x^2 + y^2} = a \sin (\tan^{-1} x/y + c)$

(C) $\sqrt{x^2 + y^2} = a \sin (\tan^{-1} y/x + c)$

(D) $x^2 + y^2 = a \sin (\tan^{-1} y/x + c)^2$

129. Solution of the differential equation $(x - y)^2\, dx + 2xy\, dy = 0$ is—

(A) $ye^{y^2/x} = A$ (B) $xe^{y^2/x} = A$

(C) $ye^{x/y^2} = A$ (D) $xe^{x/y^2} = A$

130. Solution of the differential equation $(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0$ is—

(A) $3x^4y^2 + 6xy^2 + 2y^6 = c$

(B) $3x^2y^4 + 6xy^2 + 2y^6 = c$

(C) $3x^2y^4 + 6xy^2 + y^6 = c$

(D) $3x^2y^4 + 6x^2y^2 + 2y^6 = c$

131. Solution of the differential equation $x^2 y dx - (x^3 + y^3)\, dy = 0$ is—

(A) $y^2 = ce^{y^2/x^2}$ (B) $y^3 = ce^{x^3/y^3}$

(C) $y^3 = ce^{y^3/x^3}$ (D) $y^3 = ce^{x^2/y^2}$

132. Solution of the equation $(y - xy^2)dx - (x + x^2 y)\, dy = 0$ is—

(A) $x = cye^{1/xy}$ (B) $y = cxe^{1/xy}$

(C) $xy = ce^{1/xy}$ (D) $c = xye^{1/xy}$

133. Solution of the equation $(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$ is—

(A) $y + xe^{x/y} = c$ (B) $x + ye^{x/y} = c$

(C) $y + xe^{y/x} = c$ (D) $x + ye^{y/x} = c$

134. The general solution of the differential equation $(x^2 + y^2)\, dx - 2xydy = 0$ is—

(A) $x^2 - cx - y^2 = 0$

(B) $(x - y)^2 = cx$

(C) $x + y + 2xy = c$

(D) $y = x^2 - 2x + c$

135. The solution of the equation $\frac{dy}{dx} + 2xy = 2xy^2$ is—

(A) $y = \frac{cx}{1 + e^{x^2}}$ (B) $y = \frac{1}{1 - ce^{x}}$

(C) $y = \frac{1}{1 + ce^{x^2}}$ (D) $y = \frac{cx}{1 + e^{x^2}}$

136. If $f'(x) = \tan^{-1} x$ and $f(0) = 0$, then $f(1)$ is—

(A) $\frac{\pi}{4} + \log 2$ (B) $\frac{\pi}{2} - \log 2$

(C) $\frac{1}{4}(\pi - \log 4)$ (D) $\frac{1}{4}(\pi + \log 4)$

137. Solution of differential equation $3e^x \tan y dx + (1 - e^x) \sec^2 y dy = 0$ is—

(A) $\tan y\,(1 - e^{3x}) = c$

(B) $\tan y\,(1 - e^{x})^{-3} = c$

(C) $\tan y\,(1 - e^{-3x}) = c$

(D) $\text{tam } y\,(1 - e^{-x})^3 = c$

138. Integrating factor of $(3x - 10y^3)\frac{dy}{dx} + y = 0$ is—

(A) e^{y^2} (B) e^{x^2}

(C) y^3 (D) x^2

139. Integrating factor of $\left\{ x(x-1)\frac{dy}{dx} - (x-2)y \right\} = x^3(2x-1)$ is given by—

(A) $\frac{x-1}{x^3}$ (B) $\frac{x^2}{x-1}$

(C) $\frac{x-1}{x^2}$ (D) $\frac{x^3}{2x-1}$

140. The transformation which will transform the differential equation $\frac{dz}{dx} + \frac{2zx}{x^2+1} \log z = \frac{xz}{x^2+1} (\log z)^3$ to the form $\frac{dy}{dx} + P(x)\, y = Q(x)$ is—

(A) $y = \log z$ (B) $y = \frac{1}{(\log z)^2}$

(C) $y = \frac{z}{(\log z)^2}$ (D) $y = (\log z)^2$

141. The solution of the equation $xP + 2y = Pxy$ is—

(A) $xy^2 = Ae^y$ (B) $xy^2 = Ae^x$

(C) $x^2 y = Ae^y$ (D) $xy = Ae^y$

142. The primitive of the differential equation $y = Px + f(P)$, $P = \frac{dy}{dx}$ is—

(A) $y = x + f(c)$

(B) $y = cx = f(c)$

(C) $y = cx + f(c) = 0$

(D) $y = cx + f(c)$

143. The differential equation whose primitive is $y^2 = 2cx + c^2$ is—

(A) $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y$

(B) $\frac{dy}{dx} + yx = x$

(C) $\frac{dy}{dx} = yx + x$

(D) $\left(\frac{dy}{dx}\right)^2 + y^2 x = x$

144. The general solution of the differential equation $P^2 + 2Py \cot x - y^2 = 0$ is—

(A) $y(1 \pm \cos x) = c$

(B) $y = c \pm \cos x$

(C) $y \cos x = c$

(D) $cy \pm \cos x = 0$

145. The general solution of the differential equation $xy\,(P^2 + 1) = (x^2 + y^2)\,P$ is—
(A) $(y - cx)\,(x^2 - y^2 - c) = 0$
(B) $(y - c - x)\,(xy - c) = 0$
(C) $(y - x - c)\,(x^2 + y^2 - c) = 0$
(D) $(x^2y - c)\,(y - cx^2) = 0$

146. The differential equation $x^2P^2 + yP\,(2x + y) + y^2 = 0$ can be reduced to the Clairaut's form using the transformation—
(A) $u = y, v = xy$ (B) $u = y, v = x/y$
(C) $u = x, v = y^2$ (D) $u = y, v = x^2$

147. The solution of the differential equation $P^3y^2 - 2Px + y = 0$ is—
(A) $2cx = c^3 + y$ (B) $2cx = c^3 + y^2$
(C) $2cy = c^3 + x^3$ (D) $2cy = c^3 + y^3$

148. The general solution of the differential equation $y = 2Px + y^{x-1}\,P^n$ is—
(A) $y = 2cx + c^{n-1}$
(B) $y = 2cx + c^n$
(C) $y^2 = 2cx + c^n$
(D) $y^2 = 2cx + c^{n-1}$

149. The general solution of the differential equation $y = x\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2$ is—
(A) $y = cx - c^2$ (B) $y = (x + c)$
(C) $y = cx - c$ (D) $y = cx + c^2$

150. The solution of $\frac{dy}{dx} = \log\left[x\frac{dy}{dx} - y\right]$ is—
(A) $y = cx - e^c$
(B) $c = \log\,(-cx + y)$
(C) $c = \log\,(cx + y)$
(D) None of these

151. The general solution of the differential equation $(x + a)\,P^2 + (x + y)\,P - y = 0$ is—
(A) $y = cx + \frac{ac^2}{c+1}$
(B) $y = cx + \frac{ac}{c+1}$
(C) $y = -cx + \frac{ac^2}{c+1}$
(D) $y = -cx + \frac{ac}{c+1}$

152. Solution of the equation $y - 2px = f\,(xP^2)$, where $p = \frac{dy}{dx}$ is—
(A) $y = 2cx + f\,(c^2)$
(B) $y = cx + f\,(c^2x)$
(C) $y = 2c\sqrt{x} + f(cx^2)$
(D) $y = 2c\sqrt{x} + f\,(c^2)$

153. Solution of $x^2P^2 + xyP - 6y^2 = 0$ is—
(A) $(yx - c)\,(y - c) = 0$
(B) $(y^3x - c)\,(y - cx^2) = 0$
(C) $(yx^3 - c)\,(y - cx^2) = 0$
(D) $(yx^2 - c)\,(y - cx^2) = 0$

154. The general solution of $P^3 - 4xyP + 8y^2 = 0$ is—
(A) $x = c^2\,(y - c)^2$
(B) $4x^2 = 27c^2y + c^2$
(C) $27x = 4y^2c^2 + c^2$
(D) $64y = c\,(4x - c)^2$

155. The general solution of the differential equation $(x^2 + y^2)\,(1 + P)^2 - 2\,(x + y)\,(1 + P)\,(x + Py) + (x + Py)^2 = 0$ can be obtained by substituting—
(A) $u = x^2, v = y^2$
(B) $u = xy, v = x + y$
(C) $u = x + y, v = x^2 + y^2$
(D) $u = x^2 + y^2, v = xy$

156. The differential equation $e^{3x}\,(P - 1) + P^3e^{3y} = 0$ can be reduced to Clairaut's equation by substituting—
(A) $u = e^x, v = e^y$ (B) $u = e^y, v = e^x$
(C) $u = x^2$ (D) $v = y^2$

157. The solution of the differential equation $y = Px + \log P$, is—
(A) $y = cx + \log c$
(B) $y = cx^2 + \log c$
(C) $y^2 = cx + \log c$
(D) $y^2 = cx^2 + \log c$

158. The general solution of the differential equation $P^2 - 5P + 6 = 0$ is—
(A) $(y - 3x - c_1)\,(y - 2x - c_2) = 0$
(B) $(3x + y + c_1)\,(2y + x - c_2) = 0$
(C) $(y + 3x - c_1)\,(y + 2x - c_2) = 0$
(D) $(y - x - c_1)\,(y + x - c_2) = 0$

159. If $f(D)\,y = 0$ be a linear differential equation with constant coefficients, then its auxiliary equation is—
(A) $f(m^3) = 0$ (B) $f(m) = 0$
(C) $f(D - m) = 0$ (D) $f(x) = 0$

160. A differential equation in which the dependent variable and its derivatives occur only in first degree and are not multiplied together is called a .... differential equation—
(A) Homogeneous
(B) Non-homogeneous
(C) Linear
(D) Linear with constant coefficient

161. If the general solution of the linear differential equation with constant coefficients $f(0) = 0$ be $c_1e^{a_1x} + c_2e^{a_2x} + \ldots\ldots + c_ne^{a_nx} = 0$, then the roots of the auxiliary equation are—
(A) All real
(B) All imaginary
(C) Some real and some imaginary
(D) All real and distinct

162. The auxiliary equation $aD^2 + bD + c = 0$ is obtained from the linear differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where a, b, c are constants, by substituting—
(A) $y = x^m$ (B) $y = b \sin mx$
(C) $y = m^x$ (D) $y = e^{mx}$

163. The roots of the auxiliary equation of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$ are—
(A) 1, 4 (B) –1, 4
(C) –1, –4 (D) 1, –4

164. The general solution of the differential equation $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$ is—
(A) $y = C_1e^{-x}$
(B) $y = C_2e^{4x}$
(C) $y = C_1e^{-x} + C_2e^{2x}$
(D) $y = C_1e^{-x} + C_2e^{4x}$

165. The general solution of the linear differential equation $(D^4 - 81)\,y = 0$ is given by—
(A) $(C_1 + C_2x)\,e^{3x} + (C_3 + C_4x) \sin 3x$
(B) $(C_1 + C_2x)\,e^{3x} + (C_3 + C_4x)\,e^{-3x} + (C_5 + C_6x) \cos 3x + (C_1 + C_8x) \sin 3x$
(C) $C_1e^{3x} + C_2e^{-3x} + C_3 \cos 3x + C_4 \sin 3x$
(D) $C_1e^{3x} + C_2e^{-3x} + e^{3x}(C_3 \cos x + C_4 \sin x)$

166. $\frac{1}{D - a}\,Q(x)$ is equal to—
(A) $e^{ax}\int Q(x)\,dx$ (B) $e^{-ax}\int e^{ax}\,Q(x)\,dx$
(C) $e^{-ax}\int Q(x)\,dx$ (D) $e^{ax}\int e^{-ax}\,Q(x)\,dx$

167. The general solution of the differential equation $D^2(D + 1)^2\,y = e^x$ is—
(A) $y = C_1 + C_2x + (C_3 + C_4x)\,e^x$
(B) $y = C_1 + C_2x + (C_3 + C_4x)\,e^{-x} + \frac{1}{4}e^x$
(C) $y = (C_1 + C_2e^{-x}) + (C_3 + C_4)\,e^{-x} + \frac{1}{4}e^x$
(D) None of these

168. The general solution of the differential equation $\frac{d^2y}{dx^2} - y = e^x$ is—
(A) $y = A \cos(x + B) + e^x$
(B) $y = A \cosh(x + B) + \frac{1}{2}xe^x$
(C) $y = A \cosh(x + B) + xe^x$
(D) $y = A \cos(x + B) + xe^x$

169. The general solution of the differential equations $(D^2 + D - 2)\,y = e^x$ is given by—
(A) $y = C_1e^x + C_2e^{-2x} + \frac{1}{3}xe^x$
(B) $y = C_1e^x + C_2e^{-2x}$
(C) $y = C_1e^x + C_2e^{-2x} + \frac{1}{6}x^2e^x$
(D) $y = \frac{1}{3}xe^x + (C_1 + C_2x)\,e^{-2x}$

170. A P. I. of the differential equation $(D^2 + 4)\,y = x$ is—
(A) xe^{-2x} (B) $x \cos 2x$
(C) $x \sin 2x$ (D) $x/4$

171. The solution of the differential equation $\frac{d^2y}{dx^2} + \omega^2y = 10\omega^2$ is—
(A) $y = A \cos \omega x \sin \omega x + B + 10$
(B) $y = A \sin(\omega x + B) + 10\omega^2$
(C) $y = Ax + Bx \cos \omega x + 10\omega^2$
(D) $y = A \cos(\omega x + B) + 10$

172. The general solution of the differential equation $\frac{d^2y}{dx^2} + a^2y = \sec ax$ is—

(A) $y = C_1 \cos ax + C_2 \sin ax + x \sin ax \log (\cos ax)$

(B) $y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a} \{x \sin ax + \log (\cos ax)\}$

(C) $y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a}\left\{x \sin ax + \frac{1}{a} \log (\cos ax)\right\}$

(D) $y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a}\left\{x \sin ax + \frac{1}{a} \log (\cos ax) \cos ax\right\}$

173. For the differential equation $t(t-2)^2 y'' + ty' + y = 0$, $t = 0$ is—

(A) An ordinary point
(B) A branch point
(C) An irregular point
(D) A regular singular point

174. Suppose that y_1 and y_2 form a fundamental set at solutions of a second order ordinary differential equation on the interval $-\infty < t < +\infty$, then—

(A) There is only one zero of y_1 between consecutive zeros of y_2
(B) There are two zeros of y_1 between consecutive zeros of y_2
(C) There are finite number of zeros of y_1 between consecutive zeros of y_2
(D) None of the above

175. The general solution of the system of differential equations $\frac{dX}{dt} = MX + b$, where $X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$; M, a 2×2 matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and b, a 2×1 constant vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is given by—

(A) $e^{m_1} C + b$ (B) $e^{m_1} C + bt$
(C) $e^{m_1} C - b$ (D) $e^{m_1} C - bt$

Where C is any 2×1 constant vector.

176. The general solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is of the form—

(A) $u = f(x + iy) + g(x - iy)$
(B) $u = f(x + y) + g(x - y)$
(C) $u = Cf(x - iy)$
(D) $u = g(x + iy)$

177. If $y' - x \neq 0$, a solution of the differential equation $y'(y' + y) = x(x + y)$ is given by y—

(A) $1 - x - e^{-x}$ (B) $1 - x + e^{x}$
(C) $1 + x + e^{-x}$ (D) $1 + x + e^{x}$

178. For the differential equation $4x^3y'' + 6x^2y' + y = 0$, the point at infinity is—

(A) An ordinary point
(B) A regular singular point
(C) An irregular singular point
(D) A critical point

179. For the differential equation $xy' - y = 0$ which of the following function is not an integrating factor ?

(A) $\frac{1}{x^2}$ (B) $\frac{1}{y^2}$
(C) $\frac{1}{xy}$ (D) $\frac{1}{x + y}$

180. The differential equation $(3a^2x^2 + by \cos x)\,dx + (2 \sin x - 4ay^3)\,dy = 0$ is exact for—

(A) $a = 3, b = 2$ (B) $a = 2, b = 3$
(C) $a = 3, b = 4$ (D) $a = 2, b = 5$

181. Let $u(x, t)$ be the solution of $u_{11} = u_{xx}$; $0 < x < 1$, $t > 0$, $u(x, 0) = x(1 - x)$, $u_1(x, 0) = 0$, then $u\left(\frac{1}{2}, \frac{1}{4}\right)$ is—

(A) $\frac{3}{16}$ (B) $\frac{1}{4}$
(C) $\frac{3}{4}$ (D) $\frac{1}{16}$

182. The differential equation $\frac{dy}{dx} = k(a - y)(b - y)$, when solved with the condition $y(0) = 0$, yields the result—

(A) $\frac{b(a-y)}{a(b-y)} = e^{(a-b)kx}$

(B) $\frac{b(a-x)}{a(b-x)} = e^{(b-a)ky}$

(C) $\frac{a(b-y)}{b(a-y)} = e^{(a-b)kx}$

(D) $xy = ke$

183. The equation $x^2(y-1)Z_{xx} - x(y^2-1)Z_{xy} + y(y^2-1)Z_{yy} + Z_x = 0$ is hyperbolic in the entire xy –plane except along—
(A) x-axis
(B) y-axis
(C) A line parallel to y-axis
(D) A line parallel to x-axis

184. Determine the type of the following differential equation $\frac{d^2y}{dx^2} + \sin(x+y) = \sin x$—
(A) Linear, homogeneous
(B) Non linear, homogeneous
(C) Linear, non-homogenous
(D) Non-linear, non-homogeneous

185. Which of the following is not an integrating factor of $xdy - ydx = 0$?
(A) $\frac{1}{x^2}$ (B) $\frac{1}{x^2+y^2}$
(C) $\frac{1}{xy}$ (D) $\frac{x}{y}$

186. The general solution of the differential equation $\frac{dy}{dx} + \tan y \tan x = \cos x \sec y$ is—
(A) $2\sin y = (x + c - \sin x \cos x) \sec c$
(B) $\sin y = (x + c) \cos x$
(C) $\cos y = (x + c) \sin x$
(D) $\sec y = (x + c) \cos x$

187. The differential equation whose linearly independent solutions are $\cos 2x$, $\sin 2x$ and e^{-x} is—
where $D = \frac{d}{dx}$
(A) $(D^3 + D^2 + 4D + 4)\, y = 0$
(B) $(D^3 - D^2 + 4D - 4)\, y = 0$
(C) $(D^3 + D^2 - 4D - 4)\, y = 0$
(D) $(D^3 - D^2 - 4D + 4)\, y = 0$

188. The order of the numerical differentiation formula $f''(x_0) = \frac{1}{12h^2}[-\{f(x_0 - 2h) + f(x_0 + 2h)\} + 16\{f(x_0 - h) + f(x_0 + h)\} - 30 f(x_0)]$ is—
(A) 2 (B) 3
(C) 4 (D) 1

189. The Bessel's function $\{J_0(a_k x)\}_{k=1}^{\infty}$ with a_k denoting the k-th zero of $J_0(x)$ form on orthogonal system on [0, 1] with respect to weight function—
(A) 1 (B) x^2
(C) x (D) $\sqrt{x}$

190. Linear combinations of solutions of an ordinary differential equation are also solutions if the differential equation is—
(A) Linear non-homogenous
(B) Linear homogeneous
(C) Non-linear homogeneous
(D) Non-linear non-homogeneous

191. Which of the following, concerning the solution of the Neumann problem for Laplace's equation, on a smooth bounded domain, is true ?
(A) Solution is unique
(B) Solution is unique upto an additive constant
(C) Solution is unique upto a multiplicative constant
(D) No conclusion can be drawn about uniqueness

192. Which of the following satisfies the heat equation (without sources terms and with diffusion constant 1) in one space dimension ?
(A) $\sin\left[\frac{x^2}{4t}\right]$ (B) $e^t \sin x$
(C) $x^2 - t$ (D) $\frac{e^{-x^2/4t}}{\sqrt{t}}$

193. Which of the following is elliptic ?
(A) Laplace equation
(B) Wave equation
(C) Heat equation
(D) $u_{xx} + 2u_{xy} - 4u_{yy} = 0$

194. Let $X = (0,1) \cup (2, 3)$ be an open set in R. Let f be a continuous function on X such that the derivative $f'(x) = 0$ for all x. the range of f has—
(A) Uncountable number of points
(B) Countably infinite number of points
(C) At most 2 points
(D) At most 1 point

195. The orthogonal trajectory to the family of circles $x^2 + y^2 = 2\,cx$ (c arbitrary) is described by the differential equation—

(A) $(x^2 + y^2)\,y' = 2xy$

(B) $(x^2 - y^2)\,y' = 2xy$

(C) $(y^2 - x^2)\,y' = xy$

(D) $(y^2 - x^2)\,y' = 2xy$

196. Pick the region in which the following differential equation is hyperbolic $y^{u_{xx}} + 2xyu_{xy} + xu_{xy} = u_x + u_y$—

(A) $xy \neq 1$ (B) $xy \neq 0$

(C) $xy > 1$ (D) $xy > 0$

197. Extremals $y = y\,(x)$ for the variational problem $u\,[y\,(x)] = \int_0^1 (y + y')^2\,dx$ satisfy the differential equation—

(A) $y'' + y = 0$ (B) $y'' - y = 0$

(C) $y'' + y' = 0$ (D) $y' + y = 0$

198. In a sufficiently small neighbourhood around $x = 2$, the differential equation $\frac{dy}{dx} = \frac{y}{\sqrt{x}}$, $y\,(2) = 4$ has—

(A) No solution

(B) A unique solution

(C) Exactly two solutions

(D) Infinitely many solutions

199. The set of linearly independent solutions of the differential equation $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0$ is—

(A) $\{1, x, e^x, e^{-x}\}$ (B) $\{1, x, e^{-x}, xe^{-x}\}$

(C) $\{1, x, e^x, xe^x\}$ (D) $\{1, x, e^x, xe^{-x}\}$

200. For the differential equation

$$x^2\,(1 - x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0\text{—}$$

(A) $x = 1$ is an ordinary point

(B) $x = 1$ is a regular singular point

(C) $x = 0$ is an irregular singular point

(D) $x = 0$ is an ordinary point

201. The general solution of the system of differential equations

$$y + \frac{dz}{dx} = 0,$$

$$\frac{dy}{dx} - z = 0$$

is given by—

(A) $y = \alpha e^x + \beta e^{-x}$
$z = \alpha e^x - \beta^{-x}$

(B) $y = \alpha \cos x + \beta \sin x$
$z = \alpha \sin x - \beta \cos x$

(C) $y = \alpha \sin x - \beta \cos x$
$z = \alpha \cos x + \beta \sin x$

(D) $y = \alpha e^x - \beta e^{-x}$
$z = \alpha e^x + \beta e^{-x}$

202. It is required to find the solution to the differential equation $2x\,(2 + x)\frac{d^2x}{dx^2} + 2(3 + x)\frac{dy}{dx} - xy = 0$ around the point $x = 0$. The roots of the indicial equation are—

(A) $0, \frac{1}{2}$ (B) $0, 2$

(C) $\frac{1}{2}, \frac{1}{2}$ (D) $0, \frac{1}{2}$

203. A function $u\,(x,\ t)$ satisfies the wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, $0 < x < 1$, $t > 0$. If $u\left[\frac{1}{2}, 0\right] = \frac{1}{4}$, $u\left[1, \frac{1}{2}\right] = 1$ and $u\left[0, \frac{1}{2}\right] = \frac{1}{2}$, then $u\left[\frac{1}{2}, 1\right]$ is—

(A) $\frac{1}{4}$ (B) $\frac{5}{4}$

(C) $\frac{4}{5}$ (D) $\frac{4}{7}$

204. The Fourier transform $f\,(\omega)$ of $f\,(x)$ $-\infty < x < \infty$ is defined by $f\,(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f\,(x)\,e^{-iwx}\,dx$ the Fourier transform with respect to x of the solution $u\,(x,\ y)$ of the boundary value problem $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $-\infty < x < \infty$, $y > 0$, $u\,(x,\ 0) = f\,(x)$, $-\infty < x < \infty$ which remains bounded for large y is given by $U\,(\omega,\ y) = f\,(\omega)\,e^{-|w|y}$ then, the solution $u\,(x, y)$ is given by—

(A) $u\,(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f\,(x - z)}{y^2 + z^2}\,dz$

(B) $u\,(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f\,(x + z)}{y^2 + z^2}\,dz$

(C) $u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-z)}{y^2 + z^2}\, dz$

(D) $u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x+z)}{y^2 + z^2}\, dz$

205. It is required to solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $0 < x < a$, $0 < y < b$, satisfying the boundary conditions $u(x, 0) = 0$, $u(x, b) = 0$, $u(0, y) = 0$ and $u(a, y) = f(y)$. If C_n's are constants, then the equation and the homogeneous boundary conditions determine the fundamental set of solutions of the form—

(A) $u(x, y) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi x}{b} \sin\frac{n\pi x}{b}$

(B) $u(x, y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{b} \sin\frac{n\pi y}{b}$

(C) $u(x, y) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{b} \sinh\frac{n\pi y}{b}$

(D) $u(x, y) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi x}{b} \sinh\frac{n\pi y}{b}$

Answers with Explanation

1. (B)	2. (B)	3. (C)	4. (B)	5. (C)
6. (B)	7. (D)	8. (A)	9. (A)	10. (B)
11. (A)	12. (A)	13. (C)	14. (B)	15. (D)
16. (A)	17. (A)	18. (D)	19. (A)	20. (A)
21. (C)	22. (B)	23. (A)	24. (A)	25. (D)
26. (C)	27. (D)	28. (C)		

29. (A) $(y \cos x + \sin y + y)\, dx + (\sin x + x \cos y + x)\, dy = 0$

$$\frac{\delta M}{\delta y} = \cos x + \cos y + 1 = \frac{\delta N}{\delta x}$$

The equation is exact and solution is

$\int (y \cos x + \sin y + y)\, dx + f(0)\, dx = c$

30. (A) Characteristic equation

$$m^2 + m - 2 = 0$$
$$\Rightarrow \quad (m+2)(m-1) = 0$$
$$\Rightarrow \quad m = -2, 1$$
$$\Rightarrow \quad y = c_1 e^{-2x} + c_2 e^{-x}$$

31. (C) Charactristic equation

$$m^3 + m^2 + um + u = 0$$
$$\Rightarrow \quad (m^2 + u)(m + 1) = 0,$$
$$m = -1 \pm 2i$$
$$\therefore \quad y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$$

32. (B) $y_p = \frac{1}{D^3 + 1} \cos(2x - 1)$

(Put $D^2 = -2^2 = -4$)

$$= \frac{1}{D(-4) + 1} \cos(2x - 1)$$
$$= \frac{1 + 4D}{1 - 16D^2} \cos(2x - 1)$$

(Put $D^2 = -2^2 = -4$)

$$= \frac{1}{65}[\cos(2x - 1) + 4D \cos(2x - 1)]$$

or $y_p = \frac{1}{65}[\cos(2x - 1) - 8 \sin(2x - 1)]$

33. (B)	34. (A)	35. (A)	36. (A)	37. (B)
38. (C)	39. (A)	40. (B)	41. (B)	42. (A)
43. (A)				

44. (B) $\frac{d^2y}{dx^2} + 4y = \tan x$

Corresponding homogeneous equation

$$\frac{d^2y}{dx^2} + 4y = 0$$
$$\Rightarrow \quad m^2 + 4 = 0$$
$$\therefore \quad m = \pm 2i$$
$$\therefore \quad y_1 = c_1 \cos 2x + c_2 \sin 2x$$
$$\therefore \quad w = \begin{vmatrix} y_1 & y_2 \\ y^1{}_1 & y^1{}_2 \end{vmatrix}$$
$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

45. (A) The corresponding homogeneous equation is

$$y'' - 2y' + y = 0$$
$$\Rightarrow \quad m^2 - 2m + 1 = 0$$
$$\Rightarrow \quad m = 1, 1$$
$$\therefore \quad y_n = (c_1 + c_2 x) e^x$$

Then $w = \begin{vmatrix} y_1 & y_2 \\ y^1{}_1 & y^1{}_2 \end{vmatrix}$

$$= \begin{vmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{vmatrix}$$
$$= e^{2x}$$

46. (C) $w(x, x^2, x^3) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3$

47. (A) The solution of corresponding homogeneous equation is

$$y = c_1 \cos x + c_2 \sin x$$

$$\therefore \quad w(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

48. (D) $(D^2 - 5DD' + 4D'^2)z = \sin(4x + y)$

Alternate equation

$$m^2 - 5m + 4 = 0$$

$$\Rightarrow \quad (m-1)(m-4) = 0$$

$$\Rightarrow \quad m = 1, 4$$

$$z_n = f_1(y + x) + f_2(y + 4x)$$

$$z_p = \frac{1}{D^2 - 5DD' + 4D'^2} \sin(4x + y)$$

$$= \frac{1}{(D - 4D')(D - D')} \sin(4x + y)$$

$$= \frac{1}{(D - 4D')} \cdot \frac{1}{4-1} \{-\cos(4x + y)\}$$

$$= \frac{1}{(D - 4D')} \left\{-\frac{1}{3} \cos(4x + y)\right\}$$

$$= -\frac{x}{3} \cos(4x + y)$$

$$\therefore \quad z = z_n + z_p$$

$$= f_1(y + x) + f_2(y + 4x) - \frac{1}{3} x \cos(4x + y)$$

49. (C) $(D^3 - 4D^2D' + 4DD'^2) = 0$

Alternate equation

$$m^3 - 4m^2 + 4m = 0$$

$$\Rightarrow \quad m(m-2)^2 = 0$$

$$\Rightarrow \quad m = 0, 2, 2$$

$$z = \phi(y) + f_1(y + 2x') + xf_2(y + 2x)$$

50. (B) $(D^2 + D'^2)z = 12(x + y)$

Alternate equation

$$m^2 + 1 = 0, m \pm i$$

$$z_n = f_1(y + ix) + f_2(y - ix)$$

$$z_p = \frac{1}{(D^2 + D'^2)} 12(x + y)$$

$$= \frac{1}{1^2 + 1^2} 12 \frac{(x + y)^3}{6}$$

$$= (x + y)^3$$

$$\therefore \quad z = z_n + z_p = f_1(y + ix) + f_2(y - ix) + (x + y)^3$$

51. (D) $(D^2 - D'^2 + D - D')z = 0$

$$\Rightarrow \quad (D - D')(D + D' + 1) = 0$$

$$z = f_1(y + x) + e^{-x} f_2(y - x)$$

52. (D) $DD'(D - 2D' - 3)z = 0$

$$z = e^{0.x} f_1(y) + e^{0.x} f_2(0 - x) + e^{3x} f_3(y + 3x)$$

$$= f_1(y) + f_2(-x) + e^{3x} f_3(y + 3x)$$

or $z = f_1(y) + f_2(x) + e^{3x} f_3(y + 3x)$

53. (A) $(D^2 - DD' + D)z = 1$

$$\Rightarrow \quad D(D - D' + 1)z = 1$$

$$z_n = e^{0.x} f_1(y) + e^{-x} f_2(y + x)$$

$$= f_1(y) + e^{-x} f_2(y + x)$$

$$z_p = \frac{1}{D(D - D' + 1)} \cdot 1$$

$$= \frac{1}{D}(1 + D - D')^{-1} \cdot 1$$

$$= \frac{1}{D} \cdot 1 = x$$

$$\therefore \quad z = z_n + z_p = f_1(y) + e^{-x} f_2(y + x) + x$$

54. (B) $(D^2 + 2DD' + D'^2 + 2D + 2D' + 1)z = 0$

$$\Rightarrow \quad (D + D' + 1)z = 0$$

$$z = e^{-x} f_1(y - x) + xe^{-x} f_2(y - x)$$

55. (A) Lagrange's subsidiary equation

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z/a}$$

$$\Rightarrow \quad x - y = c_1$$

and $\quad z = c_2 e^{y/a}$

$$\therefore \quad z = e^{y/a} f(x - y)$$

56. (C) By Lagrange's subsidiary equation

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

$\Rightarrow$ First and second $\quad x^2 - y^2 = c_1$

First and last $\quad x^2 - y^2 = c_2$

$$\therefore \quad f(x^2 - y^2, x^2 - z^2) = 0$$

57. (B) The equation is of the form $f(p, q) = 0$

General integral is

$$z = ax + by + c$$

where $b = 3a^2 \quad \therefore \quad z = ax + 3a^2y + c$

58. (A) Let $\quad p^2 - x = y - q^2 = a$

$$\therefore \quad p = \sqrt{(x + a)}$$

$$q = \sqrt{(y - a)}$$

Substituting in $dz = pdx + qdy$ and integrating

59. (C) $pe^{-x} = qe^{-y} = a$ (say)

and $dz = pdx + qdy$

$\Rightarrow \quad dz = ae^x dx + ae^y dy$

60. (A) The Charpit's equations are

$$\frac{dp}{-p+p} = \frac{dq}{-q+q} = \frac{dz}{-p(-x-2p)-q(-y-2p)} = \frac{dx}{+(x+2p)} = \frac{dy}{-y-2q} = \frac{dF}{0}$$

First and second $dp = 0$, $dq = 0$, gives $p = a$, $q = b$

and $z = ax + by + a^2 + b^2$

61. (C) Rewriting the equation

$$\frac{dy}{dx} = \frac{y\sin(y/x) - x}{x\sin(y/x)}$$

Substitute $y/x = 0$

We have, $v + x\dfrac{dv}{dx} = \dfrac{v\sin v - 1}{\sin v}$

$$= v - \operatorname{cosec} v$$

$\Rightarrow \quad -\sin v\,dv = \dfrac{dx}{x}$

$\Rightarrow \quad \cos y/x = \log x + c$

$\Rightarrow \quad \cos\left(\dfrac{y}{x}\right) - \log x = c$

62. (C) The given equation can be rewritten as

$$(3x^2y^4 + 2xy) + (2x^3y^3 - x^2)\frac{dy}{dx} = 0$$

Dividing by y^2, we get

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^2y - \frac{x^2}{y^2}\right)dy = 0$$

$\Rightarrow \quad d\left(x^3y^2 + \dfrac{x^2}{y}\right) = 0$

$\Rightarrow \quad x^3y^2 + \dfrac{x^2}{y} = \text{constant}$

63. (A) $y = Ae^{3x} + Be^{5x}$

$$\frac{dy}{dx} = 3Ae^{3x} + 5Be^{5x} = 3y + 2Be^{5x}$$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} = 3\dfrac{dy}{dx} + 10Be^{5x} = 3\dfrac{dy}{dx} + 5\left(\dfrac{dy}{dx} - 3y\right)$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 15y = 0$

64. (B) $y\dfrac{dy}{dx} = a$

$\Rightarrow \quad ydy = adx$

$\Rightarrow \quad \dfrac{y^2}{2} = ax + b$

$\Rightarrow \quad y^2 = 2ax + 2b$

65. (A)	66. (A)	67. (A)	68. (A)	69. (A)
70. (A)	71. (A)	72. (C)	73. (A)	74. (B)
75. (B)	76. (A)	77. (A)	78. (B)	79. (A)
80. (A)	81. (B)	82. (A)	83. (B)	84. (B)
85. (A)	86. (B)	87. (A)	88. (A)	89. (A)
90. (C)	91. (B)	92. (D)	93. (B)	94. (A)
95. (A)	96. (C)	97. (C)	98. (C)	99. (A)
100. (C)	101. (A)	102. (A)	103. (A)	

104. (B) Only the equation in (B) possesses $\dfrac{dy}{dx}$, hence (B) is ordinary differential equation.

105. (C) Rationalizing the degree of differential equation (d.e.), we see that $y + x)\left(\dfrac{d^2y}{dx^2}\right)^2 = \left(\dfrac{d^3y}{dx^3}\right)^4$. Hence the degree of given d.e. is 4.

106. (C)

107. (D) The order of a differential equation can not be negative integer.

108. (B) Here order = 2 and degree = 3

$\Rightarrow \quad m = 2$

and $\quad n = 3$

$\Rightarrow \quad m < n$

109. (B) The given equation can be written as

$$\frac{dx}{x} - \frac{dy}{y} = dx$$

$\Rightarrow \quad \left(\dfrac{1}{x} - 1\right)dx = \dfrac{dy}{y}$

$\Rightarrow \quad (\log x - x) = \log y - \log A$

$\Rightarrow \quad x = \log(Ax/y)$

$\Rightarrow \quad \dfrac{Ax}{y} = e^x$

$\Rightarrow \quad Ax = ye^x$

110. (C) $\displaystyle\int \frac{dy}{1+y} + \int \frac{dx}{1+x} = \log c$

$\Rightarrow \quad \log(1+y) + \log(1+x) = \log c$

$\Rightarrow \quad \log\{(1+x)(1+y)\} = \log c$

$\Rightarrow \quad (1+x)(1+y) = c$

111. (B) The differential equation (d. e.) in (B) is linear as the shape of linear differential equation is

$$\frac{dy}{dx} + py = Q$$

Where P and Q are functions of x.

112. (C)

113. (A) Given equation can be written as

$$\frac{dy}{dx} = \frac{y+x}{y-x}$$

Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{vx+x}{vx-x}$$

$$x\frac{dv}{dx} = \frac{-v^2+2v+1}{v-1}$$

$$\Rightarrow \quad \left(\frac{v-1}{1+2v-v^2}\right) dv = \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2}\log(1+2v-v^2) = \log x + \log c$$

$$\Rightarrow \quad \log x\sqrt{1+2v-v^2} = \log c$$

$$\Rightarrow \quad x^2 + 2xy - y^2 = c$$

114. (C) Here $M = a^2 - 2xy - y^2$

$$N = -(x+y)^2$$

$$\therefore \quad \frac{\delta M}{\delta y} = -2x - 2y,$$

$$\frac{\delta N}{\delta x} = -2(x+y)$$

$$\therefore \quad \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

Hence given differential equation is exact.

$\therefore$ Required solution is

$$\int (a^2 - 2xy - y^2)\,dx + \int -y^2 dy = c$$

$$\Rightarrow \quad a^2x - x^2y - y^2x - \frac{1}{3}y^3 = c$$

115. (C) $\quad$ I. F. $= e^{\int \frac{-x}{1-x^2}dx}$

$$= e^{\frac{1}{2}\log(1-x^2)}$$

$$= \sqrt{1-x^2}$$

116. (A) The given equation can be written as

$$e^y dy = (e^x + x^2)\,dx$$

Integrating we get

$$e^y = e^x + \frac{1}{3}x^3 + c$$

which is the required solution.

117. (B)

118. (A) Let $\quad x + y = v$

$$\Rightarrow \quad 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad 1 + \frac{a^2}{v^2} = \frac{dv}{dx}$$

$$\therefore \quad \int \frac{v^2}{a^2+v^2}dv = x + c$$

$$\text{or} \quad \int \left(1 - \frac{a^2}{a^2+v^2}\right) dv = x + c$$

$$\text{or} \quad v - \frac{a^2}{a}\tan^{-1}\frac{v}{a} = x + c$$

$$\text{or} \quad x + y - a\tan^{-1}\frac{x+y}{a} = x + c$$

$$\text{or} \quad y - a\tan^{-1}\frac{x-y}{a} = c$$

$$\text{or} \quad y + x = a\tan\left(\frac{y-c}{a}\right)$$

119. (B) Put $x + y + 1 = v$

$$\Rightarrow \quad 1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \quad 1 + \frac{1}{v} = \frac{dv}{dx}$$

$$\therefore \quad \int \frac{v}{v+1}dv = \int dx + c$$

$$\Rightarrow \quad \int \left(1 - \frac{1}{v+1}\right) dv = x + c$$

$$\text{or} \quad v - \log(v+1) = x + c$$

$$\text{or} \quad \log(v+1) = v - x - c$$

$$\therefore \quad \log(x+y+2) = y + 1 - c$$

120. (B) The given equation is

$$\frac{dx}{x} - 2\frac{dy}{y} = 0$$

$$\Rightarrow \quad \log x - 2\log y = \log c$$

$$\Rightarrow \quad \log\frac{x}{y^2} = \log c$$

$$\Rightarrow \quad x = cy^2$$

which represent a family of parabolas.

121. (A) The given equation is

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{1+y^2}dx}$$

$$= e^{\tan^{-1}y}$$

122. (A) $\quad$ 123. (D)

124. (C) Given that : $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

This is homogeneous equation putting $y = vx$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{x+vx}{x-vx}$$

$$\text{or} \quad \left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right) dv = \frac{dx}{x}$$

Integrating, we get

$$\tan^{-1} v = \frac{1}{2}\log(1+v^2)$$
$$= \log x + k$$

$$\text{or} \quad 2\tan^{-1}\frac{y}{x} = \log(y^2+x^2) + 2k\,(=c)$$

125. (C) $pdx + x\sin y \cdot dy = 0$

Here M = P, N = $x \sin y$

$$\text{For exact,} \quad \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

$$\Rightarrow \quad \frac{\delta P}{\delta y} = \sin y$$

$$\Rightarrow \quad P = -\cos y + f(x)$$

$$\text{Let} \quad f(x) = x^2$$

$$\text{then} \quad P = -\cos y + x^2$$

126. (D) $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

This is homogeneous and

$$Mx + Ny = (x^3y - 2x^2y^2) - (x^3y - 3x^2y^2)$$
$$= x^2y^2 \neq 0$$

$$\therefore \quad \text{I.F.} = \frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$$

127. (B)

128. (C) $\dfrac{xdx + ydy}{xdy - ydx} = \sqrt{\dfrac{a^2 - x^2 - y^2}{x^2+y^2}}$

$$\text{Put} \quad x = r\cos\theta$$
$$y = r\sin\theta$$
$$\Rightarrow \quad r^2 = x^2 + y^2$$
$$\frac{y}{x} = \tan\theta$$

Differentiating these relation, we get

$$rdr = xdx + ydy$$

$$\text{and} \quad \frac{-ydx + xdy}{x^2} = \sec^2\theta\, d\theta$$

$$\Rightarrow \quad -ydx + xdy = r^2\cos^2\theta\sec^2\theta\, d\theta$$
$$= r^2 d\theta$$

$\therefore$ Given problem takes the form

$$\frac{rdr}{r^2 d\theta} = \sqrt{\frac{a^2 - r^2}{r^2}}$$

$$\int \frac{1}{\sqrt{a^2 - r^2}}\,dr = \int d\theta + c$$

$$\therefore \quad r = a\sin(\theta + c)$$

$$\text{or} \quad \sqrt{x^2+y^2} = a\sin\left(\tan^{-1}\frac{y}{x} + c\right)$$

129. (B) The given equation can be written as

$$(x - y^2) + 2xy\frac{dy}{dx} = 0$$

$$\text{Let} \quad x - y^2 = v$$

$$\Rightarrow \quad 1 - 2y\frac{dv}{dx} = \frac{dv}{dx}$$

$$\therefore \quad v + x\left(1 - \frac{dv}{dx}\right) = 0$$

$$\text{or} \quad \frac{dv}{dx} - \frac{v}{x} = 1$$

which is linear equation

$$\therefore \quad \text{I. F.} = e^{\int -1/x\, dx}$$
$$= e^{-\log x}$$
$$= 1/x$$

$$\text{Solution is} \quad v \cdot \frac{1}{x} = \int 1 \cdot \frac{1}{x}\,dx + \log c$$

$$\text{or} \quad v \cdot \frac{1}{x} = \log x + \log c$$

$$\text{or} \quad cx = e^{v/x}$$
$$cx = e^{(x - y^2)/x}$$
$$\text{or} \quad cx = e^{1 - y^2/x}$$
$$\text{or} \quad cx = e \cdot e^{-y^2/x}$$
$$\text{or} \quad xe^{y^2/x} = e/c = A \text{ (say)}$$

130. (C) The given equation is

$$(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0$$

$M = xy^3 + y$, $N = 2\,(x^2y^2 + x + y^4)$

$$\frac{\delta M}{\delta y} = 3xy^2 + 1, \quad \frac{\delta N}{\delta x} = 4xy^2 + 2$$

$$\frac{1}{M}\left(\frac{\delta N}{\delta x} - \frac{\delta M}{\delta y}\right) = \frac{1}{(xy^2+1)\,y}(xy^2+1)$$
$$= \frac{1}{y}$$

$$\therefore \quad \text{I. F.} = e^{\int 1/y\, dy} = e^{\log y} = y$$

Multiplying by y in the given equation

$$(xy^4 + y^2)\,dx + 2\,(x^2y^3 + xy + y^5)\,dy = 0$$

$\therefore$ Solution is $\int (xy^4 + y^2)\,dx + \int y^5 \cdot dy = c$

$$\text{or} \quad \frac{x^2}{2}y^4 + xy^2 + \frac{1}{6}y^6 = c$$

$$\text{or} \quad 3x^2y^4 + 6xy^2 + y^6 = c$$

131. (B) The given equation is $x^2y dx - (x^3 + y^3) dy = 0$

Here
$$M = x^2y$$
$$N = -(x^3 + y^3)$$

The given equation is homogeneous

$$\therefore \quad Mx + Ny = x^3y - (x^3y + y^4) = -y^4 \neq 0$$

$$\therefore \quad I.F. = \frac{1}{Mx + Ny} = -\frac{1}{y^4}$$

Multiplying by I. F. in the given equation

$$\frac{x^2}{y^3} dx - \left(\frac{x^3}{y^4} + \frac{1}{y}\right) dy = 0 \qquad \ldots(1)$$

Solution is

$$\int \frac{x^2}{y^3} dx - \int \frac{1}{y} dx = \log k$$

or $\dfrac{x^3}{3y^3} - \log y = \log k$

or $y \cdot k = e^{x^3/3y^3}$

or $y^3k^3 = e^{x^3/y^3}$

or $y^3 = \dfrac{1}{k^3} e^{x^3/y^3}$

or $y^3 = c \cdot e^{x^3/y^3}$

132. (D) The given equation can be written as

$$(1 - xy)\, y dx - (1 + xy)\, x dy = 0$$

This is the form

$$f_1(x, y)\, y dx + f_2(x, y)\, x dy = 0$$

$$\therefore \quad Mx - Ny = (xy - x^2y^2) - (xy + x^2y^2) = -2x^2y^2$$

$$\therefore \quad I.F. = \frac{1}{Mx - Ny} = \frac{1}{-2x^2y^2} \text{ or } \frac{1}{-x^2y^2}$$

Multiplying by $\dfrac{1}{-x^2y^2}$ in the given equation we get

$$\left(\frac{-1}{x^2y} + \frac{1}{x}\right) dx - \left(\frac{1}{-xy^2} - \frac{1}{y}\right) dy = 0 \ldots(1)$$

Solution is

$$\int \left(\frac{1}{x^2y} - \frac{1}{x}\right) dx - \int \frac{1}{y} dy = \log k$$

or $-\dfrac{1}{xy} - \log x = \log k$

or $xyk = e^{-1/xy}$

or $xye^{1/xy} = \dfrac{1}{k}$

or $xye^{1/xy} = c$

133. (B) The given equation is

$$(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$$

$$M = 1 + e^{x/y}$$

$$N = e^{x/y}\left(1 - \frac{x}{y}\right)$$

$$\frac{\delta M}{\delta y} = e^{x/y}\left(-\frac{x}{y^2}\right)$$

$$\frac{\delta N}{\delta x} = e^{x/y}\left(-\frac{1}{y}\right) + \left(1 - \frac{x}{y}\right) e^{x/y}\left(\frac{1}{y}\right) = e^{x/y}\left(-\frac{x}{y^2}\right)$$

$$\therefore \quad \frac{\delta M}{\delta y} = \frac{\delta N}{\delta x}$$

Hence the given equation is exact.

$\therefore$ Solution is $\int (1 + e^{x/y})\, dx + \int 0 \cdot dy = c$

or $x + e^{x/y} \cdot y = c$

or $x + ye^{x/y} = c$

134. (A) The given equation is

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \qquad \ldots(1)$$

Put $y = vx$

$\Rightarrow \dfrac{dy}{dx} = v + x\left(\dfrac{dv}{dx}\right)$

$\Rightarrow v + x\dfrac{dv}{dx} = 1 + \dfrac{v^2}{2v}$

$\Rightarrow x\dfrac{dy}{dx} = \dfrac{1 - v^2}{2v}$

$\Rightarrow \left(\dfrac{2v}{1 - v^2}\right) dv = \dfrac{dx}{x}$

$\Rightarrow -\log(1 - v^2) = \log x - \log c$

$\Rightarrow \log x(1 - v^2) = \log c$

$\Rightarrow x^2 - y^2 = cx$

135. (C) The given equation can be written as

$$y^{-2}\frac{dy}{dx} + 2xy^{-1} = 2x$$

Let $y^{-1} = v$

$\Rightarrow -y^{-2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$

$\therefore -\dfrac{dv}{dx} + 2xv = 2x$

or $\dfrac{dv}{dx} - 2xv = -2x$

which is linear equation

$$I.F. = e^{\int -2x\, dx} = e^{-x^2}$$

Solution is $v \cdot e^{-x^2} = \int -2xe^{-x^2}\,dx + c$

or $v \cdot e^{-x^2} = e^{-x^2} + c$

or $\frac{1}{y} = 1 + ce^{x^2}$

or $y = \frac{1}{1+ce^{x^2}}$

136. (C) $f'(x) = \tan^{-1}(x)$

$$f(x) = \tan^{-1} x\,dx$$

$$= x\tan^{-1} x - \int \frac{1}{1+x^2}\,dx$$

$$f(x) = x\tan^{-1} x - \frac{1}{2}\log(1+x^2) + c$$

Put $x = 0$

Here $f(0) = 0 + c$

$\Rightarrow$ $c = 0$

$\therefore$ $f(x) = x\tan^{-1} x - \frac{1}{2}\log(1+x^2)$

$$f(1) = \frac{\pi}{4} - \frac{1}{2}\log 2$$

$$= \frac{\pi}{4} - \frac{1}{4}\log 4$$

$$= \frac{1}{4}(\pi - \log 4)$$

137. (B) Variable separable

$$3\frac{e^x}{1-e^x}\,dx = -\int \frac{\sec^2 y}{\tan y}\,dy$$

$$-3\log(1-e^x) = -\log\tan y + k$$

$$\log\tan y\,(1-e^x)^{-3} = k = \log c$$

$\therefore$ $\tan y\,(1-e^x)^{-3} = c$

138. (C) $\frac{dx}{dy} = -\frac{3x - 10y^3}{y}$

$$\frac{dx}{dy} + \frac{3}{y}x = 10y^2$$

$$\text{I. F.} = e^{3\int 1/y\,dy}$$

$$= e^{3\log y} = y^3$$

139. (C) $\frac{dy}{dx} - \frac{x-2}{x(x-1)}y = \frac{x^2(2x-1)}{x-1}$

Here $P = -\frac{x-2}{x(x-1)}$

$$= -\frac{2}{x} + \frac{1}{x-1}$$

$$e^{\int P dx} = e^{-2\log x + \log(x-1)}$$

$$= e^{\log\frac{x-1}{x-2}}$$

$$= \frac{x-1}{x^2}$$

140. (B) Divide the differential equation by $z(\log z)^3$, we get

$$\frac{1}{z(\log z)^3}\frac{dz}{dx} + \frac{2x}{x^2+1}\cdot\frac{1}{(\log z)^2} = \frac{x}{x^2+1}$$

Put $y = \frac{1}{(\log z)^2}\ \frac{dy}{dx} = -\frac{2}{(\log z)^3}\cdot\frac{1}{z}\cdot\frac{dz}{dx}$

$\therefore$ $-\frac{1}{z(\log z)^3}\frac{dz}{dx} = \frac{1}{2}\cdot\frac{dy}{dx}$

$\therefore$ Differential equation becomes

$$\frac{1}{2}\frac{dy}{dx} - \frac{2x}{x^2+1}\cdot y = -\frac{x}{x^2+1}$$

or $\frac{dy}{dx} - \frac{4x}{x^2+1}\cdot y = -\frac{2x}{x^2+1}$

which is a linear equation

$\therefore$ Transform required is

$$y = \frac{1}{(\log z)^2}$$

141. (C) The given equation is

$$Px + 2y = Pxy$$

or $Px = \frac{2y}{y-1}$

or $\frac{dy}{dx}x = \frac{2y}{y-1}$

or $2\frac{dx}{x} = \left(\frac{y-1}{y}\right)dy$

Integrating, we get

$$2\log x = (y - \log y) + \log A$$

or $\log\frac{x^2y}{A} = y$

or $\log\frac{x^2y}{A} = y$

or $x^2y = Ae^y$

142. (D) The given equation is

$$y = Px + f(P)$$

This is Clairaut's equation

$\therefore$ Its solution is

$$y = cx + f(c)$$

143. (A) Given equation is

$$y^2 = 2cx + c^2 \qquad \ldots(1)$$

$\Rightarrow$ $2y\frac{dy}{dx} = 2c$

$\Rightarrow$ $y\frac{dy}{dx} = c$

Substituting the value of c in (1), we get

$$y\left(\frac{dy}{dx}\right)^2 + 2x\left(\frac{dy}{dx}\right) - y = 0$$

which the required differential equation

144. (A) Equation is $P^2 + 2Py\cot x - y^2 = 0$

$$\therefore \quad P = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

or $\quad P = -y\cot x \pm y\,\text{cosec}\, x$

$\therefore \quad P = y\,\text{cosec}\, x - y\cot x \quad \ldots(1)$

and $\quad P = -y(\text{cosec}\, x + \cot x) \quad \ldots(2)$

From (1), $\frac{dy}{y} = (\text{cosec}\, x - \cot x)\,dx$

Integrating, we get

$$\log y = \log\tan\frac{x}{2} - \log\sin x + \log c$$
$$= -\log(1 + \cos x) + \log c$$

$\therefore \quad y(1 + \cos x) = c$

Similarly the solution of (2) is

$y(1 - \cos x) = c$

Hence the required solution of the given equation

$y(1 \pm \cos x) = c$

145. (A) The given equations can be written as

$(xP - y)(yP - x) = 0$

i.e. $\quad xP = y$

and $\quad yP = x$

or $\quad \frac{dy}{y} = \frac{dx}{x}$

and $\quad ydy = x\cdot dx$

Integrating these equation we get

$\log y = \log x + \log c$

and $\quad \frac{y^2}{2} = \frac{x^2}{2} + c$

or $\quad y = cx$

and $\quad x^2 - y^2 = c$

$\therefore$ Hence the required solution is

$(y - cx)(x^2 - y^2 - c) = 0$

146. (A) Equation is $x^2P^2 + xP(2x + y) + y^2 = 0$

Let $y = u$, and $xy = v$, or $y = u$ and $x = \frac{v}{y} = \frac{v}{u} \quad \ldots(1)$

$\therefore \quad dy = du$

and $\quad dx = \frac{(udv - vdu)}{u^2} \quad \ldots(2)$

$\therefore \quad P = \frac{dy}{dx} = \frac{u^2du}{udv - vdu}$

Substituting the values of x, y and P from (1) and (2) in the original equation

$$\frac{v^2}{u^2}\left(\frac{u^2du}{udv - vdu}\right)^2 + u\left(\frac{u^2du}{udv - vdu}\right)\left(\frac{2v}{u} + u\right) + u^2 = 0$$

or $u^2v^2(du)^2 + u^2(2v + u^2)\,du\,(udv - vdu) + u^2(udv - vdu)^2 = 0$

or $[u^2v^2 - vu^2(2v + u^2) + u^2v^2](du)^2 + [u^2(2v + u^2) - 2u^2v]\,du\cdot dv + u^4(dv)^2 = 0$

or $\quad -v(du)^2 + ududv + (dv)^2 = 0$

or $\quad v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$

or $\quad v = uP + P^2$

where $\left(P = \frac{dv}{du}\right)$. This equation is Clairaut's form

147. (B) The given equation is solvable for x and can be written as

$$2x = P^2y^2 + \frac{y}{P}$$

Differentiating with respect to y, we get

$$2\frac{dx}{dy} = P^2\cdot 2y + y^2\cdot 2P\frac{dP}{dy} + \frac{P\cdot 1 - y\frac{dP}{dy}}{P^2}$$

or $\quad \frac{2}{P} = 2yP^2 + \left(2Py^2 - \frac{y}{P^2}\right)\frac{dP}{dy} + \frac{1}{P}$

or $\quad \left(\frac{1}{P} - 2yP^2\right) - \left(2Py^2 - \frac{y}{P^2}\right)\frac{dP}{dy} = 0$

$$\frac{(1 - 2yP^3)}{P} + y\frac{(1 - 2yP^3)}{P^2}\frac{dP}{dy} = 0$$

$\Rightarrow \quad 1 + \frac{y}{P}\frac{dP}{dy} = 0$

$\therefore \quad \frac{dP}{P} + \frac{dy}{y} = 0$

Integrating, we get

$\log P + \log y = \log c$

or $\quad Py = c$

$\Rightarrow \quad P = c/y$

Substituting the value of P in the original equation, we get

$$\frac{c^3}{y} - 2\frac{c}{y}x + y = 0$$

or $\quad 2cx = c^3 + y^2$

148. (C) Equation $y = 2Px = y^{n-1}P^n$ is solvable for x and can be written as $2x = y/P - y^{n-1}P^{n-1}$

Differentiating with respect to y, we get

$$2\frac{dx}{dy} = \frac{P\cdot - y\cdot\frac{dP}{dy}}{P^2} - (y^{n-1}(n-1)\cdot P^{n-2}\,dP/dy + P^{n-1}(n-1)\,y^{n-2})$$

or $\frac{2}{P} = \frac{1}{P} - (n-1)P^{n-1}y^{n-2}$

$$-\left(\frac{y}{P^2} + (n-1)\,y^{n-1}P^{n-2}\right)\frac{dP}{dy}$$

or $\left(\frac{1+(n-1)P^n y^{n-2}}{P}\right)$

$$+\,y\left(\frac{1+(n-1)\,y^{n-2}P^n}{P^2}\right)\frac{dP}{dy} = 0$$

$\Rightarrow \quad 1 + \frac{y}{P}\frac{dP}{dy} = 0$

or $\quad \frac{dP}{P} + \frac{dy}{y} = 0$

Integrating, $\log P + \log y = \log c$

or $\quad P = \frac{c}{y}$

Substituting the value of P in the original equation

$$y = 2\frac{c}{y}x + y^{n-1}\frac{c^n}{y^n}$$

or $\quad y^2 = 2cx + c^n$

149. (D) The given equation is $y = Px + P^2$. which is the Clairaut's form

$\therefore$ Required solution is

$$y = cx + c^2$$

150. (A) The given equation is $P = \log(Px - y)$

$\Rightarrow \quad e^P = Px - y$

$\Rightarrow \quad y = Px - e^P$

which is the Clairaut's form

$\therefore$ Solution is $\quad y = cx - e^c$

151. (A) The given equation can be written as

$$(1+P)\,y = (P+P^2)\,x + aP^2$$

$\Rightarrow \quad y = Px + \frac{aP^2}{P+1}$

which is a Clairaut's form

$\therefore$ Solution is $y = cx + \frac{ac^2}{c+1}$

152. (D) Differentiating w.r. to x, we get

$dy/dx - 2P - 2x\,dP/dx = f'(xP^2)$

$$\left\{2Px\frac{dP}{dx} + P^2\right\}$$

$\Rightarrow \quad -\left(P + 2x\frac{dP}{dx}\right) = Pf'(xP^2)$

$$\left\{2x\frac{dP}{dx} + P\right\}$$

$\Rightarrow \quad \left(2x\frac{dP}{dx} + P\right)[Pf'(xP^2) + 1] = 0$

$\Rightarrow \quad 2x\frac{dP}{dx} + P = 0$

$\Rightarrow \quad 2\frac{dP}{dx} + \frac{dx}{x} = 0$

$\Rightarrow \quad 2\log P + \log x = \log c^2$

$\Rightarrow \quad P^2 x = c^2$

Substituting in the given equation, we get

$$y - 2\sqrt{c^2 x} = f(c^2)$$

$\Rightarrow \quad y = 2c\sqrt{x} + f(c^2)$

153. (C) Equation is $x^2P^2 + xyP - 6y^2 = 0$

or $\quad x^2P^2 + 3xyP - 2xyP - 6y^2 = 0$

or $\quad (xP + 3y)(xP - 2y) = 0$

or $\quad x\frac{dy}{dx} + 3y = 0$

and $\quad x\frac{dy}{dx} - 2y = 0$

or $\quad \frac{dy}{y} + 3\frac{dx}{x} = 0$

and $\quad \frac{dy}{y} - \frac{2dx}{x} = 0$

Integrating these gives

$$\log y + 3\log x = \log c$$

$\Rightarrow \quad x^3 y = c$

and $\quad \log y - 2\log x = \log c$

$\Rightarrow \quad y = cx^2$

$\therefore$ The complete solution is

$$(x^3y - c)(y - cx^2) = 0$$

154. (D) The equation $P^3 - 4xyP + 8y^2 = 0$ is solvable for x and can be written as

$$4x = \frac{P^2}{y} + \frac{8y}{P}$$

Differentiating with respect to y, we get

$$4\frac{dx}{dy} = \frac{y\cdot 2P\frac{dP}{dy} - P^2\cdot 1}{y^2}$$

$$+\,8\,\frac{\left(P\cdot 1 - y\frac{dP}{dy}\right)}{P^2}$$

or $\quad \frac{y}{P} = \left(\frac{2P}{y} - \frac{8y}{P^2}\right)\frac{dP}{dy} - \frac{P^2}{y^2} + \frac{8}{P}$

or $\quad 2\left(\frac{P^3 - 4y^2}{yP^2}\right)\frac{dP}{dy} = \left(\frac{P^3 - 4y^2}{y^2P}\right)$

or $$\frac{2}{P}\frac{dP}{dy} = \frac{1}{y}$$

or $$2\frac{dP}{P} = \frac{dy}{y}$$

Integrating gives

$$2\log P = \log y + \log c$$

or $$P^2 = yc$$

or $$P = \sqrt{yc}$$

Substituting the value of P in the original equation, we get

$$4x = c + \frac{8y}{\sqrt{cy}}$$

or $$c\,(4x - c)^2 = 64y$$

155. (C) The given equation can be written as

$(x^2 + y^2)\,(dx + dy)^2 - 2\,(x + y)\,(dx + dy)\,(x\,dx + y\,dy) + (xdx + ydy)^2 = 0$

Now let $u = x + y$ and $v = x^2 + y^2$

$\therefore$ $$du = dx + dy$$

and $$dv = 2\,(xdx + ydy)$$

From equation (1)

$$v\,(du)^2 - u\,(du)\,(dv) + \left(\frac{1}{2}\,dv\right)^2 = 0$$

or $$4v\,(du)^2 - 4vdudv + (dv)^2 = 0$$

or $$\left(\frac{dv}{du}\right)^2 - 4u\left(\frac{dv}{du}\right) + 4v = 0$$

or $$P^2 - 4uP + 4v = 0 \qquad \ldots(2)$$

where $$P = \frac{dv}{du}$$

It is a solvable for v

$$4v = 4uP - P^2$$

Differentiating with respect to u, we get

$$4\frac{dv}{du} = 4\left(u\frac{dP}{du} + P\cdot 1\right) - 2P\frac{dP}{du}$$

or $$4P = 4\left(u\frac{dP}{du} + P\right) - 2P\frac{dP}{du}$$

or $$(4u - 2P)\frac{dP}{du} = 0$$

or $$dP = 0$$

Integrating, we get $P = c$

Substituting the value of P in equation (2)

$$c^2 - 4uc + 4v = 0$$

or $c^2 - 4\,(x + y)\,c + 4\,(x^2 + y^2) = 0$

156. (B) The given equation is

$$e^{3x}\,(P - 1) + P^3\,e^{3y} = 0 \qquad \ldots(1)$$

Let $$e^x = u$$

and $$e^y = v$$

$\therefore$ $$e^x dx = du$$

and $$e^y dy = dv$$

$\therefore$ $$P = \frac{dy}{dx} = \frac{e^x du}{e^y du} = \frac{udv}{vdu}$$

$\therefore$ From (1)

$$u^3\left(\frac{udv}{vdu} - 1\right) + \left(\frac{udv}{vdu}\right)v^3 = 0$$

or $$u^3\left(u\frac{dv}{du} - v\right) + u^3\left(\frac{dv}{du}\right)^3 = 0$$

or $$u\frac{dv}{du} - v + \left(\frac{dv}{du}\right)^3 = 0 \qquad \ldots(1)$$

or $v = uP + P^3$ $\left(\text{as } P = \frac{dv}{du}\right)$ which is Clairaut's form

157. (A) $y = Px + \log P$ is the Clairaut's form

$\therefore$ The solution is

$$y = cx + \log c$$

158. (A) $$P^2 - 5P + 6 = 0$$

$\therefore$ $$(P - 3)\,(P - 2) = 0$$

$\therefore$ $$P = 3$$

$\Rightarrow$ $$\frac{dy}{dx} = 3$$

$\Rightarrow$ $$dy = 3dx$$

$\Rightarrow$ $$y = 3x + c$$

Similarly $P = 2 \Rightarrow y = 2x + c_2$

$\therefore$ The solution of the given equation is

$$(y - 3x - c_1)\,(y - 2x - c_2) = 0$$

159. (B) 160. (D) 161. (D) 162. (D)

163. (B) The auxiliary equation of the given differential equation is

$$m^3 - 3m - 4 = 0$$

$\Rightarrow$ $$(m + 1)\,(m - 4) = 0$$

$\Rightarrow$ $$m = -1, 4$$

164. (D) The given equation is $(D^2 - 3D - 4)\,y = 0$

As above roots of the auxiliary equation are -1 and 4

$\therefore$ The solution is $y = C_1e^{-x} + C_2e^{4x}$.

165. (C) $(D^4 - 81)\,y = 0$ auxiliary equation $m^4 - 81 = 0$

$\Rightarrow$ $(m + 3)\,(m - 3)\,(m + 3i)\,(m - 3i) = 0$

$\Rightarrow \quad m = \pm 3, \quad m = \pm 3i$

$\therefore$ Solution is

$y = C_1 e^{3x} + C_2 e^{-3x} + C_3 \cos 3x + C_4 \sin 3x$

166. (D)

167. (B) The A. E. of the given equation is

$$m^2 (m+1)^2 = 0$$

$$\Rightarrow \quad m = 0, 0, -1, -1$$

$\therefore$ C. F. is $y = (C_1 + C_2 x) + (C_3 + C_4 x) e^{-x}$

$$\text{P. I.} = \frac{1}{D^2 (D+1)^2} \cdot e^x = \frac{1}{1^2 (1+1)^2} e^x = \frac{1}{4} e^x$$

Hence general solution is

$$y = (C_1 + C_2 x) + (C_3 + C_4 x) e^{-x} + \frac{1}{4} e^x$$

168. (B) The A. E. of the given differential equation is

$$m^2 - 1 = 0, \quad m = \pm 1$$

$$\therefore \quad \text{C. F. is } y = c_1 e^x + c_2 e^{-x} = A \cosh (x + B)$$

$$\text{P. I.} = \frac{1}{D^2 - 1} e^x = x \cdot \frac{1}{2D} e^x = \frac{x}{2} e^x$$

$\therefore$ Required solution is

$$y = A \cosh (x + B) + \frac{1}{2} x e^x$$

169. (A) The A. E. of the given differential equation is

$$m^2 - m - 2 = 0$$

$$\Rightarrow \quad m = 1, m = -2$$

$\therefore$ C. F. is $y = C_1 e^x + C_2 e^{-2x}$

and $$\text{P. I.} = \frac{1}{D^2 + D^{-2}} e^x = x \cdot \frac{1}{2D + 1} e^x = x \cdot \frac{1}{2+1} e^x = \frac{1}{3} x e^x$$

$\therefore$ Required solution is

$$y = C_1 e^x + C_2 e^{-2x} + \frac{1}{3} x e^x$$

170. (D) $$\text{P. I.} = \frac{1}{D^2 + 4} \cdot x = \frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1} \cdot x = \frac{1}{4}\left(1 - \frac{D^2}{4} + \ldots\right) x = \frac{x}{4}$$

171. (D) A. E. is $m^2 + \omega^2 = 0$

$$\Rightarrow \quad m = \pm i\omega$$

$\therefore$ C. F. is $y = A \cos (\omega x + B)$

$$\text{P. I.} = \frac{1}{D^2 + \omega^2} \cdot 10\omega^2 = \frac{1}{\omega^2}\left(1 + \frac{D^2}{\omega^2}\right)^{-1} \cdot 10\omega^2$$

$$\frac{1}{\omega^2}\left(1 - \frac{D^2}{\omega^2} + \ldots\right) \cdot 10\omega^2 = 10$$

$\therefore$ Solution is $y = A \cos (wx + B) + 10$

172. (D) The given equation is $(D^2 + a^2) y = \sec ax$

A. E. is $m^2 + a^2 = 0$

$$\Rightarrow \quad m = \pm ai$$

$\therefore$ C. F. is $y = C_1 \cos ax + C_2 \sin ax$

and $$\text{P. I.} = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{2ai}\left[\frac{1}{D - ai} - \frac{1}{D + ai}\right] \sec ax$$

But $$\frac{1}{D - ai} \sec ax = e^{iax} \int e^{-iax} \sec ax dx = e^{iax} \int \frac{\cos ax - i \sin ax}{\cos ax} dx = e^{iax}\left(x + \frac{i}{a} \log \cos ax\right)$$

Similarly $$\frac{1}{D + ai} \sec ax = e^{iax}\left(x - \frac{i}{a} \log \cos ax\right)$$

$$\therefore \quad \text{P. I.} = \frac{1}{2ai}\left[e^{iax}\left(x + \frac{i}{a} \log \cos ax\right) - e^{-iax}\left(x - \frac{i}{a} \log \cos ax\right)\right]$$

$$= \frac{1}{a}\left[x \sin ax + \frac{1}{a} (\log \cos ax) \cos ax\right]$$

Hence the general solution of the given equation is

$$y = C_1 \cos ax + C_2 \sin ax + \frac{1}{a}\left[x \sin ax + \frac{1}{a}(\log \cos ax) \cos ax\right]$$

173. (D) 174. (D)

175. (A) $\frac{dX}{dt} = MX + b$

or $\frac{dX}{dt} - MX = b$

where, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ I. F. $= e^{-\int M dt} = e^{-Mt}$

$\therefore$ $Xe^{-Mt} = \int be^{-Mt}\, dt + c$

$= -b \cdot \frac{e^{-Mt}}{M} + c$

$\Rightarrow$ $x = \frac{-b}{M} + ce^{Mt}$

176. (A) Given $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$

A. E. is $D^2 + D'^2 = 0$

or $D^2 + 1 = 0$

or $D = \pm i$

$\therefore$ $u = f(x + iy) + g(x - iy)$

177. (A) $y'(y' + y) = x(x + y)$

$\therefore$ $y^2 + yy' - x^2 - xy = 0$

$y'(y' + x + y) - x(y' + x + y) = 0$

$(y' - x)(y' + x + y) = 0$

Since $y' \neq x$

$\therefore$ $y' + x + y = 0$

or $\frac{dy}{dx} + x + y = 0$

or $\frac{dy}{dx} + y = -x$

Now I. F. $= e^{\int P dx}$

$= e^x$

$y \times e^x = \int -xe^x dx + c$

$= -[xe^x - \int 1 \cdot e^x\, dx] + c$

or $ye^x = -xe^x + e^x + c$

or $y = -x + 1 + ce^{-x}$

where c is constant

Choosing $c = -1$, we get

$y = 1 - x - e^{-x}$

178. (B) Given, $4x^3y'' + 6x^2y' + y = 0$

We transform the independent variable x to t by the relation

$t = \frac{1}{x}$

or $x = \frac{1}{t}$ ...(1)

Now $y' = -\frac{1}{x^2}\frac{dy}{dt}$

or $y'' = \frac{1}{x^4}\frac{d^2y}{dt} + \frac{2}{x^3}\frac{dy}{dt}$

and the given differential equation transform to

$4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ ...(2)

The point at ∞ is transformed to the origin. From equation (2), we not that the origin is regular singular point.

Hence, the point at ∞ is a regular singular point of the given equation.

179. (D)

180. (A) $M = 3a^2x^2 + by \cos x$

$N = 2\sin x - 4ay^3$

For the equation to be exact,

$\frac{\delta N}{\delta x} = \frac{\delta M}{\delta y}$

or $2 \cos x = b \cos x$

or $b = 2$

181. (A) For the initial value problem

PDE : $u_{tt} - c^2u_{xx} = 0, -\infty < x < \infty, t \geq 0$

ICS : $u(x, 0) = \eta(x)$, $u_t(X, 0) = v(x)$

D' Alember solution is given by,

$$u(x, t) = \frac{1}{2}[\eta(x + ct) + \eta(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} v(x)\, dx$$

For the given problem $v(x) = 0$

$\eta(x) = x(1 - x), c = 1$

$\therefore$ The solution is of the form,

$u(x, t) = \frac{1}{2}[\eta(x + t) + \eta(x - t)]$

$= \frac{1}{2}[(x + t)(1 - x - t) + (x - t)(1 - x + t)]$

$\therefore$ $u\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{2}\left[\frac{3}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4}\right]$

$= \frac{3}{16}$

182. (A) 183. (B) 184. (B) 185. (D) 186. (B)
187. (A) 188. (A)
189. (C) Because if λ_i and λ_j are roots of equation $J_n(\lambda a) = 0$

then $\int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\, dx$

$$= \begin{cases} 0 & \text{If } i \neq J \text{ (diff. roots)} \\ \dfrac{a^2}{2} J^2_{n+1}(\lambda_i a) & \text{If } i = j \text{ (equal roots)} \end{cases}$$

For orthogonality of system on [0, 1], the weight function is x.

190. (B) A linear homogeneous equation can always be reduced to a linear differential equation with constant coefficients of the form

$$\frac{d^n y}{dx^n} + a_n \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_1 y = x \quad \ldots(1)$$

$$f(D) = x$$

We will now show that if $y_1, y_2, \ldots y_n$ are linearly independent solution of equation (1) then $c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is also a solution of equation (1),

Where $c_1, c_2, \ldots c_n$ being arbitrary constants.

Since $y_1, y_2, \ldots y_n$ are solution of (1), we get

$$f(D)\, y_1 = 0$$
$$f(D)\, y_2 = 0$$
$$\ldots\ldots\ldots \quad \ldots$$
$$\ldots\ldots\ldots \quad \ldots$$
$$f(D)\, y_n = 0$$

$f(D)\,\{c_1y_1 + c_2y_2 + \ldots + c_ny_n\}$

$= f(D)(c_1y_1) + f(D)(c_2y_2) + \ldots + f(D)(c_ny_n)$

$= c_1 f(D)\, y_1 + c_2 f(D)\, y_2 + \ldots + c_n f(D)\, y_n$

$= c_1 \cdot 0 + c_2 \cdot 0 + \ldots + c_n \cdot 0 = 0$ (using (2))

This proves that statement

191. (B)
192. (D) Heat equation is given by

$$\frac{\delta U}{\delta t} = c^2 \frac{\delta^2 U}{\delta x^2},\ c^2 = 1$$

Let $$U = \frac{e^{-x^2}}{\sqrt{t}}$$

$$\therefore \quad \frac{\delta U}{\delta t} = \frac{\sqrt{t} \cdot e^{-x^2/4t}\left(\dfrac{x^2}{4t^2}\right) - e^{\frac{-x^2}{4t}} \cdot \frac{1}{2\sqrt{t}}}{t}$$

$$= \frac{e^{-x^2/4t}}{\sqrt{t}} \left[\frac{x^2}{4t^2} - \frac{1}{2t}\right]$$

$$= u\left(\frac{x^2}{4t^2} - \frac{1}{2t}\right)$$

$$\frac{\delta U}{\delta x} = \frac{1}{\sqrt{t}} \cdot e^{-x^2/4t}\left(\frac{2x}{4t}\right) = U - \left(\frac{x}{2t}\right)$$

$$\frac{\delta^2 U}{\delta x^2} = \frac{\delta U}{\delta x}\left(\frac{-x}{2t}\right) - U\left(\frac{1}{2t}\right) = U\left\{\frac{x^2}{4t^2} - \frac{1}{2t}\right\}$$

$$\therefore \quad \frac{\delta U}{\delta t} = \frac{\delta^2 U}{\delta x^2}$$

193. (A) Laplace equation $\Delta^2\psi = 0$ is the elliptic equation occurring most frequently in physical problems.

194. (D)

195. (B) Given, $x^2 + y^2 = 2cx$

$$\Rightarrow \quad c = \frac{x^2 + y^2}{2x}$$

Differentiating with respect to x, we get

$$2x + 2y\frac{dy}{dx} = 2c$$

$$\Rightarrow \quad x + y\frac{dy}{dx} = c$$

$$= \frac{x^2 + y^2}{2x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\dfrac{x^2 + y^2}{2x} - x}{y}$$

$$= \frac{y^2 - x^2}{2xy}$$

Its orthogonal trajectory is given by $-\dfrac{dx}{dy}$

$$-\frac{dx}{dy} = \frac{y^2 - x^2}{2xy}$$

$$\Rightarrow \quad \frac{dy}{dx}(y^2 - x^2) = -2xy$$

$$\Rightarrow \quad y'(x^2 - y^2) = 2xy$$

196. (C) 197. (B)

198. (B) $$\frac{dy}{dx} = \frac{y}{\sqrt{x}}$$

$$\Rightarrow \quad \frac{1}{y}\, dy = x^{-1/2}\, dx$$

$$\Rightarrow \quad y = e^{2\sqrt{x}} \cdot e^c$$

At $x = 2, y = 4$

$$\therefore \quad e^c = 4e^{-2\sqrt{2}}$$

$\therefore \quad y = e^{2\sqrt{x}} \cdot 4e^{-2\sqrt{2}}$

Hence differential equation has unique solution.

199. (A) Given : $D^4 - D^2 = 0$

$\Rightarrow \quad D^2 (D-1)(D+1) = 0$

$\Rightarrow \quad D = 0, 0, 1, -1$

$\therefore \quad y = 1 + c_1 x + c_2 e^x + c_3 e^{-x}$

Hence, the set of linearly independent solution is $\{1, x, e^{-x}, e^x\}$

200. (B) Given : $x^2 (1-x)\dfrac{d^2y}{dx} + x\dfrac{dy}{dx} + y = 0$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + \frac{1}{x(1-x)}\frac{dy}{dx} + \frac{1}{x^2(1-x)}y = 0$$

$$\therefore \quad P = \frac{1}{x(1-x)}$$

$$Q = \frac{1}{x^2(1-x)}$$

$\Rightarrow \quad P - xQ = 0$

Hence, solution of C. F. is $u = x$. So, the $x = 1$ is a regular singular point.

201. (C) $y = \alpha \sin x + \beta \cos x$ ...(1)

$z = \alpha \cos x + \beta \sin x$ ...(2)

Differentiate equation (1) with respect to x, we get

$$\frac{dy}{dx} = \alpha \cos x + \beta \sin x = z$$

$$\frac{dy}{dx} - z = 0 \qquad ...(3)$$

Differentiate equation (2) with respect to x, we get

$$\frac{dy}{dx} = -\alpha \sin x + \beta \cos x = -y$$

$$\Rightarrow \quad y + \frac{dy}{dx} = 0$$

Hence, equation (1) and (2) are general solution of differential equation (3) and (4).

202. (B) $2x(2+x)\dfrac{d^2y}{dx^2} + 2(3+x)\dfrac{dy}{dx} - xy = 0$

Let $2x(2+x) = v^2$

$\Rightarrow \quad 2x^2 + 4x - v^2 = 0$

$$\Rightarrow \quad x = \frac{-4 \pm \sqrt{16 + 8v^2}}{4}$$

$$x = \frac{-2 \pm \sqrt{4 + 2v^2}}{2}$$

203. (C) 204. (B) 205. (C)

●●

Algebra

Group—A non empty set elements, G is said to form a group if in G, there is defined a binary operator called product such that

(a) $a, b \in G \Rightarrow a * b \in G$ (closed)

(b) $a, b, c \in G \Rightarrow a * (b * c) = (a * b) * c$ (Associative)

(c) There exist $e \in G : a * e = e * a = a$, for all $a \in G$. (Existence of identity)

(d) For every $a \in G$, there exist $a^{-1} \in G$.

$a * a^{-1} = a^{-1} * a = e$

(Existence of inverse)

Abelian Group—A group G is Abelian (commutative) if for every $a, b \in G, a * b = b * a$.

If G is a non empty set and * is any binary operation defined on G, then $(G, *)$ is—

(a) Quasi-group—$a, b \in G \Rightarrow a * b \in G$.

(b) Semi-group—$a, b \in G \Rightarrow a * b \in$ and $(a * b) * c = a * (b * c), a, b, c \in G$.

(c) Monoid—$a, b \in G \Rightarrow a \cdot b \in G, (a * b) * c = a * (b * c), a, b, c \in G$ and there exist $e \in G$ identity : $a * e = e * a = a$.

i.e., semi-group is a quasi-group with associativity.

Monoid is a semi-group with identity. Group is a monoid with inverse. Abelian group is a group with commutativity.

Order of group—The number of elements in G, o (G).

Cyclic group—$a^i \in G$ and $o(G) = n$:

$$a^i = \begin{cases} a^0 = a^n = e, i = 0, n \\ a^i = i < n \\ a^{i-n} = i > n \end{cases}$$

Lemma—(a) The identity element of G is unique.

(b) $\forall a \in G$, its inverse a^{-1} is unique

(c) $a \in G \Rightarrow (a^{-1}) = a$

(d) $a, b \in G \Rightarrow (a * b)^{-1} = b^{-1} * a^{-1}$

(e) $a * b = a * c \Rightarrow b = c$

$b * a = c * a \Rightarrow b = c$

Sub-group—A non empty subset H of G, is a sub-group of group G, if H is a group on the operator of G.

Right and Left Co-sets—H is a sub-group of group G, $a \in G$, then

Right co-set of H in G is $Ha = \{ha : h \in H\}$

Left co-set of H in G is $aH = \{ah : h \in H\}$

Index of Sub-group—If H is a sub-group of G, the index of H in G is the number of distinct right co-set of H in G.

Order (period) of Element—If G is a group $a \in G$, the order of a (period of a), is the least positive integer $m : a^m = e$, $o(a) = m$.

Product Sub-groups—$HK = \{x \in G : x = hk; h \in H, K \in K\}$ H, G are sub-group of G.

Lemma—A non empty subset H of a group G is a subgroup of G iff

(a) $a, b \in H \Rightarrow ab \in H$

(b) $a \in H \Rightarrow a^{-1} \in H$.

Lemma—If H is a non empty finite subset of G and H is closed, then H is a subgroup of G.

Lemma—$\forall a \in G, Ha = \{x \in G : a \equiv x \bmod H\}$

Lemma—There is one to one corresponding between two right cosets of H in G.

Normal Subgroups and Quotient Groups

Normal Subgroup—A subgroup N of G is normal subgroup is $\forall g \in G$ and $n \in N, gng^{-1} \in N$.

Quotient Group—If G is a group, N is normal subgroup of G, then group G/N is called quotient (factor) group.

Lemma—N is a normal subgroup of G iff $g N g^{-1} = N$, for $\forall g \in G$.

Lemma—N is a normal subgroup of G iff $Na = aN, \forall a \in G$.

Lemma—N is a normal subgroup of G iff $(Na)(Nb) = Nab$.

Lemma —N is a normal subgroup of G, G is finite group then $o(G/N) = o(G)/o(N)$.

Homomorphism

Homorphism—A mapping ϕ from group $\bar{G}$ into a group G is said to be a homomorphism if $\forall a, b \in G, \phi(ab) = \phi(a)\phi(b)$.

Kernel—If ϕ is a homomorphism of G into $\bar{G}$, then Kernel of ϕ, $k\phi$ is defined by $k\phi = \{x \in G : \phi(x) = \bar{e} \cdot \bar{e}$ an identity element in $\bar{G}\}$.

Isomorphism—A homomorphism ϕ from G into $\bar{G}$ is an isomorphism if ϕ is one-to-one.

Isomorphic—Two groups G and G* are isomorphic if there is an isomorphism of G into G* ($G \simeq G^*$).

Lemma—N is a normal subgroup of G; $\phi : G \to G/N : \phi(x) = Nx, \forall x \in G$. Then ϕ is homomorphism of G on to G/N.

Lemma—If ϕ is homomorphism of G into $\bar{G}$, then

(a) $\phi(e) = \bar{e}$, the identity element of $\bar{G}$.
(b) $\phi(x^{-1}) = \phi(x)^{-1}, \forall x \in G$.

Lemma—If ϕ is homomorphism of G into $\bar{G}$ with Kernel K, then K is a normal subgroup of G.

Lemma—If ϕ is a homomorphism of G into $\bar{G}$ with Kernel k, then the set of all inverse images of $\bar{g} \in \bar{G}$ under ϕ is given by kx, where x is any particular inverse image of $\bar{g} \in \bar{G}$.

Lemma—A homomorphism ϕ of G into $\bar{G}$ with Kernel $k\phi$ is isomorphism of G into $\bar{G}$ iff $k\phi = (e)$.

Some Important Theorems

1. If ϕ is a homomorphism of G onto $\bar{G}$ with Kernel k, then $G/K = \bar{G}$.

2. **Cauchy's theorem for Abelian group**—If G is a finite Abelian group and any prime number $P/o(G)$ there exist $a \neq e \in G'$. $oP = e$.

3. **Sylow's theorem for Abelian group**—If G is a finite Abelian group and P any prime such that $P^2/o(G)$, $P^{\alpha+1}/o(G)$, the G has a subgroup of order P^{α}.

4. If G is Abelian group of order $o(G)$ and $P^{\propto} | o(G)$, $P^{\alpha+1}/o(G)$, then there is unique subgroup of G of order P^{α}.

5. If ϕ is homomorphism of G into $\bar{G}$, with Kernel k, and $\bar{N}$ is a normal subgroup of $\bar{G}$, $N = \{x \in G \mid \phi(x) = \bar{N}\}$

Then $G/N \approx \bar{G}|\bar{N}$ and $G|N = (G|K)(N|K)$

Automorphism—A homomorphism of a group G onto itself.

Theorems—1. If G is a group, A (G), a set of automorphism is also a group.

2. Let G be a group and ϕ an automorphism of G.

If $a \in G$ and $o(a) > 0$, then $o(\phi(a)) = o(a)$.

Sylowis theorem—1. If P is a prime number and $P^2|o(G)$, then G has a subgroup of order P^{α}.

2. If $P^m|o(G)$, $P^{m+1}|o(G)$, then G is a subgroup of order P^m.

3. If A and B are finite subgroup of G, then

$$o(AxB) = \frac{o(A)\, o(B)}{o(A \cap xBx^{-1})}$$

Direct Product

Internal direct product—If G is a group and $N, N_2, \ldots N_n$ are normal subgroup of G :

(a) $G = N_1, N_2, \ldots N_n$

(b) $g \in G$, $g = m_1 m_2 \ldots m_n$, $m_i \in N_i$ in a unique way, then G is internal direct product of $N_1 N_2, \ldots N_n$.

1. If G is internal direct product of $N_1, \ldots N_n$, then for $i \neq j$, $N_i \cap N_j = (e)$ and $a \in N_i$, $b \in N_j$ then $ab = ba$.

2. If G is internal direct product of $N_1, \ldots N_n$ and if $T = N_1 \times N_2 \times \ldots \times N_n$, then G and T are isomorphic.

Finite Abelian Group

Invariants of G—If G is an Abelian group of order P^n, P a prime $G = A_1 \times A_2 \times \ldots \times A_n, \forall A_j$ is cyclic of order P^{n_i}, $n_i \leq n_{i+1}$, then $n_{11} \ldots n_{1\propto}$ are invariant of G.

Theorem—The number of non-isomorphic Abelian groups of order P^n are equals to the number of partitions of n.

Rings

Associative Ring—A non empty set R is said to be a ring if in R there are defined two operators + and. respectively : $a, b, c \in R$.

(a) $a + b \in R$
(b) $a + b = b + a$
(c) $(a + b) + c = a + (b + c)$
(d) $o \in R : a + o = a, \forall a \in R$
} Abelian group with o on addition

(e) $-a \in R : a + (-a) = 0$ (closed under)

(f) $a \cdot b \in R$ (Associative under)

(g) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (Left distribution)

(h) $(a + b) \cdot c = a \cdot c + b \cdot c$ (Right distribution)
$c \cdot (a + b) = c \cdot a + c \cdot b$

Ring with Unity—$1 \in R : a \cdot 1 = 1 \cdot a = a\ \forall \cdot a \in R$

Commutative Ring—If $a \cdot b = b \cdot a, \forall a, b \in R$

Zero Divisor—R is commutative ring, $a \neq a \in R$, is zero divisor if there exist $b \in R, b \neq 0 : ab = 0$.

Integral Domain—A commutative ring is an integral domain if it has no zero divisor.

Division Ring (skew field)—A ring is called a division ring if its non-zero elements from a group under multiplication.

Characteristic Zero—An integral domain D is of characteristic zero if $ma = 0, a \neq 0, \in P, \Rightarrow m = 0$.

Finite Characteristic—An integral domain D is a finite characteristic if there exist a positive integer $m : ma = 0, \forall a \in D$.

Null (zero) ring—$\{0\}, + :) : 0 + 0 = 0$ and $0 \cdot 0 = 0$

Field—A field is a commutative division ring.

Homomorphism

Homomorphism—A mapping ϕ from ring R into ring R' is homomorphism if $\forall a, b \in R$.

(a) $\phi(a + b) = \phi(a) + \phi(b)$

(b) $\phi(ab) = \phi(a)\,\phi(b)$

Kernel—If ϕ is a homomorphism of R into R', then the Kernel of ϕ, $I(\phi)$ is the set of all $a \in R : \phi(a) = 0$, the zero element of R.

Zero Homomorphism—$\phi(a) = 0$ for all $a \in R$ and $I(\phi) = R$.

Isomorphism—A homomorphism of R into R' if it is also one-to-one mapping.

Isomorphic—A and B are isomorphic, if there is a isomorphism from one onto another.

Some Important Theorems

1. If ϕ is homomorphism of R into R', then (i) $\phi(a) = 0$ (ii) $\phi(-a) = -\phi(a), \forall a \in R$.

2. If ϕ is homomorphism of R into R' with Kernel $I(\phi)$, then

(a) $I(\phi)$ is a subgroup of R under addition

(b) If $a \in I(\phi)$ and $r \in R$, then both $ar \in R$ and $ra \in R$.

3. The homomorphism ϕ of R into R' is an isomorphism iff $I(\phi) = 0$.

4. If integral domain is of finite characteristic then its characteristic is a prime number.

Ideals and Quotient Rings

Ideal—A non-empty subset $\cup$ of R is ideal if

(a) U is a subgroup under addition

(b) $\forall u \in U$ and $r \in R$, $ur, ru \in U$.

Quotient Ring—If U is an ideal of ring R, then R/U is a quotient ring and is homomorphic image of R.

Maximal Ideal—An ideal $M \neq R$ in a ring R is maximal ideal of R whenever U is an ideal of $R : M \subset U \subset R$ then either $R = U$ or $M = U$.

Some Important Theorems

1. If R is a commutative ring with unit element and M is an ideal of R then M is maximum ideal of R iff R/M is a field.

2. If R is a commutative ring with unit element whose only ideals are (0) and R, itself, then R is a field.

Euclidean Ring

Euclidean Ring—An integral domain R is an Euclidean ring if for every $a \neq 0 \in R$ there is defined a non-negative integer $d/(a)$.

(a) $\forall a, b \in R, a \neq 0, b \neq 0 \Rightarrow d(a) \leq d(ab)$

(b) For any $a, b \in R, a \neq 0, b \neq 0$ there exist $t, r \in R : a = tb + r$ where either $r = 0$ or $d(r) < d(b)$.

Principal Ideal—An integral domain R with unit element is a principal ideal ring if every ideal $A \in R$ is of the form $A = (a) = \{xa | x \in R\}$ for some $a \in R$.

Unit (elements)—$a \in R$ is unit element in R if there exist $b \leq R : ab = 1$.

Unit—If R is commutative ring with unit element.

Prime Element—In Euclidean ring R a non unit π is said to be prime element of R if when ever $\pi = ab$, $a, b \in R$, then one of a or b is a unit in R.

Relatively Prime—In the Euclidean ring R a, $b \in$ R are relatively prime if their greatest common divisor is a unit of R.

Some Important Theorems

1. If R is an Euclidean ring and A an ideal of R. Then there exist an element $a_0 \in$ R : A consists exactly of all a_0x as range over R.

2. A Euclidean ring possesses a unit element.

3. If R is an Euclidean ring. Then any two elements a, $b \in$ R have a greatest common divisor d. Moreover $d = \lambda a + ub$ for some λ, $u \in$ R.

4. If R is an integral domain with unit element and suppose for a, $b \in$ R, a/b and b/a are true. Then $a = ub$, where u is a unit in R.

5. If R is an Euclidean ring and a, $b \in$ R. If $b \neq 0$ is not a unit in R, then $\alpha(a) < d(ab)$

6. If R is an Euclidean ring, then every element in R is either a unit in R or can be written as the product of a finite number of prime elements of R.

7. If R is an Euclidean ring. Suppose for a, b, $c \in$ R, a/bc but $(a, b) = 1$, then a/c.

8. If π is a prime element in the Euclidean ring R and π/ab, where a, $b \in$ R, then π divides atleast one of a or b.

9. If π is a prime element in the Euclidean ring R and $\pi/a_1, a_2, \ldots a_n$, then π divides atleast one $a_1, a_2, \ldots a_n$.

10. **Unique factorization theorem**—If R is an Euclidean ring and $a \neq 0$ a non unit in R suppose $a = \pi, \pi_2 \ldots \pi_n = \pi'_1 \pi'_2 \ldots \pi'_n$ where π, and π'_1 are prime elements in R. Then $n = m$ and each π_1, $i = 1, 2, \ldots n$ is an associate of same π'_j, $j = 1 \ldots n$ and conversely each π'_j is associated with same π_j.

11. Every non–zero element in a Euclidean ring R can be uniquely written as a product of prime elements or is a unit in R.

12. The ideal A = (a_0) is a maximal ideal of the Euclidean ring R iff a_0 is a prime element of R.

Polynomial Rings Over Commutative Rings—F $(x_n \ldots x_n)$: The field of aration functions in $x_1 \ldots, x_n$ over F.

Unique factorization domain—An integral domain R, with unit element is a unique factorization domain if—

(a) Any non-zero element in R is either a unit or can be written as the product of a finite number of irreduciable elements of R.

(b) The decomposition in part (a) is unique upto the order and associates of the irreduciable elements.

Some Important Theorems

1. If R is an integral domain, then so in R $[x]$.

2. If R is an integral domain, then so is R $[x_1 \ldots x_n]$

3. If R is unique factorization domain and if a, $b \in$ R, then a and b have the greatest common divisor $(a, b) \in$ R. Moreover if a, b are relatively prime $(a, b) = 1$, whenever $a|bc$ then a/c.

4. If $a \in$ R is an irreducible element and $a|bc$ then $a|b$ or $a|c$.

5. If R is a unique factorization domain, then the product of two primitive polynomials in R $[x]$ is again a primitive polynomial in R $[x]$.

6. If R is a unique factorization domain, and if $f(x)$, $g(x) \in$ R $[x]$, then $c(fg) = c(g) = c(f)\, c(g)$.

7. If $f(x) \in$ R (x) is both primitive and irreducible as an element of R $[x]$, then it is irreducible as an element of F $[x]$. Conversely, if the primitive element $f(x) \in$ R $[x]$ is irreducible as an element of F $[x]$, it is also irreducible as an element of R $[x]$.

8. If R is a unique of factorization domain and if P (x) is a primitive polynomial in R $[x]$, then it can be factored in a unique way as the product of irreducible elements in R $[x]$.

9. If R is a unique factorization domain, then so is R $[x]$.

10. If R is a unique factorization domain the n so is R $[x_1 \ldots, x_n]$.

11. If F is a field then F $[x_1, \ldots, x_n]$ is a unique factorization domain.

Fields

Fields (F) is a non empty set, F is a field,

(a) (F, +) is an Abelian group

(b) (F, $\cdot$) is semi-Abelian *i.e.*, (F – {0} $\cdot$) is Abelian group

(c) Multiplication is distributive over addition.

$$a(b + c) = ab + ac$$
$$(b + c)a = bc + ca$$

Extension—K, F are fields K is an extension of F if F $\subset$ K $\Leftrightarrow$ F is a subfield of K.

Degree of Extension—The degree of extension K over F, [K', F] is the dimension of K as a vector space of F.

Algebraic Over F—K is an extension of F, $a \in$ K is algebraic over F, if there exist elements α_0, α_1, ... $\alpha_n \in$ F, not all zero, such that $\alpha_0\ a^n + \alpha_1\ \alpha^{n-1} + \ldots + \alpha_n = 0$.

Sub-field Obtained by Adjoining *a* to F—If K is an extension of F, $a \in$ K, then F (a) is the smallest subfield containing both F and a.

Algebraic of Degree *n*—The element $a \in$ F is algebraic of degree n over F if it satisfies a non-zero polynomial over F of degree n but no, non-zero polynomial of lower degree.

Algebraic Extension—The extension K of F is called an algebraic extension of F if every element in K is algebraic over F.

Algebraic Number—A complex number is algebraic number if it is algebraic over the field of rational number.

Some Important Theorems

1. If L is a finite extension of K and if K is a finite extension of F, then L is a finite extension of F and [L : F] = [L : K] [K : F].

2. If L is a finite extension of F and K is a sub-field of L which contains F, then [K : F]\ [L : F].

3. The element $a \in k$ is a algebraic over F iff F (a) is a finite extension of F.

4. If $a \in k$ is algebraic of degree over F, then [F (a)' F] = n.

5. If a, $b \in k$ are algebraic over F then $a \pm b$, ab and a/b $(b \pm a)$ are all algebraic over F *i.e.*, the element in k which are algebraic over F from a subfield of K.

6. If a, $b \in k$ are algebraic over, F of degrees m and n respectively, then $a \pm b$, ab and a/b $(b \neq 0)$ are algebraic over F of degree at most mn.

7. If L is an algebraic extension of K and if K is an algebraic extension of F, then L is an algebraic extension of F.

Some Solved Examples

Example 1. If order of a group G is a prime number P, then G does not possess proper sub-group.

Solution : P is a prime number, its divisors are only $\pm$ 1 and $\pm$ P.

Let H be a sub-group of G, by Lagrange's theorem o (H) $\cdot$ o (G) is adivisor of P.

$$\Rightarrow \qquad o(\text{H}) = \text{P or } 1$$

$$\Rightarrow \qquad \text{H} = \text{G or H} = \{e\}$$

$\Rightarrow$ H is not a sub-group of G in either case.

Hence G does not possess proper subgroup.

Example 2. If G = {1, –1} is a group (G;), the order of 1 and –1 is ?

Solution : G have an identity element 1.

$$(1)^1 = 1$$

$$\Rightarrow \qquad 0\,(1) = 1$$

$$\text{and} \qquad (-1)^2 = 1$$

$$\Rightarrow \qquad 0\,(-1) = 2$$

Thus, order 1 is 1 and (–1) is 2.

Example 3. If a, b are any two elements of a group G and H is any sub-group of G, then $a \in$ Hb iff Ha = Hb.

Solution : $a \in$ Hb

$$\Rightarrow \qquad ab^{-1} \in \text{H}bb^{-1}$$

$$\Rightarrow \qquad ab^{-1} \in \text{H}e$$

$$\Rightarrow \qquad ab^{-1} \in \text{H}$$

$$\Rightarrow \qquad \text{H}ab^{-1} = \text{H}$$

$\Rightarrow$ H$ab^{-1}b$ = Hb (Multiplying both sides an the left by b)

$$\Rightarrow \qquad \text{H}ae = \text{H}b \qquad (\because\ b^{-1}b = e)$$

$$\Rightarrow \qquad \text{H}a = \text{H}b \qquad (\because\ ae = a)$$

Conversely let Ha = Hb. Since $a \in$ Ha.

Therefore, $a \in$ Hb.

Example 4. Prove that if R is a Euclidean ring and $b \in$ R is not a unit, then $\alpha\,(a) < \alpha\,(ab)$ $\forall a \in$ R.

Solution : Let $a\,(\pm 0) \in$ R, consider the ideal U = (a). By the definition of Euclidean ring $\alpha\,(a) \leq \alpha\,(xa)$ for any non-zero $x \in$ R.

Also $ab \in$ U and if $\alpha\,(a) = \alpha\,(a\alpha)$, then $(a) = (ab) =$ U

i.e., every element of U is a multiple of ab. In particular, a is also a multiple of ab, *i.e*, $a = abx$ for some x in R

$$\text{Now} \qquad a = abx$$

$$\Rightarrow \qquad a \cdot 1 = abx$$

$$\Rightarrow \qquad 1 = bx \qquad (\because\ a \neq 0)$$

$\Rightarrow$ b is a unit in R.

Thus, if b is not unit in R, then $\alpha\,(a) \neq (ab)$ and hence we conclude that $\alpha\,(a) < \alpha\,(ab)$

Example 5. A non-empty subset H of a group G is a sub-group iff $a \in$ H, $b \in$ H $\Rightarrow aob^{-1} \in$ H where b^{-1} is the inverse of b in G.

Solution : Suppose H is a subgroup of G and $a \in$ H, $b \in$ H.

Now each element of H must possess inverse because H itself is a group (assumption) $b \in$ H $\Rightarrow$ $b^{-1} \in$ H.

Also H is closed under the composition (o say) in G, therefore

$$a \in \text{H}, b^{-1} \in \text{H} \;\Rightarrow\; aob^{-1} \in \text{H}$$

Let $a \in$ H, $b \in$ H $\Rightarrow aob^{-1} \in$ H, then to show that H is a subgroup it has to satisfy group postulate.

(i) **Closure Property**—Let a, $b \in$ H, then $b \in$ H $\Rightarrow b^{-1} \in$ H.

$\therefore$ By the given condition $a \in$ H, $b^{-1} \in$ H $\Rightarrow$ $a_0\,(b^{-1})^{-1} \in$ H $\Rightarrow aob \in$ H.

Thus H is closed with respect to the composition in G.

(ii) **Associatively**—Since the elements of H are also the elements of G the composition is associative in H.

(iii) **Existence of Identity**—Since $a \in$ H, $a^{-1} \in$ H $\Rightarrow aoa^{-1} \in$ H.

$\Rightarrow$ $e \in$ H, identity element $e \in$ H.

(iv) **Existence of Inverse**—Let $a \in$ H, then $e \in$ H, $a \in$ H $\Rightarrow eoa^{-1} \in$ H.

$\Rightarrow$ $a^{-1} \in$ H each element of H possesses inverse

Hence, H itself is a group for the composition o in group G.

Example 6. If $a^2 = a$, then $a = e$, a being an element of a group G.

Solution :
$$\begin{aligned} a^2 &= a \\ \Rightarrow \quad a \cdot a &= a \\ \Rightarrow \quad (a \cdot a)\, a^{-1} &= a \cdot a^{-1} \\ \Rightarrow \quad a\,(a \cdot a^{-1}) &= e \\ \Rightarrow \quad a \cdot e &= e \\ \Rightarrow \quad a &= e \end{aligned}$$

Example 7. If G is a group of even order, then it has an element $a \neq e$: $a^2 = e$, e being identity element.

Solution : $a \in$ G $\Rightarrow \exists a^{-1} \in$ G and $e = e^{-1}$, e an identity element.

Since o (G) is even, there exist at least one element $a \in$ G, which is its own inverse.

$$\begin{aligned} \textit{i.e.,} \quad a &= a^{-1}, a \neq e \\ \Rightarrow \quad a \cdot a &= a^{-1} \cdot a = e \\ \Rightarrow \quad a^2 &= e \end{aligned}$$

Example 8. In the additive group of integers, the order of every element a ($\neq 0$) is infinite.

Solution : For additive group of integers, 0 is an identity element and

$$0\,(0) = 1 \text{ as } 0' = 0$$

There exist no positive integer n which gives $na = 0$ for ($a \neq 0$)

Hence, order of every element except 0 is infinite.

Example 9. A group (G, *) is commutative iff $(a * b)^{-1} = a^{-1} * b^{-1}$, $\forall a, b \in$ G.

Solution : (G, *) is commutative, a^{-1}, $b^{-1} \in$ G

$$\Rightarrow \quad a^{-1} * b^{-1} = b^{-1} * a^{-1} \quad \ldots(1)$$

$$\text{and} \quad (a * b)^{-1} = b^{-1} * a^{-1} \quad \ldots(2)$$

By (1) and (2)

$$(a * b)^{-1} = a^{-1} * b^{-1}$$

Conversely let

$$(a * b)^{-1} = a^{-1} * b^{-1}, \forall a, b \in \text{G} \quad \ldots(3)$$

By (2) and (3), we have

$$a^{-1} * b^{-1} = b^{-1} * a^{-1}, \forall a, b \in \text{G}$$

$\Rightarrow$ * is commutative

$\Rightarrow$ The group (G, *) is Abelian.

Example 10. Is the set of integer z with binary operation $a \cdot b = a - b$, $\forall a, b \in z$ a group ?

Solution : (a) Closure : $a - b \in z$ for $\forall a, b \in z$

(b) Associatively : a, b, $c \in z$

$$(a - b) - c \neq a - (b - c)$$

$\therefore$ Set of integer z is not a group on defined operation.

Example 11. H $\subseteq$ K be two subgroup of a finite group G, then [G : H] = [G : K] [K : H].

Solution : H $\subseteq$ K are subgroups of a group G.

$\therefore$ H is also a subgroup of K

Since H is a subgroup of finite group G

By Lagrange's theorem

$$[\text{G} : \text{H}] = \frac{o\,(\text{G})}{o\,(\text{H})} = \frac{o\,(\text{G})}{o\,(\text{H})} = \frac{o\,(\text{G})}{o\,(\text{K})} \cdot \frac{o\,(\text{K})}{o\,(\text{H})} = [\text{G} : \text{K}]\,[\text{K} : \text{H}]$$

Example 12. If H is a subgroup of group G. For $x \in$ G, $x\text{H}x^{-1} = \{xhx^{-1} : h \in \text{H}\}$ is a subgroup of G.

Solution : Let xh_1x^{-1}, xh_2x^{-1} are two elements of xhx^{-1} then $h_1, h_2 \in$ H.

$$\begin{aligned}(xh_1x^{-1})(xh_2x^{-1})^{-1} &= xh_1x^{-1}x(xh_2)^{-1}\\ &= xh_1(xh_2)^{-1}\\ &= xh_1h_2^{-1}x^{-1}\\ &= x(h_1h_2^{-1})x^{-1}\end{aligned}$$

where $h_1h_2^{-1} \in$ H

$\Rightarrow x(h_1h_2)x^{-1} \in x\text{H}x^{-1}$ is a subgroup of G.

Example 13. If a, b belongs to a ring R, and $(a+b)^2 = a^2 + 2ab + b^2$, then R is a commutative ring.

Solution :
$$\begin{aligned}(a+b)^2 &= (a+b)\cdot(a+b)\\ &= a\cdot(a+b)+b.(a+b)\\ &= a\cdot a + a\cdot b + b\cdot a + b\cdot b\\ &= a^2 + ab + ba + b^2 \qquad \ldots(1)\end{aligned}$$

Given $\quad (a+b)^2 = a^2 + 2ab + b^2 \quad \ldots(2)$

By (1) and (2), we conclude

$$ab + ba = 2ab$$

$$\Rightarrow \quad ba = ab$$

i.e., R is a commutative ring.

Example 14. If a non zero element x of a ring R with unity has a multiplicative inverse, then x can not be a zero divisor.

Solution : Given $x \neq 0$ and x^{-1} exists. Also let $y \neq 0$ but $xy = 0$ so

$xy = 0 \Rightarrow x^{-1}(xy) = x^{-1}(0) \Rightarrow (x^{-1}x)y = 0 \Rightarrow 1\cdot y = 0 \Rightarrow y = 0$

Thus, $x \neq 0$, $xy = 0 \Rightarrow y = 0$, which is against our assumptions that $y = 0$.

Hence x can not be a zero divisor.

Example 15. If a, b are any elements of the ring (R, +;) m, n are integers, then

$$(na)(mb) = (nm)(ab)$$

Solution :
$$\begin{aligned}a(mb) &= a(b + b + \ldots m \text{ times})\\ &= ab + ab + \ldots m \text{ times}\\ &= m(ab)\\ (na)(mb) &= (a + a + \ldots n \text{ times})(mb)\\ &= amb + amb + \ldots n \text{ times}\\ &= m(ab) + m(ab) + \ldots n \text{ times}\\ &= n\, m(ab))\\ &= (nm)(ab)\end{aligned}$$

Example 16. The set of integer is a subring of the ring of rational numbers.

Solution : z is a set of integers, Q is a set of rational numbers.

$z \subset$ Q and $a, b \in z \Rightarrow a - b \in z$ and $ab \in z$

$\therefore$ z is a subring of Q.

Example 17. $s = \{\pm ma : m = 0, , 2, \ldots$ and 'a' any fixed integer} then s is a subring of (1, + ;) the ring of integers over addition and multiplication.

Solution : Let $ra, sa \in$ S and $s, r \in$ I

$ra - sa = (r - s)a \in s$ as $(r - s) \in$ I

$(ra)(sa) = (rsa)a \in s$ as $(rs)a \in$ I

$\therefore$ s is a subring of I.

Example 18. Any group of prime order can have no proper subgroup.

Solution : Let o (G) = P, P is a prime number. Let H be a subgroup of G and let o (H) $= m$,

By Lagrange's theorem o (H)|o (G) $\Rightarrow m|p$

$\because$ P is a prime number.

$\therefore$ $m = 1$ or p. If $m = 1 \Rightarrow o(\text{H}) = 1 \Rightarrow$ H $\{e\}$, if $m = p \Rightarrow o$ (H) $= o$ (G)

$\Rightarrow$ H = G.

$\therefore$ Either H = $\{e\}$ or H = G, *i.e.*, H is not a proper subgroup of G.

Hence, any group of prime order can have no proper subgroup.

Example 19. Two right cosets Ha and Hb are distinct iff two left cosets a^{-1} H and b^{-1} H are distinct.

Solution : Let a^{-1} H $= b^{-1}$ H, then a^{-1} H $=$ b^{-1} H $\Leftrightarrow (b^{-1})^{-1} a^{-1} \in$ H

$\Leftrightarrow ba^{-1} \in \text{H} \Leftrightarrow (ba^{-1})^{-1} \in \text{H} \Leftrightarrow (a^{-1})^{-1} b^{-1} \in$ H

$\Leftrightarrow ab^{-1} \in \text{H} \Leftrightarrow \text{H}a = \text{H}b$

$\because$ Ha = Hb $\Leftrightarrow ab^{-1} \in$ H, which is a contraction

Hence, H$a \neq$ H$b \Rightarrow a^{-1}\text{H} \neq b^{-1}\text{H}$

Similarly $a^{-1}\text{H} \neq b^{-1}\text{H} \Rightarrow \text{H}a \neq \text{H}b$

$\therefore$ H$a \neq$ Hb $\Leftrightarrow a^{-1}\text{H} \neq b^{-1}\text{H}$

Example 20. Prove that in a field :

(a) $\dfrac{a}{b} - \dfrac{c}{d} = \dfrac{ad - bc}{bd}$

(b) $(-a)^{-1} = -(a^{-1})$

(c) $\dfrac{(-a)}{(-b)} = \dfrac{a}{b}$

Solution : (a) $\dfrac{a}{b} - \dfrac{c}{d} = ab^{-1} - cd^{-1}$

$= b^{-1}a - cd^{-1}$ ($\because$ commutative law holds)

$= b^{-1}\,add^{-1} - b^{-1}\,bcd^{-1} \qquad (\because b^{-1}b = 1\ dd^{-1})$

$= (b^{-1}ad - b^{-1}bc)\,d^{-1}$

($\because$ right distributive law holds)

$= b^{-1}\,(ad - bc)\,d^{-1}$

($\because$ left distributive law holds)

$= \dfrac{ad - bc}{bd}$

(b) Let $(-a)^{-1} = x, \Rightarrow (-a)\,x = 1$, (unit element of the field)

$\Rightarrow\ ax = -1,\ \Rightarrow\ x = a^{-1}\,(-1)$

(a^{-1} is the multiplicative inverse of a)

$\Rightarrow x = -(a^{-1}\cdot 1) = -(a^{-1}) \qquad \because a^{-1}\cdot 1 = a^{-1}$

$\Rightarrow\ (-a)^{-1} = -(a^{-1})$

(c) $\left(\dfrac{-a}{-b}\right) = (-a)\cdot(-b)^{-1} = (-a)\,[(-b)^{-1}]$

$= ab^{-1}$

$= \dfrac{a}{b}$

Example 21. If I is an additive group of integers and E the additive group of even integers with zero, then the map $f : \mathrm{I} \to \mathrm{E}$ given by $f(x) = 2x$, where $x \in \mathrm{I}$, is an isomorphism.

Solution : Let $m, n \in \mathrm{I}$, then $f(m) = f(n)$

$\Rightarrow \qquad 2m = 2n$

$\Rightarrow \qquad m = n$

i.e., mapping f is one-one.

Let $z = \mathrm{E}, \exists\, x \in \mathrm{I} : x = \dfrac{z}{2}$, z even

i.e., mapping f is onto

Again $f(m + n) = 2\,(m + n)$

$= 2m + 2n$

$= f(m) + f(n)$

$\therefore$ The mapping f preserves the group composition and hence the mapping f is an isomorphism.

Example 22. Again G is Abelian if $b^{-1}a^{-1}ba = e\,\forall a, b \in \mathrm{G}$.

Solution : We have

$(ab)^{-1} = b^{-1}a^{-1}, \forall a, b \in \mathrm{G} \quad \ldots(1)$

If e is an identity element, then

$(ab)^{-1}\,(ab) = e$

$\Rightarrow \qquad b^{-1}a^{-1}ab = e \qquad \ldots(2)$

given $\qquad b^{-1}a^{-1}ba = e \qquad \ldots(3)$

By (2) and (3)

$b^{-1}a^{-1}ab = b^{-1}a^{-1}ba$

$\Rightarrow \qquad ab = ba$

$\Rightarrow$ G is Abelian.

Example 23. Prove that if G is a group and for $a, b \in \mathrm{G}$

$(a\cdot b)^2 = a^2\cdot b^2$ iff G be Abelian.

Solution : Let

$(a\cdot b)^2 = a^2\cdot b^2$

$\Rightarrow \qquad (a\cdot b)\cdot(a\cdot b) = (a\cdot a)\,(b\cdot b)$

$\Rightarrow \qquad a\cdot(b\cdot a)\cdot b = a\,(a\cdot b)\cdot b$

(By associative law)

$\Rightarrow \qquad a\cdot(b\cdot a) = a\cdot(a\cdot b)$

(By right cancellation law)

$\Rightarrow \qquad b\cdot a = a\cdot b$

(By left cancellation law)

$\Rightarrow$ G is Abelian.

Conversely

G is Abelian

$\Rightarrow \qquad b\cdot a = a\cdot b, \forall a, b \in \mathrm{G}$

$\Rightarrow \qquad a\cdot(b\cdot a) = a\cdot(ab)$

(Multiplying both sides on the left by a)

$\Rightarrow \qquad (ab)\cdot(ab) = (a\cdot a)\,(b\cdot b)$

(By associative law)

$\Rightarrow \qquad (a\cdot b)^2 = a^2\cdot b^2$

Hence, $(a\cdot b)^2 = a^2\cdot b^2\ \forall\, a, b \in \mathrm{G}$ iff G is Abelian.

Example 24. Prove that the centre of a group is always a normal subgroup of the group.

Proof : Let z be the centre of G so that

$z = \{z \in \mathrm{G} : zx = xz\ \forall x \in \mathrm{G}\}$

Let $z_1, z_2 \in z$ then, $z_1, z_2 \in z$

$\Rightarrow \qquad z_1x = xz_1$

and $\qquad z_2x = xz_2 \forall x \in \mathrm{G}$

Now $\qquad z_2x = xz_2 \forall x \in \mathrm{G}$

$\Rightarrow \qquad z_2^{-1}z_2xz_2^{-1} = z_2^{-1}xz_2z_2^{-1}\ \forall x \in \mathrm{G}$

$\Rightarrow \qquad xz_2^{-1} = z_2^{-1}\,x \forall\ x \in \mathrm{G}$

But $\qquad (z_1z_2^{-1})\,x = z_2\,(z_2^{-1}x)$

(Associativity)

$= z_1\,(xz_2^{-1})$

$(\because z_2^{-1}x = xz_2^{-1})$

$= (z_1x)\,z_2^{-1}$ (Associativity)

$= (xz_1)\,z_2^{-1} \quad (\because z_1x = xz_1)$

$= x\,(z_1z_2^{-1})\ \forall x \in \mathrm{G}$

$\therefore\ z_1z_2^{-1} \in z$

Hence $z_1 \in z \cdot z_2 \in z \to z_1 z_2^{-1} \in z$ and therefore, z is a subgroup of G.

Again let $x \in G$ and $z \in z$, then

$$\begin{aligned} xzx^{-1} &= (xz)\,x^{-1} && \text{(Associativity)} \\ &= (zx)\,x^{-1} && (\because xz = zx) \\ &= z(xx^{-1}) && \text{(Associativity)} \\ &= ze = z \in z \end{aligned}$$

$\therefore\ xzx^{-1} \in z\ \forall x \in G$ and $z \in z$.

Hence, z is a normal in G. Thus, z is a normal subgroup of G.

Example 25. If a, b are any two elements of a group G and H is any subgroup of G, then

$$a \in Hb \text{ iff } Ha = Hb.$$

Solution : $a \in Hb$

$$\begin{aligned} \Rightarrow\quad ab^{-1} &\in Hbb^{-1} \\ \Rightarrow\quad ab^{-1} &\in He && (\because bb^{-1} = e) \\ \Rightarrow\quad ab^{-1} &\in H && (\because He = H) \\ \Rightarrow\quad Hab^{-1} &= H \\ \Rightarrow\quad Hab^{-1}b &= Hb \end{aligned}$$

(Multiplying both sides on the left by b)

$$\begin{aligned} \Rightarrow\quad Hae &= Hb && (\because b^{-1}b = e) \\ \Rightarrow\quad H \cdot a &= Hb && (\because ac = a) \end{aligned}$$

Conversely let $Ha = Hb$. Since $a \in Ha$, therefore $a \in Hb$.

Example 26. Prove that if H is any subgroup of G, then $H^{-1} = H$.

Solution : Let $h^{-1} \in H^{-1}$, then $h \in H$

Since H is a subgroup of G

$$h \in H \to h^{-1} \in H$$

(By inverse axiom)

Therefore, $h^{-1} \in H^{-1} \to h^{-1} \in H$

$$\therefore\quad H^{-1} \le H$$

Again $\quad h \in H \to h^{-1} \in H$

[$\because$ H itself is a group]

$$\begin{aligned} \Rightarrow\quad (h^{-1})^{-1} &\in H^{-1} \\ \Rightarrow\quad h &\in H^{-1} \\ H &\le H^{-1} \end{aligned}$$

Hence, from (1) and (2), we get

$$H^{-1} = H$$

Example 27. Prove that the normalizer of any element of a group is always a subgroup of the same.

Solution : Let G be any group and N (a) the normalizer of a $(\in G)$ in G. Let $x, y \in N(a)$, then

$$x \in N(a) \text{ and } z \in N(a)$$

$$\begin{aligned} \Rightarrow\quad ax &= xa \text{ and } ay = ya \\ \text{Now}\quad ay &= ya \to y^{-1}ayy^{-1} = y^{-1}yay^{-1} \\ \Rightarrow\quad y^{-1}a &= ay^{-1} \\ y^{-1}a &= ay^{-1} \end{aligned}$$

and therefore,

$$\begin{aligned} a\,(xy^{-1}) &= (ax)\,y^{-1} \\ &= (xa)\,y^{-1} \\ &= x\,(ay^{-1}) \\ x\,(y^{-1}a) &= (xy^{-1})\,a \end{aligned}$$

Which shows that $xy^{-1} \in N(a)$

Hence, N (a) is a subgroup of G.

Example 28. Prove that if R is a Euclidean ring and $b \in R$ is not unit, then $d(a) < d(ab)\ \forall a \in R$.

Solution : Let

$$a\,(\neq 0) \in R$$

Consider the ideal $U = (a)$ by the definition of Euclidean ring $d(a) \le d(xa)$ for any non zero $x \in R$

$$\begin{aligned} \text{Also}\quad ab &\in U \\ \text{and if}\quad d(a) &= d(ad) \\ \text{then}\quad (a) &= (ab) = U \end{aligned}$$

i.e., every element of U is a multiple of ab. In particular a is also a multiple of ab, *i.e.*,

$$a = abx \qquad \text{for some } x \text{ in R}$$

Now $\quad a = abx$

$\Rightarrow a \cdot 1 = abx \to 1 = bx \qquad (\because a \neq 0)$

$\Rightarrow b$ is a unit in R

Thus if b is not unit in R then

$$d(a) \neq d(ab)$$

and hence we conclude that

$$d(a) < d(ab)$$

Example 29. Prove that the union of two subgroups is a subgroup iff one is contained in other. The union of two subgroups H_1 and H_2 is a subgroup if and only if one is contained in the other.

Solution : Let H_1 and H_2 are two subgroups of a group.

(i) Let $\quad H_1 \subseteq H_2$

or $\quad H_2 \subseteq H_2$

then $\quad H_1 \cup H_2 = H_2$ or H_1

$\therefore\ H_1 \cup H_2$ is a subgroup as $H_1 \cdot H_2$ are subgroups

(ii) Suppose $H_1 \cup H_2$ is a subgroup

If possible let us assume that H_1 is not contained in H or H_2 is not contained in H_1.

If H_1 is not contained in H_2

$\Rightarrow$ $a \in H_1$, and $a \notin H_2$ ...(1)

and H_2 is not contained in H_1

$\Rightarrow$ $b \notin H_2$ and $b \notin H_1$ ...(2)

$\therefore$ From (1) and (2), we get

$$a \in H_1 \cup H_2 \text{ and } b \in H_1 \cup H_2$$

As $H_1 \cup H_2$ is a subgroup so ab is also an element of $H_1 \cup H_2$

But $ab \in H_1 \cup H_2$

$\Rightarrow$ $ab \in H_1$

or $ab \in H_2$

Let $ab \in H_1$ then $a^{-1}\,ab \in H_1$ as $a \in H_1$ and H_1 is a subgroup. So that $a^{-1}\,ab = b \in H_1$, which is a contradiction of the assumption that

$$a \notin H_1$$

Hence either

$$H_1 \subseteq H_2$$

or $H_2 \subseteq H_1$

Example 30. If G is a group such that $(ab)^m = a^m b^m$, for three consecutive integers $m \forall a, b \in G$ then prove that the group G is Abelian.

Solution : Let the three consecutive integral valves of m be $n-1$, n and $n+1$, then for

$$\forall a, b \in G$$

$$(ab)^{n-1} = a^{n-1}\,b^{n-1}$$

$$(ab)^n = a^n b^n$$

and $(ab)^{n+1} = a^{n+1}\,b^{n+1}$

Now $(ab)^{n+1} = (ab)^n\,(ab)$

$\Rightarrow$ $a^{n+1}\,b^{n+1} = a^n b^n\,(ab)$

$(\because (ab)^m = a^m b^m)$

For $m = n, n+1$

$\Rightarrow$ $a^n a b^n = a^n b^n ab$

$\Rightarrow$ $ab^n b = b^n ab$

(By left cancellation law)

$\Rightarrow$ $ab^n = b^n a$

(By right cancellation law)

$\Rightarrow$ $a^{n-1}\,(ab^n) = a^{n-1}\,(b^n a)$

Multiplying both sides

(by a^{n-1} on the left)

$\Rightarrow$ $(a^{n-1}\,a)b^n = a^{n-1}\,(b^{n-1}\,ba)$

(By associative law)

$\Rightarrow$ $a^n b^n = a^{n-1}\,b^{n-1}\,(ba)$

$\Rightarrow$ $(ab)^n = (ab)^{n-1}\,(ba)$

$(\because (ab)^m = a^m b^m)$

(for $m = n, n-1$)

$\Rightarrow$ $(ab)^{n-1}\,(ab) = (ab)^{n-1}\,(ba)$

$\Rightarrow$ $ab = ba$

(By left cancellation law)

$\Rightarrow$ G is an Abelian group.

Example 31. Let f be a homomorphism from G into G, then prove that

(i) $f(e) = e$ where e is the unit element of G

(ii) $f(a^{-1}) = f(a)^{-1}$ for each $a \in G$.

Solution : (i) For each $a \in G$

$$f(a)\bar{e} = f(a) = f(ae) = f(a)\,f(e)$$

and hence $f(e) = e$

(ii) For each $a \in G$

$$f(a)\,f(a^{-1}) = f(aa^{-1}) = f(e) = \bar{e}$$

$$= f(a)\,f(a)^{-1}$$

and hence $f(a^{-1}) = f(a)^{-1}$

Example 32. Prove that a non empty subset H of a group G is a subgroup iff $HH^{-1} \subseteq H$.

Solution : Let H is subgroup of G.

Let ab^{-1} be any element of HH^{-1}

then $a \in H$ and $b \in H$

Also $b \in H \rightarrow b^{-1} \in H$ as H is a subgroup.

Thus $a \in H$, $b^{-1} \in H \rightarrow ab^{-1} \in H$

(By closure property)

$\therefore$ $ab^{-1} \in HH^{-1} \rightarrow ab^{-1} \in H$

Hence $HH^{-1} \subseteq H$

Let $HH^{-1} \subseteq H$

Let $a \in H$, $b \subseteq H$, then $ab^{-1} \in HH^{-1}$

$\because$ $HH^{-1} \subseteq H$ so $ab^{-1} \in HH^{-1}$

$\Rightarrow$ $ab^{-1} \in H$

i.e. $a \in H$, $b \in H \rightarrow ab^{-1} \in H$

$\therefore$ H is a subgroup of G.

Example 33. The mapping $I \rightarrow I(n)$ defined by $f(a) = \{a\}\ \forall a \in I$ is a homomorphism of I onto $I(n)$ I being the set of all integers and $I(n)$ is the set of residue classes modulo n.

Solution : Let x, y be any two element of I then

$$x + y \in I \text{ and } xy \in I \text{ and we have}$$

$$f(a+b) = \{a+b\}$$

$(\because f(a) = \{a\}\ \forall a \in I)$

$$= \{a\}_n\,\{b\}$$

or $\quad f(ab) = f(a)f(b) \quad \ldots(2)$

$\therefore$ From (1) and (2) we conclude that the mapping f is a homomorphism of I into I (n), as $\forall \{a\} \in$ I (n) there exists an element $a \in$ I such that

$$f(a) = \{a\}$$

Example 34. Show that a set

S = $\{am : am \in z$ and m any fixed integer$\}$ is an additive group.

Solution : Here

$$S = \{am : m \text{ fixed integer } a \in z\}$$
$$= \{\ldots -3m - 2m - m, 0m\ 2m\ 3m, \ldots\}$$

To prove that S is a group it satisfies group postulate.

Closure : If and bm be any two elements of S where $a, b \in z$, then we get $am + bm = (a + b)m$ $(\because a + b \in z)$

Associativity : $\forall abc \in z$, we get

$$am + (bm + cm) = am + (b + c)m$$
$$= (a + b + c)m \quad \ldots(1)$$

and $(am + bm) + cm = (a + b)m + cm$

$$= (a + b + c)m \quad \ldots(2)$$

$\therefore$ From (1) and (2) we find that

$$(m + (bm + cm)) = (am + bm) + cm$$

Hence, the operation of addition is associative in S.

Existence of identity : Since

$$0 + am = am = am + 0 \forall\ am \in s$$

So 0 is the identity of S for addition.

Existence of inverse : $am = 0 = (am) + (-am)$

Hence, the inverse of each elements of s exists.

Since all group postulates are satisfied and so the set S form a group under addition.

Example 35. The order of an element of a group is the same as that of its inverse.

Solution : Let $a \in$ G and let n and m be the orders of a and a^{-1} respectively.

Therefore, $\quad a^n = e, (a^{-1})^m = c \quad \ldots(1)$

Now $\quad a^n = e \Rightarrow (a^n)^{-1} = e^{-1}$

$\Rightarrow \quad (a^{-1})^n = e$

$\Rightarrow \quad m \leq n \quad$ (as m is order of a^{-1})

Again $(a^{-1})^m = e \Rightarrow (a^m)^{-1} = e$

$\Rightarrow \quad a^m = e \quad [\because x^{-1} = e \Rightarrow x = e]$

$\Rightarrow \quad n \leq m \quad$ [as n in order of a]

Thus $m \leq n$ and $n \leq m \Rightarrow m = n$

Example 36. Use Lagrange's theorem to show that any group of prime order can have no proper subgroups.

Solution : Let o (G) = P where P is a prime number

Let H be a subgroup of G and let o (H) = m

By Lagrange's theorem we know o (H)/o (G)

$\therefore$ m is a divisor of P.

Also P being prime, we find that either

$$m = 1 \text{ or } m = P$$

Now $\quad m = 1 \Rightarrow o(H) = 1$

$\Rightarrow \quad H = \{e\}$

and $\quad m = P \Rightarrow o(H) = o(G)$

$\Rightarrow \quad H = G$

$\therefore$ Either H = $\{e\}$ or H = G *i.e.* H is not a proper subgroup of G.

Hence, any group of prime order can have no proper subgroups.

Example 37. If a and b are any elements of a group G, then $(bab^{-1})^n = ba^nb^{-1}$, for any integer n.

Solution : (i) Let $n = 0$

If e be the identity element, then by definition, we have $(bab^{-1})^0 = e$

Also $\quad ba^0b^{-1} = bb^{-1} = bb^{-1} = e$

$\therefore \quad (bab^{-1})^0 = ba^0b^{-1}$

(ii) Let $n > 0$

Here we get $(bab^{-1})^1 = bab^{-1} = ba^1b^{-1}$ $\quad (\because a^1 = a)$

$\Rightarrow$ $(bab^{-1})^n = ba^nb^{-1}$ is true for $n = 1$

Let us now suppose that this result is true for $n = k$, *i.e.* suppose

$$(bab^{-1})^k = ba^kb^{-1} \quad \ldots(1)$$

then $(bab^{-1})^{k+1} = (bab^{-1})^k (bab^{-1})^1$

$$= (ba^kb^{-1})(bab^{-1}) \text{ from (1)}$$
$$= ba^kb - 1\ bab^{-1}$$
$$= ba^k\ eab^{-1}$$
$$= ba^kab^{-1}$$
$$= ba^k + 1\ b - 1$$

$\therefore$ The result is true for $n = k + 1$ also if it is true for $n = k$ also the result is true for $n = 1$

$\therefore$ By mathematical induction it is true for all values $opn > 0$.

(iii) Let $\quad n < 0$

Let $n = -k$, where $k > 0$

then $(bab^{-1})^n = (bab^{-1})^{-k}$

$$= [(bab^{-1})^k]^{-1}$$

$= [ba^k b^{-1}]^{-1}$

$= (b^{-1})^{-1} (a^k)^{-1} (b)^{-1}$

(By reversal rule)

$= ba^{-k}b^{-1}$

$= ba^n b^{-1}$ $\quad (\because -k = n)$

Hence, $(bab-1)^n = ba^n b^{-1}$

Example 38. If R is an integral domain, then prove that :

(i) R $[x]$ is also an integral domain.

(ii) R $[x]$ is not a field.

Solution : (i) Let R be an integral domain and let P $(x) \neq 0$ and $q\ (x) \neq 0$ be in R $[x]$ since R is an integral domain we have

deg (P (x) q (x)) = *deg* P (x) + *deg* q (x) now it is impossible to have P (x) q $(x) = 0$ and therefore, R $[x]$ is an integral domain.

(ii) Let P $(x) \in$ R $[x]$ and *deg* P $(x) > 0$. Since degree of the identity polynomial 1 is 0 there exists no $q\ (x) \in$ R $[x]$ such that *deg* P (x) + *deg* q (x) = *deg* 1 = 0. Thus no polynomial of non-zero degree has a multiplicative inverse and thus R $[x]$ in not a field.

Example 39. Write down the cyclic groups of order 2, 3 and 4 from the symmetric groups 4 on four symbols.

Solution : The required cyclic groups of order 2 are

$H_1 = \{(12)\ (1)\ (2)\ (3)\ (4) = 1\}$

Here if $a = (1, 2)$, then

$a^2 = (1)\ (2)\ (3)\ (4)$

$H_2 = \{(13)\ (1)\ (2)\ (3)\ (4) = I\}$

$H_3 = \{(14)\ (1)\ (2)\ (3)\ (4) = I\}$

$H_4 = \{(23)\ (1)\ (2)\ (3)\ (4) = I\}$

$H_5 = \{(24)\ (1)\ (2)\ (3)\ (4) = I\}$

$H_6 = \{(34)\ (1)\ (2)\ (3)\ (4) = I\}$

The required cyclic groups of order 3 are

$H_1 = \{(123), (132)\ (1)\ (2)\ (3)\ (4) = I\}$

IP$a = (123)$ then $a^2 = (132)$ and $a^3 = I$

$H_2 = \{(124)\ (142)\ (1)\ (2)\ (3)\ (4) = I\}$

$H_3 = \{(134)\ (143)\ (1)\ (2)\ (3)\ (4) = I\}$

$H_4 = \{(234)\ (243)\ (1)\ (2)\ (3)\ (4) = I\}$

and the required cyclic groups of order 4 are

$H_1 = \{(1234)\ (13)\ (24)\ (1432)$

$(1)\ (2)\ (3)\ (4) = I]$

($\because$ IP$a = (1234)$, then $a^2 = (13)\ (24)$ $a^3 = (1432)$ and $a^4 = I$)

$H_2 = \{(1243)\ (14)\ (23)\ (1342)$

$(1)\ (2)\ (3)\ (4) = I\}$

$H_3 = \{(1324)\ (12)\ (34)\ (1423)$

$(1)\ (2)\ (3)\ (4) = I\}$

Example 40. If R be commutative ring with unity and I an ideal in R, then prove that R/I is a field iff I is a maximal ideal in R.

Solution : Let I be a maximal ideal in R, then I < R and R/I is a commutative ring with unity.

Let x be any element of R/I and let us consider a subset S of R where

$$S = \{a + xb : a \in I\ b \in I\}$$

Now $\quad Y \in R$

$a + xb \in S$

$\Rightarrow \quad (a + xb)\ y \in S$

$\Rightarrow \quad ay + x\ (by) \in S$

Similarly $y\ (a + xb) \in S$

$\therefore$ S is an ideal in R

Also as $I \subset S$ so S = R.

Hence any element $z \in R$ can be written as

$$z = a + xc \text{ where } c \in R$$

Let c be the unity in R and $c = a + x, d$, where $d \in R$

Again as $e + I = (a + I) + (xc + I)$

$= (a + I) + (x + I)\ (c + I)$

$= (x + I)\ (c + I)$

So $(c + I)$ is the multiplicative inverse of $(x + I)$

$\therefore$ The ring of co-sets R/I is a field as x is any element of R/I

On the contrary assume let R/I be a field and suppose I is not maximal in R.

Let I' be another ideal in R such that

$$I \subset I \subset R$$

Let z be any element of R and let

$$q \in I' - I$$

Then if, we define

$(q + I)^{-1}\ (z + I) = (p + I)$

We get $\quad (z + I) = (q + I\ (p + I)$

Now as $z - qp \in I$ and $I \subset I$ so $z - qp \in I$

But $P \in I'$, so $z \in I$ also $z \in R$

$\therefore$ I = R which contradicts that fact $I \subset R$

Hence, 0 or supposition is wrong and so I is a maximal in R.

Example 41. Prove that if R is a commutative ring and $a \in$ R, then

R$a = \{ra : r \in$ R$\}$ is an ideal of R

Solution : If x, y be any two arbitrary elements of Ra

then $x = r_1 a$ and $y = r_2 a$

For some $r_1 r_2 \in$ R

$\therefore \quad x - y = r_1 a - r_2 a = (r_1 - r_2)\, a \in$ Ra

(Since $r_1 - r_2 \in$ R as $r_1 - r_2 \in$ R)

$\therefore\ x \in$ Ra $y \in$ R$a \Rightarrow x - y \in$ Ra ...(1)

Again $x - y = (r_2 a)\,(r_2 a) = (r_1 r_2)\, a \in$ Ra

(Since $r_1 r_2\, a \in$ R as $a, r_1 r_2 \in$ R)

$\therefore\ x \in$ Ra $y \in$ R$a \Rightarrow xy \in$ Ra ...(2)

From (1) and (2) we conclude that Ra is a Subring of R

Again if $u \in$ R

then $ux = u\,(r_1 a) = (ur_1)\, a \in$ Ra

(Since $ur_1 \in$ R as $ur_1 \in$ R)

Similarly we can prove that

$$\begin{aligned} xa &= (r_1 a)^4 \\ &= u\,(r_1 a) \\ &\quad \text{as R is a commutative ring} \\ &= (ur_1)\, a \in \text{R}a \text{ as above} \end{aligned}$$

Thus, we find that Ra is a subring of R and for each element $x \in$ Ra and $u \in$ R we find that $ux \in$ Ra and

$$xu \in \text{R}a$$

$\because$ Ru is an ideal.

Example 42. Prove that a non-empty finite subset H of a group G is a subgroup of G.

If $\quad a, b \in$ H

$\Rightarrow \quad ab \in$ H

being the composition in G.

Solution : The non-empty subset H of G is a subgroup of G if it satisfies group axioms.

Closure : Closure property is already given because of the given condition.

$$a, b \in \text{H} \Rightarrow ab \in \text{H}$$

Associative : Since H is a subset of the group G, the associative law holds for all elements of H.

Existence of Inverse & Identity : To prove existence of identity and inverse.

Let $a \in$ H then by closure law, we have

$a^2 \in$ H, $a^3 \in$ H, ... $a^n \in$ H ... *i.e.* $a, a^2, \ldots a^3 \ldots a^n \ldots \in$ H.

Since H is a finite subset of G, there must be repetitions in the elements stated above otherwise H will become infinite.

Let $\quad a^r = a^s$ for $r \neq s$ and $r > s > 0$

then $\quad a^{r-s} = e \quad$ (by cancellation law)

Also $\quad a^{r-s} \in$ H

$\therefore\ e$, the identity element $\in$ H

Clearly $\quad r - s \geq 1$

Therefore, $\quad r - s - 1 \geq 0$

Hence, $\quad a^{r-s-1} \in$ H

$\Rightarrow \quad a^r - soa^{-1} \in$ H

$\Rightarrow \quad eoa^{-1} \in$ H

$\Rightarrow \quad a^{-1} \in$ H

($\because a^{r-s} = e$ already proved)

All the axioms of a group are satisfied and H $\subset$ G is a subgroup of G.

Example 43. Let G be a group H a subgroup of G. Let for $x \in$ G such that

$$x\text{H}x^{-1} = \{ax^{-1} : a \in \text{H}\}$$

Prove that xHx^{-1} is a subgroup of G.

Solution : Let $x a_1 x^{-1}$ and $x a_2 x^{-1}$ be any two elements of xHx^{-1}, then $a_1 a_2 \in$ H.

Now, we have

$$\begin{aligned} (xa_1x^{-1})\,(xa_2x^{-1})^{-1} &= (xa_1x^{-1})\,(xa_2{}^{-1}x^{-1}) \\ &= xa_1\,(x^{-1}x)\,a_2{}^{-1}\,x^{-1} \\ &= xa_1a_2{}^{-1}x^{-1} \\ &= xax^{-1} \end{aligned}$$

if $\quad a = a_1a_2{}^{-1} \in$ H

Evidently $\quad xax^{-1} \in x$Hx^{-1}

Hence, xHx^{-1} is a subgroup of G.

Example 44. Prove that

(i) Every cyclic group is Abelian group.

(ii) The order of a cyclic group is same as that of its generator.

Solution : (i) Let G = $(a) = \{e, a, a^2, a^3, \ldots\}$ be a cyclic group with a as the generator then for any

$$m, n \in z, a^m, a^n \in \text{G}$$

and
$$\begin{aligned} a^m a^n &= a^{m+n} \\ &= a^{n+m} \\ &= a^n a^m \end{aligned}$$

and hence, G is Abelian, in the additive rotation we write

$$\text{G} = \{1, a, 2a, 3a \ldots\}$$

In this case

$$ma + na = (m+n)a$$
$$= (n+m)a$$
$$= na + ma$$

and thus G is Abelian.

(ii) Let G be a cyclic group with a generator

$$a \in G$$

Let $o(a) = n$

i.e. $a^n = e$ and $a^m \neq e$ for $0 < m < n$

If $m > n$ let $m = q^{n+r}$

$$0 \leq r < N$$

so $a^m = a^{q^n + r} = a^{q^n} \cdot a^r$

$$= (a^n)^q \cdot a^r$$
$$= e^q \cdot a^r = e \cdot a^r$$
$$= a^r$$

Therefore, there are exactly n elements in the group and they are

$$\{e, aa^2 \ldots a^{n-1}\} \Rightarrow o(G) - n$$

Example 45. Let G be a finite group and $a \in G$, show that order of a, $o(a)$ is equal to the o (H) where H is the subgroup of G generated by a deduce that $o(a)$ divides o (G).

Solution : The subgroup H of G generated by a is given by

$$H = \{a^r : r \in I\}$$

Where I is the set of integers

Let $o(H) = n$

We can show that H has exactly m distinct elements $a, a^2, a^3 \ldots a^3$ and that every element of H is equal to one of these m elements.

so $o(H) = o(a) = m$

By Lagrange's theorem

$$o(H) \mid o(G)$$

Hence $o(a) \mid o(G)$

Example 46. Prove that the mapping

$$f : V_3(F) \to V_2(F) \text{ defined by}$$
$$f(a_1 a_2 a_3) = (a_1, a_2)$$

is a homomorphism ?

Solution : Let $\alpha = (a_1, a_2, a_3)$ and $\beta = (b_1, b_2, b_3)$ be any two elements of V_3 (V)

Let a, b be any two elements of F, then

$$f(a\alpha + b\beta) = f[(a_1, a_2, a_3) + b(b_1, b_2, b_3)]$$
$$= f[aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3)]$$
$$= (aa_1 + bb_1, aa_2 + bb_2)$$
$$= a(a_1, a_2) + b(b_1, b_2)$$
$$= af(a_1, a_2, a_3) + bf(b_1, b_2, b_3)$$
$$= af(\alpha) + bf(\beta)$$

$\therefore f$ is a linear transformation

Let $(a_1 a_2)$ be an element of V (F), then $(a_1, a_2, 0) \in (-V_3 F)$ and $f(a_1, a_2, 0) = (a_1, a_2)$

Hence f is onto

Thus f is a homomorphism of V_3 (F) onto V_2 (F)

If W is the Kernel of this homomorphism then

$$W = \{0, 0, a) : a \in F\}$$

Thus $f(0, 0, a) = (0\,0)$

The zero vector of V_2 (F) $\forall a \in F$

Also if $f(a_1, a_2, a_3) = (0, 0)$

Then $f(a_1, a_2, a_3) = (a_1, a_2) = (0, 0)$

i.e., $a_1 = 0 = a_2$

Hence $(a_1, a_2, a_3) \in W$

The W is the Kernel of f.

Example 47. Prove that the relation of isomorphism in the set of all groups in an equivalence relation.

Solution : If G and $\bar{G}$ are isomorphic

i.e. $G = \bar{G}$

The $G \approx \bar{G}$ is a equivalence a relation if its satisfies the following axioms :

(i) $G \approx \bar{G}$ (reflexivity)

(ii) $G \approx G\bar{G} \approx \bar{\bar{G}}$ implies $\bar{G} \approx \bar{\bar{G}}$ (symmetry)

(iii) $G \approx G\bar{G} \approx \bar{\bar{G}}$ implies $\bar{G} \approx \bar{\bar{G}}$ (Transitivity)

For each $x \in G$ define $f : G - G$ by $f(x) = x$

Then f is an isomorphism of G onto itself and hence G = G.

Let $G \approx \bar{G}$, then there exists an isomorphism.

f from G onto $\bar{G}$ since f is 1 – 1 and onto, the map

$f^{-1} : \bar{G} \to G$ exists and is 1 – 1 and onto

Let

$a, b \in \bar{G}$ and let $f(x) = 4$

$f(y) = b$ for $x, y \in G$

Now $x = f^{-1}(a)$, $y = f^{-1}(b)$

$$f(xy) = f(x)f(y) = ab$$

and so $f^{-1}(ab) = xy = f^{-1}(a)f^{-1}(b)$

Then f^{-1} is also a homomorphism and so $\bar{G} \approx G$.

Let $G \approx \bar{G}$ and $\bar{G} \approx \bar{\bar{G}}$ thus there exist isomorphisms f and g from G onto $\bar{G}$ and from $\bar{G}$ onto $\bar{G}$ respectively since f and g are one-one and onto, the composition g of : $G \Rightarrow \bar{\bar{G}}$ is $1-1$ and onto

Further from $x, y = \in G$

$$\begin{aligned}(gof)(xy) &= g(f(xy)) \\ &= g(f(x)f(y)) \\ &= g(f(x))\,g(f(y))\end{aligned}$$

Thus g of is a homomorphism and hence $G \approx \bar{G}$

Example 48. Prove that the left cosets of a subgroup are either disjoint or identical.

Solution : Let H be a subgroup of a group G and $a, b \in G$.

If there is no element common to aH and bH, then $aH \cap bH = \phi$, the null set.

i.e. aH and bH are disjoint.

If there is an element c (say) common to aH then $aH \cap bH$ = then we have

$$\begin{aligned} c &= ah_1, h_1 \in H \\ c &= bh_2, h_2 \in H \\ ah_1 &= bh_2 \\ \Rightarrow \quad (ah_1)h_2^{-1} &= (bh_2)\,h_2^{-1} \\ \Rightarrow \quad a\,(h_1h_2^{-1}) &= b\,(h_2h_2^{-1}) \\ \Rightarrow \quad a\,(h_1h_2^{-1}) &= b \\ \Rightarrow \quad a^{-1}\,a\,(h_1h_2^{-1}) &= a^{-1}b \\ \Rightarrow \quad (a^{-1}a)\,(h_1h_2^{-1}) &= a^{-1}b \\ h_1h_2^{-1} &= a^{-1}b \end{aligned}$$

$\therefore\ a^{-1}b \in H$ since $h_1h_2^{-1} \in H$

Hence, $aH = bH$

Hence, two left cosets which are not disjoint are identical.

Example 49. In any group G show that $e^n = e$ for any integer n.

Solution : (i) Let $n = 0$

If $n = 0$, then $e^n = e^0 = e$

(ii) Let $n > 0$ (*i.e.* n is a positive integer)

Let $e^n = e$ be true for $n = m$ *i.e.* $e^m = e$

Now $e^{m+1} = e^m \cdot e^1 = e^m e = e \cdot e \qquad (\because\ e^m = e)$

or $\qquad e^{n+1} = e \quad (\because\ a \cdot e = a \forall a \in G)$

This shows that $e^n = e$ is true for $n = m + 1$ if it is true for $n = m$.

Obviously $e^1 = e$ *i.e.* $e^n = e$ is true for $n = 1$

Hence if is true for $n = 1 + 1$ *i.e.* 2 if is true for $n = 1$ and proceeding in this way we can show that it is true for $n = 3, 4$ etc.

Hence $e^n = e$ is for all positive integral values of n.

(iii) Let $n < 0$ (*i.e.* n is a negative integer)

Let $n = k$ where k is a positive integer

Then $e^n = e^{-k} = (e^k)^{-1} = (e)^{-1} \qquad (\because\ ek = e)$

For $\qquad k > 0 = e^{-1} = e$

Hence, in any group $e^n = e$ for any integer n.

Example 50. Prove that the set {1, 2, 3, 4} form a group under multiplication modul.

Solution : Under multiplication modulo the composition table is

s	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Closure—This table shows that the product of any two elements of the given set under multiplication modulos as composition belongs to the set and therefore under the given operation the set is closed.

Associative—$(ab).\ c$ and $a\ (b \cdot c)$ both denote zero or that least non-negative integer obtained on dividing the ordinary multiplication of a, b and c by 5. *i.e.* the composition is associative.

Existence of Identity—From the table it is evident that 1 is the identity.

Existence of Inverse—The inverse of 1, 2, 3 and 4 are 1, 3, 2 and 4 respectively because $1 \cdot 1 = 1$, $2 \cdot 3 = 6 = 1 \pmod 5$ $3 \cdot 2 = 1 \pmod 5$ and $4 \cdot 4 = 1 \pmod 5$ this is also evident from the composition table.

Commutative—$ab = ba$ any two elements a, b in the given set.

Hence, all the group postulates are satisfied and so the given set is a finite Abelian group of order 4 under multiplication.

Example 51. If a is of order n and p is prime to n, then the order of ap is also n.

Solution : Let n' be the order of ap

then we have $\quad n' \leq n \quad \ldots(1)$

Again p is prime to n, so there exist integers α and β such that

$$\alpha n + \beta n = 1$$

Hence, we can write

$$a = \alpha = \alpha^{an} + \beta^{n}$$
$$= a^{ap} \cdot \alpha\beta^{n}$$
$$= \alpha^{ap}$$

as $\quad a^{n} = e$ then identity element

$$= (ap)^{a}$$

a is a power of a', hence, we have

$$n \leq n$$

$\therefore$ a is a power of ap hence, we have

From (1) and (2) whaven $n' = n$

Example 52. Let G be the group of all real numbers under addition and G be the group of all non-zero real numbers under multiplication. Prove that the mapping ϕ: G $\rightarrow$ G is such that $\phi(x) = z^{x}$, $x \in$ G, then show that ϕ is homomorphism.

Solution : Let $m, n \in$ G, then $\phi(m) = 2^{m}$

$$\phi(n) = 2^{n}$$

Since $\quad \phi(x) = 2^{x}\, x \in G$

Now $\quad \phi(m+n) = 2^{m+n} = 2^{m}\, 2^{n}$

$$= \phi(m)\,\phi(n)$$

i.e. $\quad \phi(m+n) = \phi(m)\,\phi(n)$

Hence, the mapping ϕ is a homomorphism.

Example 53. Prove that if S is any ideal of ring R and T any subring of R then S is ideal of S + T.

Solution : If $a, b \in$ S and $a\beta \in$ T

$\Rightarrow \quad a + a \in$ S + T and $b + \beta \in$ S + T

Also $(a + \alpha)(b + \beta) \equiv ab + a\beta + \alpha b + \alpha\beta$

$\Rightarrow \quad (a + a)(b + \beta) - a\beta = ab + a\beta + \alpha b \in$ S

S being an idea, we get

$ab \in$ S, $ab \in$ S and $a\beta \equiv$ S

Also $\quad a\beta \in$ T

$\therefore \quad (a + a)(b + \beta) \in$ S + T

Again $\quad (a + a) - (b + \beta) = (a - b) + (\alpha\beta) \in$ S + T

$\therefore$ S + T is a subring of R

Also we have

$$S \subset S + T \subset R$$

Hence, S is an ideal of S + T

Example 54. Prove that the order of every element of a finite group G is finite.

Solution : Let o (G) = n and let $a \in$ G be arbitrary.

The element e, a, a^{2} … a^{n} are $(n + 1)$ in number and o (G) = n and hence they are not all $\leq$ S $\leq n$ such that $a^{r} = a^{s}$. Hence, $e = a^{r}a^{-r} = a^{sa-r} = as - r = a^{k}$ where $k = s - r$, $1 \leq k \leq n$.

Therefore, o (a) is finite.

Example 55. Prove that if H, K are two subgroup of a group G, then HK is a subgroup of G iff HK = KH.

Solution : Suppose that HK is a subgroup then

Then $\quad (HK)^{-1} = HK$

$\Rightarrow \quad K^{-1}H^{-1} = HK$

$$KH = HK$$

[$\because$ H is subgroup $\Rightarrow$ H^{-1} = H and K is a subgroup $\Rightarrow$ K^{-1} = K]

Let HK = KH then, we have to prove that HK is a subgroup of G for this it is sufficient to prove

$$(HK)\,(HK)^{-1} = HK$$

Now $\quad (HK)\,(HK^{-1}) = HK\,(K^{-1}H^{-1})$ (Reversal law)

$$= H\,(KK^{-1})\,H^{-1}$$ (Associativity)

$$= (HK)\,H^{-1}$$

[$\because$ K is a subgroup $\because$ KK^{-1} = K]

$$= (KH) - H^{-1}$$

[$\because$ HK = KH]

$$= K\,[HH^{-1}]$$
$$= KH$$

[$\because$ H is a subgroup $\therefore$ HM^{-1} = H]

$$HK = KH$$

$\Rightarrow$ HK is subgroup

Example 56. Prove that the only element of order one in a group. Then by the definition of order of an element of a group.

Solution : We have

$$a^{-1} = e \text{ or } a = e$$

which is the possible let $a \neq e$ be an element of order of an element of a group, we have

$$a^{-1} = e \text{ or } a = e$$

which is the contradiction to assumption that $a \neq e$

Hence, the identity element e is the only element of order one in any group.

Example 57. Let R be a system satisfying all the conditions of a ring except commulativity of addition. If there exists an element $x \in$ R which can be right cancelled in the sense $a{\cdot}x = b{\cdot}x \Rightarrow a = b$ them show that (R + , .) is a ring.

Solution : Here

$(a + b) \cdot (x + x) = (a + b)\, x + (a + b) \cdot x$ and also

$(a + b) \cdot (x + x) = a{\cdot}x + b{\cdot}x + a{\cdot}x + bx$

$\because \; ax + b{\cdot}x + ax + b{\cdot}x$

$= a{\cdot}x + ax + b{\cdot}x + b{\cdot}x$

$\Rightarrow \quad b{\cdot}x + ax = a{\cdot}x + b{\cdot}x$

$\Rightarrow \quad (b + a)\, x = (a + b)\, x$

$\Rightarrow \quad b + a = a + b$

$(\because \; ax = bx \Rightarrow a = b)$

Hence, commutative law of addition also holds in R besides the other conditions of a ring.

Hence (R + •) is a ring.

Example 58. Prove that a field has no proper ideal.

Solution : Let I be an ideal of a field F

Now if F has no proper ideal, then by definition either I is the null ideal {0} or I is the unit ideal F

i.e. either I = {0} or I = F

If I = {0} the null ideal, then nothing is left to be proved

If I ≠ {0} then there exists atleast one non zero element a in such that

$1 = a^{-1} a \in$ I

Also $\forall \; b \in$ F, we have $b = b\,1 \in$ F

i.e. all element of F $\in$ I

$\Rightarrow \quad$ F $\subseteq$ I $\quad$ ...(i)

But by definition of idea, we know that every ideal of F $\subseteq$ F

i.e. $\quad$ I $\subseteq$ F $\quad$ ...(ii)

$\because$ From (i) and (ii) we get

I = F

Example 59. Prove that intersection of two subgroup is again a subgroup.

Solution : Let H_1 and H_2 or e two subgroups of group G

Let $\quad a \in H_1 \cap H_2$

and $\quad b \in H_1 \cap H_2$

then $\quad a \in H_1 \cap H_2$

$b \in H_1 \cap H_2$

$\Rightarrow \quad a \in H_1\, a \in H_2$

and $\quad b \in H_1\, b \in H_2$

$\Rightarrow \quad a \in H_1\, b \in H_1$

and $\quad a \in H_2\, b \in H_2$

$\Rightarrow \quad ab^{-1} \in H_1$

and $\quad ab^{-1} \in H_2$

[$\because$ H_1 and H_2 are subgroups]

$\Rightarrow \quad ab^{-1} \in H_1 \cap H_2$

Hence, $H_1 \cap H_2$ is a subgroup.

OBJECTIVE TYPE QUESTIONS

1. Which of the following is false ?
   (A) $(z, +)$ is a group
   (B) $(z, +, \bullet)$ is a ring
   (C) $(z, +, \bullet)$ is a commutative ring
   (D) $(z, +, \bullet)$ is not an integral domain
2. Every finite integral domain is a—
   (A) Group (B) Ring
   (C) Field (D) Both ring and field
3. Which of the following is not an integral domain ?
   (A) (N, +, •) (B) (C, +, •)
   (C) (Q, +, •) (D) (R, +, •)
4. The set of residue classes (modulo n) is a ring without zero divisor w.r.t. addition and multiplication, iff—
   (A) n is prime (B) n is even
   (C) n is add (D) None of these
5. A boolean ring—
   (A) Is commutative (B) Has a unity
   (C) Has zero divisor (D) None of these
6. An integral domain S is—
   (A) Field when S is finite
   (B) Always a field
   (C) Never field
   (D) None of these
7. A divisor ring has at least—
   (A) Two elements
   (B) Three elements
   (C) One element
   (D) None of these

8. If (R, +, •) is a ring such that $x \cdot x = x \ \forall \ x \in$ R, then—
   (A) $x + y = 0 \Rightarrow x = y$
   (B) $x + x \neq 0$
   (C) $x \neq y \Rightarrow x + y = 0$
   (D) None of these
9. An integral domain—
   (A) Necessarily possesses multiplicative inverse of its non-zero elements
   (B) Is a commutative ring
   (C) Is a division ring
   (D) None of these
10. The ring of integers is also a—
   (A) Field (B) Integral domain
   (C) Division ring (D) None of these
11. The ring of even integers is also a—
   (A) Field
   (B) Division ring
   (C) Integral domain
   (D) None of these
12. Which of the following is false ?
   (A) Every field is an integral domain
   (B) Every integral domain is a commutative ring
   (C) Every commutative ring is a ring
   (D) At least one of the above is false
13. The set $\{14r : r \in z\}$ is—
   (A) Maximal ideal of z
   (B) Just a principal ideal of z
   (C) Prime ideal of z
   (D) None of these
14. The condition for none existence of zero divisor is—
   (A) $a^2 = a \ \forall \ a \in$ R
   (B) The cancellation laws holds for multiplication in R
   (C) $(a + b)^2 = a^2 + b^2 + 2ab, \forall \ a, b \in$ R
   (D) None of these
15. An ideal P = $\{Pr : r \in z\}$ which is a proper ideal of ring z is a prime ideal iff—
   (A) P is prime
   (B) P is odd
   (C) P is a multiple of 3
   (D) P is even
16. The set of residue classes mod m ($m \in$ N) is a ring without zero divisors under addition and multiplication for—
   (A) m prime (B) m odd
   (C) m any integer (D) m composite
17. If r is a system such that it is a group under addition and multiplication obeys the closure and distributive laws, then—
   (A) R need not be a ring
   (B) R has to be a ring
   (C) R is not a ring
   (D) R is necessarily a field
18. Which of the following integral domains is not ordered ?
   (A) The integers
   (B) The rational numbers
   (C) The real numbers
   (D) The complex numbers
19. Which of the following is not a prime field ?
   (A) The set of rational numbers
   (B) The set of residue classes mod 5
   (C) The set of residue classes mod 6
   (D) All of the above
20. Which of the following statement is correct ?
   (A) In a ring $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$
   (B) Every finite ring is an integral domain
   (C) Every finite integral domain is a field
   (D) The set of natural numbers is a ring with respect to the usual addition and multiplication
21. Let R {0, 1, 2, 3}, under addition and multiplication modulo 4 is—
   (A) A field
   (B) A ring with zero divisors
   (C) A ring without zero divisors
   (D) A division ring
22. Let (R = {0, 1, 2, 3, 4, 5} + 6, × 6). The R is—
   (A) A ring with zero divisors
   (B) A field
   (C) A division ring
   (D) A ring without zero divisors
23. Set residue classes modulo P, where P is prime, under addition and multiplication of residue classes is—
   (A) Field

(B) Skew field under
(C) Integral domain
(D) None of these

24. Let P be a prime number set of integers $I_A = \{0, 1, 2, \dots P^{-1}\}$ under addition and multiplication modulo P forms—
(A) Ring without zero divisors
(B) Field
(C) Integral domain
(D) All above are correct

25. A mapping f of a ring R onto a ring R' is called homomorphism if for each $a \in$ R, $b \in$ R—
(A) $f(a+b) = f(a) + f(b)$
(B) $f(a+b) = f(a) - f(b)$
(C) $f(ab) = f(a) - f(b)$
(D) $f(a+b) = f(a) + b$ and $f(ab) = f(a) \cdot f(b)$

26. Let M be the ring 2×2 matrices over the set of integers, let L = $\left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle| \, a, b \in z \right\}$, and K = $\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} a, b \in z \right\}$, then—
(A) L and K both left ideal of M
(B) L and K both right ideals of M
(C) L is left ideal of M and K is right ideal of M
(D) L is right ideal of M and K is left ideal of M

27. The set of all 2×2 matrices over the field of real number under the usual addition and multiplication of matrices is—
(A) Not a ring
(B) A ring with unity
(C) A commutative ring
(D) An integral domain

28. If Q and z are the sets of rational numbers and integers respective then which one of the following triples is a field ?
(A) $(Q, +, \times)$ (B) $(Q, -, \times)$
(C) $(Z, +, \times)$ (D) $(Z, -, \times)$

29. The set of integers under ordinary addition and multiplication—
(A) Forms a group
(B) Forms a ring
(C) Forms a field
(D) Does not form integral domain

30. Which of the following algebric operations is a field ?
(A) $(R, +, \bullet)$ (B) $(R, -, \bullet)$
(C) $(R, \bullet, \div)$ (D) $(R, \div, \bullet)$

31. The set of all rational numbers is—
(A) An additive group
(B) A multiplication group
(C) A cyclic group
(D) A finite group

32. The supremum of the function $f(x) = x - \frac{1}{x}$ in the internal $\left[\frac{1}{2}, 2\right]$ is—
(A) 2 (B) 1
(C) 3/2 (D) Does not exist

33. To form a 'Ring' we required, at least—
(A) One element
(B) Two elements
(C) Three elements
(D) One element which is additive identity

34. $(z_P, +P, P)$ is a field, if and only if—
(A) P is composite number
(B) P is prime number
(C) P is an even number
(D) P is add number

35. Consider the statements—
(a) The product of two rational numbers is always a rational number
(b) The product of two irrational numbers is always an irrational number. Then—
(A) Both (a) and (b) are correct
(B) (a) is incorrect and (b) is correct
(C) (a) is correct (b) is in correct
(D) (a) and (b) are in correct

36. The set S of square matrices of same order with respect to matrix addition is a—
(A) Quasi-group (B) Semi-group
(C) Group (D) Abelian group

37. The set of square matrices order 2, with respect to matrix multiplication is a—

(A) Quasi-group (B) Semi-group
(C) Monoid (D) Group

38. The set of all non-singular square matrices of same order with respect to matrix multiplication is—
(A) Quasi-group (B) Monoid
(C) Group (D) Abelian group

39. If order of group G is P^2, where P is prime then—
(A) G is Abelian (B) G is not Abelian
(C) G is ring (D) None of these

40. (a) If G is group, for $a \in$ G, N (a) is the normalizer of a, then $\forall x \in$ N (a)—
(A) $xa = ax$ (B) $xa = e$
(C) $ax = e$ (D) $xa \neq ax$

41. If G is a group, then for all $a, b \in$ G—
(A) $(ab)^{-1} = a^{-1}b^{-1}$ (B) $(ab)^{-1} = b^{-1}a^{-1}$
(C) $(ab)^{-1} = ab$ (D) $(ab)^{-1} = ba$

42. If G is a set of integers and $a \cdot b \equiv a - b$, then G is—
(A) Quasi group (B) Semi-group
(C) Monoid (D) Group

43. In a group G, for each element $a \in$ G, there is—
(A) No inverse
(B) A unique inverse $a^{-1} \in$ G
(C) More than one inverse
(D) None of these

44. If $a, b \in$ G, a group then b is conjugate to a if exist $c \in$ G…
(A) $b = c^{-1}ac$ (B) $a = cb$
(C) $b = ac^{-1}$ (D) $b = ac^{-1}a$

45. If P is prime number and P|o (G), then $a \in$ G—
(A) $ap \in$ G (B) $ap \notin$ G
(C) $ap \subset$ G (D) $ap \supset$ G

46. If G is a group of order n then, order of identity elements is—
(A) One (B) Greater than one
(C) n (D) None of these

47. If $a \in$ G is order n and P is prime to n, then the order of ap is—
(A) n (B) One
(C) Less than n (D) Greater than n

48. If the orders of elements a, $a^{-1} \in$ G are m and n respectively then—
(A) $m = n$ (B) $m \neq n$
(C) $m = n = 0$ (D) None of these

49. If in a group G, $a \in$ G the order of a is n and order of ap is m, then—
(A) $m \leq n$ (B) $m \geq n$
(C) $m = 0$ (D) None of these

50. The identity permutation is—
(A) Even permutation
(B) Odd permutation
(C) Neither even nor odd
(D) None of these

51. The product of even permutation is—
(A) Even permutation
(B) Odd permutation
(C) Neither even nor odd
(D) None of these

52. The inverse of an even permutation is—
(A) Odd permutation
(B) Even permutation
(C) Even or odd permutation
(D) None of these

53. The product of (1245) (32154) is—
(A) (23) (B) (15)
(C) (341) (D) (1531)

54. The inverse of an odd permutation is—
(A) Odd permutation
(B) Even permutation
(C) Even or odd
(D) None of these

55. If b and c are the inverse of some element $a \in$ G, then—
(A) $b = c$
(B) $b \neq c$
(C) $b = ac$ for same a
(D) None of these

56. Let Z be a set of integers, then under ordinary multiplication (Z) is—
(A) Monoid
(B) Semi-group
(C) Quasi-group
(D) Group

57. If N is a set of natural numbers then under binary operation $a - b = a - b$, (N) is—
(A) Quasi-group (B) Semi-group
(C) Monoid (D) Group

58. If G is a finite group and order of group is m then $\forall\, a \in$ G—
(A) $a^m = e$ an identity
(B) $a^n \neq e$
(C) $a^m = a$
(D) $a^m = a^{-1}$

59. HK is a sub-group of G iff—
(A) HK = KH (B) HK $\subset$ KH
(C) HK $\supset$ KH (D) HK $\neq$ KH

60. If G is group and $a \in$ G such that $a^2 = a$, then a is equal to—
(A) Identity element (B) Inverse
(C) Zero element (D) None of these

61. The generators of a group G = $\{a, a^2, a^3, a^4, a^5, a^6 = e\}$ are—
(A) a and a^5 (B) a^2 and a^4
(C) a^3 and a^5 (D) a^2 and a^3

62. If G = $\{1 - 1, i - i\}$ is a multiplicative group then order of i is—
(A) One (B) Two
(C) Three (D) Four

63. If G = {0, 1, 2, 3, 4) + 5}, the order of 2 is—
(A) One (B) Two
(C) Three (D) Five

64. If G is a group of even order, $\forall a \neq e$ if $a^2 = e$, then G is—
(A) Abelian group (B) Sub-group
(C) Normal group (D) None of these

65. Every group of prime order is—
(A) Cyclic (B) A belian
(C) Sub-group (D) Normal group

66. If H_1 and H_2 are two right coset sets of sub-group H, then—
(A) $H_1 \cap H_2\ \phi$ or $H_1 = H_2$
(B) $H_1 \cap H_2 \neq \phi$
(C) $H_1 \cup H_2 = \phi$
(D) $H_1 \neq H_2$ and $H_1 \cap H_2 \neq \phi$

67. The number of elements in a group is—
(A) Identity of group
(B) Order of group
(C) Inverse of group
(D) None of these

68. A one-one mapping of a finite group onto itself is—
(A) Isomorphism (B) Homomorphism
(C) Automorphism (D) None of these

69. If in a group G, $\forall\, a \in$ G—
(A) $(a^{-1})^{-1} = a$ (B) $(a^{-1})^{-1} = a^{-1}$
(C) $(a^{-1})^{-1} = a^2$ (D) None of these

70. If $f = (23)$ and $g = (45)$ be two permutation of five symbols 1, 2, 3, 4, 5 then gf is—
(A) $\begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,3\,5\,4\,2 \end{pmatrix}$ (B) $\begin{pmatrix} 1\,2\,3\,5\,6 \\ 1\,4\,6\,5\,4 \end{pmatrix}$
(C) $\begin{pmatrix} 1\,2\,3\,5\,7 \\ 1\,4\,6\,4\,1 \end{pmatrix}$ (D) $\begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,3\,2\,5\,4 \end{pmatrix}$

71. Given permutation $\begin{pmatrix} 1\,2\,3\,4\,5\,6 \\ 6\,1\,2\,5\,4\,6 \end{pmatrix}$ is equivalent to—
(A) (1632) (21) (B) (1632) (11)
(C) (1632) (45) (D) (1632) (54)

72. If given permutation are A = $\begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,3\,1\,5\,4 \end{pmatrix}$, B = $\begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,3\,4\,5\,2 \end{pmatrix}$, find BA—
(A) $\begin{pmatrix} 1\,2\,3\,4\,5 \\ 2\,1\,5\,4\,3 \end{pmatrix}$ (B) $\begin{pmatrix} 2\,1\,5\,3\,5 \\ 1\,6\,4\,2\,1 \end{pmatrix}$
(C) $\begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,5\,4\,1 \end{pmatrix}$ (D) $\begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,3\,4\,5 \end{pmatrix}$

73. If number of left cosets of H in G are n and the number of right cosets of H in G are m then—
(A) $m = n$ (B) $m \geq n$
(C) $m \leq n$ (D) None of these

74. If H is a subgroup of finite group G and order of H and G are respectively m and n, then—
(A) $m|n$ (B) $n|m$
(C) $m \times n$ (D) None of these

75. If G is a finite group of order n then for every $a \in$ G, we have—
(A) $a^n = e$, an identity element
(B) $a^n = a^1$
(C) $a^n = a$
(D) None of these

76. If H_1 and H_2 are two subgroups of G, then following is also a subgroup of G—
(A) $H_1 \cap H_2$ (B) $H_1 \cup H_2$
(C) H_1H_2 (D) None of these

77. The set M of square matrices (of same order) with respect to matrix multiplication is—
(A) Group (B) Semi-group
(C) Monoid (D) Quasi-group

78. If (G, *) is a group and $\forall\, a, b \in G\; b^{-1} * a^{-1} * b * a = e$, then G is—
(A) Abelian group (B) Non-Abelian
(C) Ring (D) Field

79. If G is a group such that $a^2 = e$, $\forall\, a \in G$, then G is—
(A) Abelian group
(B) Non-Abelian group
(C) Ring
(D) Field

80. If $f = (23)$ and $g = (45)$ are two permutations on 1, 2, 3, 4, 5 then fg is—
(A) $\begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,3\,2\,5\,4 \end{pmatrix}$ (B) $\begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,2\,5\,6 \end{pmatrix}$
(C) $\begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,4\,5\,6 \end{pmatrix}$ (D) $\begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,3\,4\,5 \end{pmatrix}$

81. If n is the order of element a of group G, then $a^m = e$, an identity element iff—
(A) $m|n$ (B) $n|n$
(C) $m \times n$ (D) $n \times m$

82. The order of identity element in a group G is—
(A) One
(B) Zero
(C) Order of group
(D) Less than order of group

83. If $a, a^{-1} \in$ G a group and order of a and a^{-1} are m and n respectively, then—
(A) $m > n$ (B) $m < n$
(C) $m = n$ (D) None of these

84. If $a, b \in$ G a group of order m, then order of ab and ba are—
(A) Same (B) Equal to m
(C) Unequal (D) None of these

85. If G = {1, –1} is a group, then order of 1 is—
(A) One (B) Two
(C) Zero (D) None of these

86. The product of permutations (123) (243) (134) is equal to—
(A) I (B) $\begin{pmatrix} 1\,2\,3\,4 \\ 5\,6\,2\,1 \end{pmatrix}$
(C) $\begin{pmatrix} 1\,2\,5\,1 \\ 1\,6\,5\,1 \end{pmatrix}$ (D) $\begin{pmatrix} 1\,2\,5\,3 \\ 1\,2\,3\,4 \end{pmatrix}$

87. The permutation $\begin{pmatrix} 1\,2\,5\,3\,4 \\ 3\,4\,1\,5\,2 \end{pmatrix}$ is equal to—
(A) (15) (13) (24) (B) (1) (2) (3)
(C) (135) (56) (D) (142) (53)

88. Given the permutation c = (1234567), then c^3 is—
(A) (135724) (B) (1473625)
(C) (1765432) (D) I

89. If c = (1234), then c^3 is—
(A) (13) (24) (B) (13)
(C) (24) (D) (23) (31)

90. Statement A : All cyclic group are Abelian statement B : The order of cyclic group is same as the order of its generator—
(A) A and B are false
(B) A is true, B is false
(C) B is true, A is false
(D) A and B are true

91. Statement A : Every isomorphic image of a cyclic group is cyclic.
Statement B : Every homomorphic image of a cyclic group is cyclic
(A) Both A and B are true
(B) Both A and B are false
(C) A is true only
(D) B is true only

92. A element ap of a finite cyclic group G of order n is a generator of G iff $0 < p < n$ and also—
(A) P is prime to n
(B) P is the multiple of n
(C) n is the multiple of n
(D) None of these

93. If G is a finite group of order n, $a \in$ G and order of a is m, if is cyclic, then—

(A) $m = n$ (B) $m > n$
(C) $m < n$ (D) None of these

94. If $a \in$ G is a generator of a cyclic group and order of a is $n < \infty$, then order of a cyclic group m is—
(A) Infinity (B) $m = n$
(C) $m > n$ (D) $m < n$

95. If e_1 and e_2 are two identity elements of a group G, then—
(A) $e_1 = e_2$
(B) $e_1 \neq e_2$
(C) $e_1 = ce_2$, for some
(D) None of these

96. The idempotent element in a group are—
(A) Inverse elements
(B) Identity element of a group
(C) Any element of a group
(D) None of these

97. Let G = {1 – 1}, then under ordinary multiplication (G) is—
(A) Monoid (B) Semi-group
(C) Quasi-group (D) Group

98. Let G be a set of rational numbers then under ordinary addition (Q +) is—
(A) Monoid (B) Semi-group
(C) Quasi-group (D) Group

99. Let G be a group of square matrices of same order with respect to matrix multiplication then it is not a—
(A) Quasi-group (B) Abelian group
(C) Semi-group (D) None of these

100. If G is a finite group, then for every $a \in$ G the order of a is—
(A) Finite (B) Infinite
(C) Zero (D) None of these

101. In the additive of integers, the order of every element $a \neq 0$ is—
(A) Infinity (B) One
(C) Zero (D) None of these

102. In the additive group of integers, the order of identity element is—
(A) Zero (B) One
(C) Infinity (D) None of these

103. In the additive group G of integers, the order of inverse element a^{-1}, $\forall a \in$ G is—
(A) Zero (B) One
(C) Infinity (D) None of these

104. The singleton {0} with binary operations additive and multiplication is ring and it is called—
(A) Zero ring (B) Division ring
(C) Singleton ring (D) None of these

105. The element $a \neq 0 \in$ R, the commutative ring is an integral domain if—
(A) $ab = 0$, $b \in$ R and $b = 0$
(B) $ab = 0$, $b \in$ R and $b \neq 0$
(C) $ab \neq 0$ $b \in$ R and $b = 0$
(D) $ab = 0$, $b \in$ R and $b = 0$

106. A ring R is an integral domain if—
(A) R is commutative ring
(B) R is commutative ring with zero divisor
(C) R is commutative ring with non-zero divisor
(D) R is a ring with zero divisor

107. A ring R with binary operation addition is an Abelian group. It with binary operation multiplication, $\forall a, b \in$ R, $a{\cdot}b = b{\cdot}a$, then R is—
(A) Commutative ring
(B) Integral domain
(C) Field
(D) Null ring

108. An integral domain D is of characteristic zero if—
(A) $ma = 0$, $a \neq 0 \in$ D $\rightarrow m = 0$
(B) $ma = 0{\cdot}a \neq 0 \in$ D $\rightarrow m \neq 0$
(C) $ma = 0$ $a \neq 0 \in$ D $\rightarrow m = a$
(D) $ma = 0$ $a \neq 0 \in$ D $\rightarrow m \neq a$

109. E is the set of even integers under ordinary addition and multiplication, then E is a ring, E is also a—
(A) Commutative ring
(B) Integral domain
(C) Field
(D) None of these

110. I is the set of integers and define $a \oplus b = a + b + 1$ and $a \odot b = a + b + ab$, then the ring {I $\oplus \cdot \odot$}is—

(A) Commutative (B) Integral domain
(C) Field (D) None of these

111. If the ring R has left identity e_1 and right, identity e_2, then—
(A) $e_1 = e_2$ (B) $e_1 \neq e_2$
(C) $e_1 = me_2$ (D) None of these

112. If a ring R $\neq \{0\}$ has the multiplicative identity, then—
(A) $1 > 0$ (B) $1 = 0$
(C) $1 \neq 0$ (D) None of these

113. If the ring R has unites e_1 and e_2, then—
(A) $e_1 = e_2$ (B) $e_1 = me_2$
(C) $e_1 = e_2$ (D) None of these

114. A ring (R, +,) is said to have zero divisor if—
(A) $a, b \in$ R, $ab = 0 \rightarrow a \neq 0$ or $b \neq 0$
(B) $a, b \in$ R $a \cdot b = 0 \rightarrow a \neq 0$ and $b \neq 0$
(C) $a, b \in$ R $a \cdot b = 0 \rightarrow a = 0$ or $b = 0$
(D) $a, b \in$ R, $a \cdot b = 0 \rightarrow a = 0$ and $b = 0$

115. A ring (R +) is said to have a ring without rero divisor if—
(A) $ab \in$ R, $a \cdot b = 0 \rightarrow a \neq 0$ or $b \neq 0$
(B) $a, b \in$ R, $a \cdot b = 0 \rightarrow a \neq 0$ and $b \neq 0$
(C) $a, b \in$ R $= 0 \rightarrow a = 0$ or $b = 0$ or both are zero
(D) $a, b \in$ R, $a \cdot b = 0 \rightarrow a = 0$ and $b \neq 0$

116. An element $a \in$ (R, +,) a ring is nilpotent if for some positive integer n—
(A) $a^n = 0$ (B) $a^n = a$
(C) $a^n = 1$ (D) None of these

117. The non zero elements a, b of ring (R, +) are called zero divisors if—
(A) $a \cdot b = 0$ (B) $a \cdot b = 1$
(C) $a \cdot b \neq 0$ (D) $a \cdot b \neq 1$

118. A skew field have—
(A) Non-zero divisors
(B) Zero divisors
(C) a, b $a \cdot b = 0 : a \neq 0, b \neq 0$
(D) None of these

119. The following statement is false—
(A) The intersection of two non-empty sub-ring is a sub-ring
(B) The intersection of two non-empty sub-group is a sub-group
(C) A skew field have zero divisors
(D) An integral domain have zero divisors

120. If f is an isomorphism of a ring (R_1 +,) onto a ring (R_1 + , ·) and
(a) Isomorphic image of a field is a field
(b) Isomorphic image of a division ring is a division ring
(c) Isomorphic image of a ring with unity, is a ring with unity

then—
(A) a, b, c are true
(B) a and b are false
(C) b and c are false
(D) a is false

121. A field having no proper subfield is—
(A) Prime field
(B) Division ring
(C) Integral domain
(D) None of these

122. Every finite integral of domain is—
(A) Of finite characteristic
(B) Of not finite characteristic
(C) Not a field
(D) None of these

123. Let (D, + ·) is an integral domain D is a field if—
(A) For $\forall a \in$ D, there exist $a^{-1} \in$ D : $aa^{-1} = 1$
(B) For $\forall a \in$ D, there exist $a^{-1} \in$ D : $a + a^{-1} = 0$
(C) For $\forall a \in$ D, there exist $b \in$ D : $ab = 0$
(D) None of these

124. The set of rational numbers Q with ordinary addition and multiplication is a commutative ring with unit element it is a field if for $\frac{a}{b} \in$ G
There exist—
(A) $a + b \in$ Q (B) $\frac{b}{a} \in$ Q
(C) $ab \in$ Q (D) None of these

125. The set C of complex number of the form $x + iy$ is a field with respect to ordinary addition and multiplication, then the unit and zero elements are respectively—

(A) $1 + i0$ and $0 + i0$
(B) $0 + i$ and $1 + i0$
(C) 0 and 1
(D) i and $-i$

126. If $C = \{x + iy : xy \in R, i = \sqrt{-1}\}$ is a field with respect to ordinary addition and multiplication, then the multiplication inverse of non-zero element of $a + ib \in C$ is—
(A) $a + b$
(B) $\left(\frac{a}{a^2 + b^2}\right) + i\left(\frac{-b}{a^2 + b^2}\right)$
(C) $\frac{a + ib}{a^2 + b^2}$
(D) None of these

127. If in a ring with unity $(xy)^2 = x^2y^2, \forall x, y = \in R$, then—
(A) R is a commutative ring
(B) R is an integral domain
(C) R is field
(D) None of these

128. Every non-zero nilpotent element of the ring R is—
(A) Zero divisor
(B) Non-zero divisor
(C) Unity
(D) $AB \neq 0$

129. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ are—
(A) Zero-divisors
(B) Non-zero divisors
(C) $A = 0$ or $B = 0$
(D) $AB \neq 0$

130. If a_1^{-1}, a_2^{-1} are two multiplicative inverse of non-zero elements $a \in F$, a field then—
(A) $a_1^{-1} \neq a_2^{-1}$ (B) $a_1^{-1} = a_2^{-1}$
(C) $a_1^{-1} < a_2^{-1}$ (D) None of these

131. The following statement is false—
(A) Every field is an integral domain
(B) Every integral domain is a field
(C) Every field is a ring
(D) Every ring is a group

132. The following statement is false—
(A) Every field is an integral domain
(B) Every finite integral domain is a field
(C) Every field is a ring
(D) Every integral domain is a field

133. A field is defined as—
(A) Division ring
(B) Commutative ring
(C) Integral domain
(D) Finite integral domain

134. A commutative ring R with unity is called integral domain if $a, b \in R$—
(A) $ab = 0 \to a \neq 0, b \neq 0$
(B) $ab = 0 \to a = 0 \cdot b = 0$
(C) $ab = 0 \to a = b$
(D) None of these

135. If $a, b \in D$, D is an ordered set than following is true—
(A) $a > b \to a > -b$
(B) $a > 0 \to ab > ac \to b < c$
(C) $a > b \to -a < -b$
(D) $a \geq |a|$

136. The generators of the group $G = \{a, a^2, a^3, a^4 = e\}$ are—
(A) a only (B) a and a^2
(C) a and a^3 (D) a and a^4

137. If $G = \{1 \ -1\}$ is a group with ordinary multiplication the order of -1 is—
(A) One (B) Two
(C) Zero (D) None of these

138. The ring of complex numbers $C = \{x + iy : x\, y$ are real number $i = \sqrt{-1}\}$ is—
(A) Not an integral domain
(B) An integral domain
(C) Ordered set
(D) None of these

139. If is an integral domain and $a \neq 0 \in I$, then—
(A) $a^2 = 0$ (B) $a^2 \geq 0$
(C) $a^2 \neq 0$ (D) None of these

140. If $a, b \in D$, where D is an ordered set, then the following is false—
(A) $a < b \to -a > -b$
(B) $(ab) = |a|\,|b|$
(C) $-|a| \leq a \leq |a|$
(D) $|a + b| \geq |a| + |b|$

141. Let R and R be two arbitrary ring $\phi : R \rightarrow R$ defined as $\phi(a) = 0$ for all $a \in R$, then—
(A) ϕ is homomorphism
(B) ϕ is automorphism
(C) ϕ is isomorphism
(D) None of these

142. The homomorphism ϕ of rings R into R is an isomorphism iff the Kernel I (ϕ) is—
(A) $I(\phi) = (0)$ (B) $I(\phi) = R$
(C) $I(\phi) \neq R$ (D) None of these

143. Let R be a ring $U \neq \phi \subset R$ is ideal of then—
A : U is a subgroup of R under addition
B : $\forall u \in U$ and $r \in R$, $ur, ru \in U$
(A) A and B both are true
(B) Only A is true
(C) Only B is true
(D) Both A and B are false

144. If U is an ideal of ring R, then—
(A) U/R is a ring (B) R/U is a ring
(C) RU is a ring (D) None of these

145. An ideal $M \neq R$ in a ring R is maximal ideal of R if U is an ideal of Q and $M \subset U \subset R$, then—
(A) Either M = U or R = U
(B) M = U = R
(C) $M = U \neq R$
(D) $M \neq U + R$

146. A field is a—
(A) Vector space
(B) Integral domain
(C) Division ring
(D) Commutative division ring

147. An integral domain D is of finite characteristic, if $\forall a \in D$ there exist m a positive integer such that—
(A) $ma = 1$ (B) $ma = 0$
(C) $ma = a$ (D) None of these

148. If integral domain D is of finite characteristic, then its characteristic is—
(A) Odd number (B) Even number
(C) Prime number (D) Natural number

149. If integral domain I is of finite characteristic, then—
(A) I is finite only
(B) I is infinite only
(C) I is finite or infinite
(D) None of these

150. A : F is a field
B : F an integral domain—
(A) $A \rightarrow B$ (B) $B \rightarrow A$
(C) $A \leftrightarrow B$ (D) $A \nleftrightarrow B$

151. A commutative division ring is—
(A) Vector space
(B) Group
(C) Integral domain
(D) Field

152. A commutative division ring is—
(A) Finite integral domain
(B) Integral domain
(C) Zero ring
(D) None of these

153. If R is a commutative ring with unit element M is maximum ideal of iff—
(A) R/M is a field (B) M/R is a field
(C) RM is a field (D) None of these

154. If F is field then its only ideals are—
(a) F, a field itself
(b) (0)
(A) (a) and (b) are true
(B) (a) is true (b) is false
(C) (a) is false (b) is true
(D) (a) and (b) false

155. If U is an ideal of ring R and $1 \in U$, then—
(A) U is a proper subset of R
(B) U is equal to R
(C) U is a super set of R
(D) $U = \phi$

156. Let R is commutative ring with unit element whose only ideals are (0) and R itself, then—
(A) R is finite integral domain
(B) R is integral domain
(C) Division ring
(D) None of these

157. If R is a commutative ring with unit element M is an ideal of and R/M is finite integral domain, then—

(A) M is a maximal ideal of R
(B) M is not a maximal ideal of R
(C) M is minimal ideal or R
(D) None of these

158. If R is an Eudidean ring and $a, b \in$ R. If $b \neq 0$ is not a unit in R, then—
(A) $d(a) < d(ab)$ (B) $d(a) > d(ab)$
(C) $d(a) = d(ab)$ (D) None of these

159. If r is a prime element in Eudidean ring R and r/ab $a, b \in$ R then—
(A) $x \times a$ or $r \times b$
(B) $r \times a$ and $x \times b$
(C) $ab \neq mr$ for some $m \in$ R
(D) None of these

160. If $f x)$ and $g(x)$ are two polynomials, then—
(A) deg $(f(x)\, g(x)) \leq \deg f(x)$, $g(x) \neq 0$
(B) deg $(f(x)\, g(x) \geq \deg f(x)$, $g(x) \neq 0$
(C) deg $(f(x)\, g(x) = \deg f(x) \deg g(x)$, $g(x) \neq 0$
(D) deg $(f(x)\, g(x) = \deg f(x) - \deg g(x)$ $g(x) \neq 0$

161. Given a polynomial $f(x) = a_0 + a_1x + \ldots + a_nx^n$ where as are integer, then content of $f(x)$ is—
(A) $g.c.d.$ of integers $a_0 \ldots a_n$
(B) Mean of integers $a_0 \ldots, a_n$
(C) Mode of integers $a_0 \ldots a_n$
(D) None of these

162. If R is a commutative ring, with unit element then—
(A) Every maximal ideal is prime ideal
(B) Every prime ideal is maximal ideal
(C) Every ideal is prime ideal
(D) Every ideal is maximal

163. If R is an integral domain with unit element, then—
(A) R $[x]$ is not a commutative ring
(B) R $[x]$ have a unit element
(C) Any unit in R $[x]$ is unit in R
(D) Any unit in R $[x]$ is not an unit in R

164. If K is an extension of F, then degree of K over F is—
(A) Dimension of K as vector space over F
(B) Number of element in K as vector space over F
(C) Degree of K
(D) Order of K

165. The set of all even integers is ring it is also a—
(A) Commutative ring
(B) Integral domain
(C) Field
(D) None of these

166. Every integral domain is not a—
(A) Field
(B) Commutative ring
(C) Ring
(D) Abelian group with respect to addition

167. I is an ordered integral domain and $a, b \in$ I, if $a > b$, then—
(A) $a + c \geq b + c$, $\forall c \in$ I
(B) $a + c \leq b + c$, $\forall c \in$ I
(C) $a + c < b + c$, $\forall c \in$ I
(D) $a + c > b + c$, $\forall c \in$ I

168. If R is ring in which $a^4 = a$, $\forall a \in$ R, then—
(A) R is commutative
(B) R is not commutative
(C) R is zero ring
(D) None of these

169. If the ring R is such that $(ab^2) = a^2b^2$, $ab \in$ R, then—
(A) R is commutative
(B) R is non commutative
(C) R is zero ring
(D) None of these

170. Given the polynomial P $(x) = a_0 + a_1x + \ldots + a_mx^m$ its degree is m if—
(A) $a_m = 0$ (B) $a_m \neq 0$
(C) $a_m{}^{-1} = 0$ (D) $a_m{}^{-1} \neq 0$

171. If $f(x)$ and $g(x)$ are two non-zero polynomials of $f[x]$ then—
(A) deg $(f(x)\, g(x)) = \deg = (f(x) \deg (g(x))$
(B) deg $(f(x)) = \deg f(x) + \deg g(x)$
(C) deg $(f(x) = \deg f(x) - \deg g(x)$
(D) deg $(f(x) = \deg f(x) + \deg g(x)$

172. The polynomial $f(x) = a_0 + a_1x + a_2x + \ldots + a_nx^n$ where $a_0, a_1 \ldots\ldots, a_n$ are integers, is primitive polynomial if—

(A) $g.c.\ d$ of $(a_0 \ldots\ldots a_n) = 0$
(B) $g.c.d$ of $(a_0 \ldots\ldots a_n) = 1$
(C) $g.c.d.$ of $(a_0 \ldots\ldots a_n) = 2$
(D) $g.c.d.$ of $(a_0 \ldots\ldots a_n) =$ prime number

173. A polynomial is said to be integer monic if—
(A) Its coefficients are integers and highest coefficient is 1
(B) Its coefficients are real numbers and highest coefficient is 0
(C) Its coefficients are real numbers and highest coefficient 0
(D) Its coefficients are integers and highest coefficients is 2

174. An integers monic polynomial is a—
(A) Primitive polynomial
(B) Non primitive polynomial
(C) Polynomial with lowest coefficient as 1
(D) Polynomial without lowest coefficient as 1

175. Let R is a commutative ring with unit element whose only ideals are (0) and R itself, then—
(A) R is integral domain
(B) R is field
(C) Division ring
(D) None of these

176. Which of the following statement is false ?
(A) F [x] is an integral domain
(B) F [x] is Eudidean ring
(C) F [x] principal ideal ring
(D) F [x] is not a group

177. A polynomials $f(x)$ and $g(x)$ are primitive polynomial, then—
(A) $f(x)\, g(x)$ is a primitive polynomial
(B) $f(x) + g(x)$ is primitive polynomial
(C) $f(x) - g(x)$ is a primitive polynomial
(D) $f(x)/g(x)$ is a primitive polynomial

178. If the ring R is an integral domain, then—
(A) R [x] is an integral domain
(B) R [x] is not an integral domain
(C) R [x] is a field
(D) None of these

179. If the ring R is finite and commutative with unit element, then—
(A) Every prime ideal is a maximal ideal
(B) Every ideal is maximal ideal
(C) Every maximal ideal is prime ideal
(D) (A) and (C) are both true

180. If is a finite extension of K and K is a finite extension of field F, then—
(A) F is a finite extension of L
(B) L is a finite extension of F
(C) K is a subfield of F
(D) L is subfield of K

181. If is a finite extension of F and K is a sub field of L which contains F, then—
(A) [L : F] [K : F] (B) [K : F]/[L : F]
(C) [K : F]/[F : L] (D) [F : K]/[L : F]

182. If K is an extension of F and $a \in$ K, F (a) is a finite extension, then—
(A) a is algebraic over F
(B) a is not algebraic over F
(C) F (a) is the largest subfield of K
(D) None of these

183. If a and b in K (where K is an extension of field F) are algebraic over F of degree m and n respectively, the following is false—
(A) ab is algebraic over F
(B) $a + b$ is algebraic over F
(C) $a - b$ is algebraic over F
(D) None of these

184. If F is a subfield of L and K is a subfield of L which contains F, then—
(A) [K : F]/[L : F] (B) [K : F] $\times$ [L : F]
(C) [F : K]/[F : L] (D) [F : K] $\times$ [F : L]

185. An element $a \in$ K is algebraic over field F if for $a_0 \ldots a_n \in$ F all not zero—
(A) $a_0a^n + a_1a^{n-1} + \ldots + a_n \neq 0$
(B) $a_0a^n + a_1a^{n-1} + \ldots + a_na^{n-1} = 0$
(C) $a_0a^n + a_1a^{n-1} + \ldots + a_n = 0$
(D) $a_0a^n + a_1a^{n-1} + \ldots + a_n = 0$

186. If K is an extension of field F and $a \in$ K is algebraic over F if—
(A) F (a) is an extension of F
(B) F (a) is a subfield of F
(C) F (a) is a finite extension of F
(D) F (a) is infinite of F

187. If K is an extension of field F and $a \in$ K is algebraic of degree n over F, then—
(A) $[F(a) : F] = n$ (B) $[F(a) : K] = n$
(C) $[K : F(a)] = n$ (D) $[F : F(a)] = n$

188. The extension K of field F is an algebraic extension of F—
(A) Every element $a \in$ K is algebraic over F
(B) For some $a \in$ K is algebraic over F
(C) Zero element is algebraic over F
(D) None of these

189. A complex number is algebraic number if—
(A) It is algebraic over real numbers
(B) It is algebraic over rational numbers
(C) It is algebraic over integers
(D) It is algebraic over natural numbers

190. If $f(x) \in F[x]$, then there is a finite extension E of F such that—
(A) $[E : F] \leq$ deg, $f(x)$
(B) $[E : F] \geq \deg f(x)$
(C) $[F : E] \geq \deg f(x)$
(D) $[F : E] \leq \deg f(x)$

191. If I is a ideal in a ring R them—
(A) R/I is a ring (B) RI is a ring
(C) R + I is a ring (D) RI is a ring

192. Let S be a non-empty set. Any function o from S × S to S is called a.........., if $o : S \times S \to S$, defined as $o(ab) = a_o b \in S$, $\forall a, b \in S$.
(A) Unary (B) Binary
(C) Quardraut (D) None of these

193. Let S be a non empty set. Let o be an operation on S then (S, o) is a .... structure—
(A) Mathematical (B) Trigonometrical
(C) Differential (D) None of these

194. Mathematical structure (S, o) is said to be … if o is binary operation *i.e.*, $\forall a, b \in S \Rightarrow aob \in S$—
(A) Groupoid (B) Monoid
(C) Semi-group (D) None of these

195. A groupoid (S, o) is … if o is associative *i.e.*, $(aob)\, oc = ao\,(boc)$, $\forall a, b, c \in S$—
(A) Group (B) Monoid
(C) Semi-group (D) None of these

196. If identity element $e \in$ S exists in a semi-group (S, o), then it is …, *i.e.*, $\forall a \in S, \exists e \in S : aoe = a = e^o a$—
(A) Group (B) Monoid
(C) Groupoid (D) None of these

197. If inverse element exists for every element in a monoid (S, o), then it is a …, *i.e.*, $\forall a \in S, \exists a^{-1} \in S : aoa^{-1} = e = a^{-1}\, {}^o a$.
(A) Group (B) Monoid
(C) Semi-group (D) None of these

198. A group (S, o), is a … if $\forall a, b \in S$, $aob = boa$—
(A) Commutative group
(B) Monoid
(C) Semi-group
(D) None of these

199. A commutative group is known as—
(A) Abelian group (B) Monoid
(C) Semi-group (D) None of these

200. 1. Identity element in a group is unique.
2. Inverse of each element of a group is unique
(A) 1 is true only
(B) 2 is true only
(C) Both 1 and 2 are true
(D) None 1 and 2 are true

201. If $a, b \in$ G a group, then $(ab)^{-1} = b^{-1} a^{-1}$. This law is known as—
(A) Reversal rule
(B) Closure rule
(C) Associative rule
(D) None of these

202. Let G be a group. Let $a \in$ G, then n is called … denoted by $o(a) = n$, if $a^n = e$, where n is least positive integer—
(A) Order of element a
(B) Order of group G
(C) Both (A) and (B)
(D) None of these

203. 1. The order of every element of finite group is infinite.
2. If their is no positive integer n such that $a^n = e$, then order of a $o(a)$ is infinite or zero.

(A) Only 1 is true
(B) Both 1 and 2 are true
(C) Only 2 is true
(D) None of 1 and 2 are true

204. 1. The order of every element of finite group is less than or equal to the order of the group.
2. The order of an element of a group is same as that of its inverse.
3. Order of any integral power of an element $a \in G$ cannot exceed the order of a.
(A) Only 1 is true (B) Only 3 is true
(C) Only 2 is true (D) All 1, 2, 3 are true

205. If $a, b \ldots \in G$ a group and $(abc \ldots z)^{-1} = z^{-1} \ldots c^{-1}b^{-1}a^{-1}$. This law is known as—
(A) Reversal rule
(B) Closure rule
(C) Associative rule
(D) None of these

206. A group G is called …… if for some $a \in G$ every element $x = G$ is of the form $x = a^n$, where n is some integer—
(A) Cyclic (B) Ring
(C) Abelian (D) None of these

207. There may be …… generator of a cyclic group—
(A) More than one (B) No
(C) Only one (D) Maximum two

208. 1. Every cyclic group is an abelian group.
2. The order of cyclic group is same as the order of its generator.
(A) Only 1 is true
(B) Both 1 and 2 are true
(C) Only 2 is true
(D) None of 1 and 2 is true

209. 1. If a is a generator of a cyclic group G, then a^{-1} is also generator of G.
2. The generator of cyclic group of order n are all elements a^p, p being prime to n and
O < P < n.
(A) Only 1 is true
(B) Both 1 and 2 are true
(C) Only 2 is true
(D) None of 1 and 2 is true

210. Every group of …… order is cyclic—
(A) Prime (B) Odd
(C) Even (D) Any

211. 1. If G is finite group of order n and contains a such that $o(a) = n$, then G is cyclic group.
2. If a cyclic group G is generated by an element a of order n, then a^m is a generator of G, iff ged of m and n is 1.
3. Every group of order 3 is cyclic.
(A) Only 1 is true (B) Only 3 is true
(C) Only 2 is true (D) All 1, 2, 3 are true

212. 1. An Abelian group of order six is cyclic.
2. If a cyclic group G is generated by an element a, $o(G) = n$, then

$$a' \in G, a' = \begin{cases} a^0 - a^n = e, & i = 0, n \\ a' & i < n \\ a^{1-n} & i > n \end{cases}$$

(A) Only 1 is true
(B) Both 1 and 2 are true
(C) Only 2 is true
(D) None of 1 and 2 is true

213. A permutation is said to be an … permutation if it can be expressed as a product of an even number of transpositions, otherwise it is an … permutation—
(A) Even, even (B) Even, odd
(C) Odd, odd (D) Odd, even

214. Every permutation can be expressed as a product of transpositions in … many ways—
(A) Infinitely
(B) Finitely
(C) Both (A) and (B)
(D) None of these

215. 1. A permutation can not be both even or odd *i.e.*, permutation f is expressed as product of transpositions, then the number of transpositions is either always even or always odd.
2. Identity permutation is always an even permutation.
(A) Only 1 is true
(B) Both 1 and 2 are true
(C) Only 2 is true
(D) None of 1 and 2 is true

216. 1. The product of two even permutations is an even permutations.
   2. The product of two odd permutations is an even permutation.
   (A) Only 1 is true
   (B) Both 1 and 2 are true
   (C) Only 2 is true
   (D) None of 1 and 2 is true

217. The product of an even permutation and an odd permutation is …… permutation.
   (A) Even
   (B) Odd
   (C) Both (A) and (B)
   (D) None of these

218. The inverse of an even permutation is …… permutation.
   (A) Even
   (B) Odd
   (C) Both (A) and (B)
   (D) None of these

219. The inverse of an odd permutation is an …… permutation.
   (A) Even
   (B) Odd
   (C) Both (A) and (B)
   (D) None of these

220. 1. A cycle of length n can be expressed as the product of $n - 1$ transpositions.
   2. Out of n ! permutation on n symbols $\frac{1}{2}n$! are even and $\frac{1}{2}n$! are odd—
   (A) Only 1 is true
   (B) Both 1 and 2 are true
   (C) Only 2 is true
   (D) None of 1 and 2 is true

221. The product of two odd permutations is an ………… permutation—
   (A) Even
   (B) Odd
   (C) Both (A) and (B)
   (D) None of these

222. A permutation is even if it can be expressed as a product of an ………… number of transpositions—
   (A) Even
   (B) Odd
   (C) Both (A) and (B)
   (D) None of these

223. A permutation is odd if it can be expressed as a product of an …… number of transpositions—
   (A) Even
   (B) Odd
   (C) Both (A) and (B)
   (D) None of these

224. Identity permutation is always …… permutation—
   (A) Even
   (B) Odd
   (C) Both (A) and (B)
   (D) None of these

225. 1. If G and G' are isomorphic groups, the identity elements in the groups does not correspond to one another.
   2. If a and b are corresponding elements of two isomorphic groups, then a^{-1} corresponds to b^{-1}.
   (A) Only 1 is true
   (B) Both 1 and 2 are true
   (C) Only 2 is true
   (D) None of 1 and 2 are true

226. 1. All groups which are isomorphic to a given group are isomorphic to each other.
   2. All cyclic groups of the same order are isomorphic to each other.
   3. A cyclic group G with generator of finite order n is isomorphic to the multiplicative group of n, nth root of unity.
   (A) Only 1 is true
   (B) Only 3 is true
   (C) Only 2 is true
   (D) All 1, 2 and 3 are true

227. A non-empty subset H of a group G is said to be a …… of G. If under the operation defined on G. H itself forms a group—
   (A) Subgroup (B) Abelian group
   (C) Cyclic group (D) None of these

228. 1. Since every set is a subset of itself group G is a subgroup of itself called trivial subgroup.
   2. If e is the identity of G. then the subset of G containing only one element e is also a subgroup of G.
   3. If H is a subgroup of G and K is a subgroup of H, then K is also a subgroup of G and transitive law acts here.
   (A) Only 1 is true
   (B) Only 3 is true
   (C) Only 2 is true
   (D) All 1, 2 and 3 are true

229. A subset H of a group G is a subgroup of G if and only if (i) $a, b \in H \Rightarrow ab \in H$ (ii) $a \in H \Rightarrow a^{-1} \in H$, where a^{-1}, is the inverse of a in G—
   (A) (i) is true only
   (B) Both (i) and (ii) are true
   (C) (ii) is true only
   (D) None of these

230. 1. A necessary and sufficient condition for an on empty subset H of a group G to be a subgroup is that $a, b \in H \Rightarrow ab^{-1} \in H$ where b^{-1} the inverse of b in G.
   2. A necessary and sufficient condition for a non-empty finite subset H of a group G to be a sub-group is that H must be closed with respect to multiplication *i.e.*, $a, b \in H \Rightarrow ab \in H$.
   (A) 1 is true only
   (B) Both 1 and 2 are true
   (C) 2 is true only
   (D) None of these is true

231. 1. The identity of a subgroup is the same as that of the group.
   2. The inverse of any element of subgroup is the same as the inverse of the same regarded as an element of the group.
   3. The intersection of two subgroup of a group G is a subgroup of G.
   (A) 1 is true only
   (B) 3 is true only
   (C) 2 is true only
   (D) All 1, 2 and 3 are true

232. 1. The union of two subgroups is not necessarily a subgroup.
   2. The union of two subgroups is a subgroup. If and only if one is contained in the other.
   3. A subgroup of a cyclic group is also cyclic.
   (A) 1 is true only
   (B) 3 is true only
   (C) 2 is true only
   (D) All 1, 2 and 3 are true

233. 1. Every proper subgroup of an infinite cyclic group is finite.
   2. A subgroup of an abelian group is not abelian.
   3. If G if finite cyclic group of order n and m is divisor of n, then there exists one and only one subgroup of order m which is also cyclic.
   (A) 1 is true only
   (B) 3 is true only
   (C) 2 is true only
   (D) All 1, 2 and 3 are true

234. 1. Every abelian group has abelian subgroups.
   2. Every group G has two trivial subgroup *viz* $\{e\}$ and G itself.
   (A) 1 is true only
   (B) Both 1 and 2 are true
   (C) 2 is true only
   (D) None of these is true

235. 1. A non abelian group can have an abelian subgroup.
   2. A non abelian group can have a non abelian subgroup.
   3. Every finite group of composite order possesses proper subgroups.
   (A) 1 is true only
   (B) 3 is true only
   (C) 2 is true only
   (D) All 1, 2 and 3 are true

236. Let H be subgroup of G, then $\forall a \in G$. $Ha = \{ha : h \in H\}$ is called …… coset of in G and $aH = \{ah : h \in H\}$ is called left coset of H in G—

(A) Right, left (B) Left, right
(C) Right, right (D) Left, left

237. 1. If G is abelian group then $Ha = aH\ \forall a = \in G$.
2. H is also a left (right) coset of G as H = He, where e is identity of G.
(A) 1 is true only
(B) Both 1 and 2 are true
(C) 2 is true only
(D) None of these is true

238. (1) Let $a, b \in G$ (group) and $H \subseteq G$ (subgroup of G), then $a \in Hb \Leftrightarrow Ha = Hb$ and $a \in bH \Leftrightarrow aH = bH$.
(2) Any two right cosets of a subgroup are either disjoint or identical.
(A) 1 is true only
(B) Both 1 and 2 are true
(C) 2 is true only
(D) None of these is true

239. Cayley Theorem states—
(A) A finite group G is isomorphic to a permutation group
(B) Order of each subgroup of finite group is a divisor of the order of the group
(C) If P is prime number and a is any integer, then $a^p = a\ (mod p)$
(D) None of these

240. Fermat Theorem states—
(A) A finite group G is isomorphic to a permutation group
(B) Order of each subgroup of a finite group is a divisor of the order of the group
(C) If P is prime number and a any integer the $a^p = a\ (mod\ p)$
(D) None of these

241. Lagrange's Theorem states—
(A) A finite group G is isomorphic to a permutation group
(B) Order of each subgroup of a finite group is a divisor of the order of the group
(C) If P is prime number and a is any integer, then $a^p = a\ (mod p)$
(D) None of these

242. 1. If G is finite group and $a \in G$, then order of a divides the order of G.
2. If G is a finite group and $a \in G$, then $a^{\circ\ (G)} = e$.
(A) 1 is true only
(B) Both 1 and 2 are true
(C) 2 is true only
(D) None of these is true

243. 1. If G is a finite group where order is a prime number then G is a Cyclic group.
2. The order of every element of a finite group is a divisor of the order of the group.
(A) 1 is true only
(B) Both 1 and 2 are true
(C) 2 is true only
(D) None of these

244. A subgroup H of a group G is said to be a …… of G, if for every $x \in G$ and for every $h \in H$, $xhx^{-1} \in H$.
(A) Normal subgroup
(B) Simple group
(C) Abelian subgroup
(D) None of these

245. A group having no proper normal subgroup is called a—
(A) Normal subgroup
(B) Abelian subgroup
(C) Simple group
(D) None of these

246. 1. Every group of prime order is not simple.
2. A subgroup H of a group G is normal, iff $\forall x \in G, xHx^{-1} = H$.
(A) 1 is true only
(B) Both 1 and 2 are true
(C) 2 is true only
(D) None of these is true

247. A subgroup H of a group G is a normal subgroup of G, iff the product of two right cosets of H in G is …… of H is G.
(A) Right coset
(B) Left coset
(C) Both (A) and (B)
(D) None of these

248. The intersection of any two normal subgroup of a group is—
 (A) Normal subgroup
 (B) Abelian subgroup
 (C) Simple group
 (D) None of these

249. Every subgroup of an Abelian group is—
 (A) Normal subgroup
 (B) Abelian subgroup
 (C) Simple subgroup
 (D) None of these

250. Every subgroup of a cyclic group is—
 (A) Normal subgroup
 (B) Abelian subgroup
 (C) Simple group
 (D) None of these

251. If H is a subgroup of G and N is a normal subgroup of G, then $H \cap N$ is …… of G.
 (A) Normal subgroup
 (B) Abelian subgroup
 (C) Simple group
 (D) None of these

252. If N is a …… of G and H is a subgroup of G, then NH is a subgroup of G.
 (A) Normal subgroup
 (B) Abelian subgroup
 (C) Simple group
 (D) None of these

253. If M and N are normal subgroup of G, then MN is also …… of G.
 (A) Normal subgroup
 (B) Abelian subgroup
 (C) Simple group
 (D) None of these

254. If H is the only subgroup of finite order m in the group G, then H is a …… of G.
 (A) Normal subgroup
 (B) Abelian subgroup
 (C) Simple group
 (D) None of these

255. If a cyclic subgroup N of G is normal in G, then every subgroup of N is …… in G.
 (A) Normal subgroup
 (B) Abelian subgroup
 (C) Simple group
 (D) None of these

256. 1. Every quotient group of an abelian group is abelian but converse is not true.
 2. Every quotient group of a cyclic group is cyclic but the converse is not true.
 3. If N is normal in G and $a \in$ G is of order n, then order of Na in G/N is a divisor of n.
 (A) 1 is true only (B) 3 is true only
 (C) 2 is true only (D) 1, 2, 3 are true

257. 1. A mapping f from a group G into a group G' is said to be homomorphism of G into G' if $f(ab) = f(ab) = f(a) f(b)$ $\forall a, b \in$ G.
 2. A mapping f from a group G onto a group G' is said to be homomorphism of G onto G' if $f(ab) = f(a) f(b)$ $\forall a, b \in$ G.
 (A) 1 is true only
 (B) Both 1 and 2 are true
 (C) 2 is true only
 (D) Now of 1 and 2 are true

258. Every ...... image of a group G is isomorphic to some quotient group of G—
 (A) Homomorphic (B) Automorphic
 (C) Isomorphic (D) None of these

259. Every .... image of an abelian group is abelian but converse is not true.
 (A) Homomorphic (B) Automorphic
 (C) Isomorphic (D) None of these

260. If f is .... mapping of a group G into a group G and $f(a)$ is homomorphic image of G in G′, then f(G) is a subgroup of G.
 (A) Homomorphic (B) Automorphic
 (C) Isomorphic (D) None of these

261. The necessary and sufficient condition for a .... f of a group G into G' with Kernel K to be isomorphic is that $K = \{e\}$.
 (A) Homomorphic (B) Automorphic
 (C) Isomorphic (D) None of these

262. A .... from a simple group is either trivial or one to one.

(A) Homomorphism
(B) Automorphism
(C) Isomorphism
(D) None of these

263. The set of all .... of a group forms a group with respect to composite of functions as the composition.
(A) Homomorphism
(B) Automorphism
(C) Isomorphism
(D) None of these

264. A normal subgroup H of a group G is said to be ...., if there exists no normal subgroup K of G which properly contains H.
(A) Normal subgroup
(B) Maximal subgroup
(C) Simple group
(D) None of these

265. Cauchy Theorem for Abelian group states—
(A) Suppose G is a finite abelian group and P/o (G) where P is a prime number. Then there is an element $a \neq e \in$ G, such that $a^P = e$
(B) A normal subgroup H of a group G is said to be maximal subgroup, if there exists no normal subgroup K of G which properly contains H
(C) Every homomorphic image of a group G is isomorphic to some quotient group of G
(D) None of these

266. General Cauchy theorem—
(A) Suppose G is a finite group and P/o (G), where P is a prime number. Then there is an element a in G such that o (a) = P
(B) Suppose G is a finite abelian group and P/o (G), where P is a prime number. Then there is an element $a \neq e \in$ G, such that $a^P = e$
(C) A normal subgroup H of a group G is said to be maximal subgroup, if there exists no normal subgroup K of G which properly contains H
(D) Every homomorphic image of a group G is isomorphic to some quotient group of G

267. For (R, + ;) closure law states—
(A) $\forall a, b \in R \Rightarrow a + b \in R$
(B) $(a + b) + c = a + (b + c)\ \forall a,b, c \in R$
(C) $\forall a \in R, \exists 0 \in R : a + 0 = a = 0 + a$
(D) None of these

268. For (R, + , .) associative law states—
(A) $\forall a, b \in R \Rightarrow a + b \in R$
(B) $(a + b) + c = a + (b + c), \forall a, b, c \in R$
(C) $\forall a \in R, \exists 0 \in R : a + 0 = a = 0 + a$
(D) None of these

269. For (R, +, .) existence of additive identity 0 states—
(A) $\forall a \cdot b \in R \Rightarrow a + b \in R$
(B) $(a + b) + c = a + (b + c)\ \forall a, b \cdot c \in R$
(C) $\forall a \in R, \exists 0 \in R : a + 0 = a = 0 + a$
(D) None of these

270. For (R, +, .) existence of additive inverse, states—
(A) $\forall a \in R, \exists -a \in R : a + (-a) = 0 = (-a) + a$
(B) $(a + b) + c = a + (b + c), \forall a, b, c \in R$
(C) $\forall a \in R, \exists 0 \in R : a + 0 = a = 0 + a$
(D) None of these

271. For (R, + , .) commutative law, states—
(A) $\forall a \in R, \exists -a \in R : a + (-a) = 0 = (-a) + a$
(B) $(a + b) + c = a + (b + c), \forall a, b, c \in R$
(C) $\forall a \in R, \exists 0 \in R : a + 0 = a = 0 + a$
(D) $\forall a, b \in R, a + b = b + a$

272. The ring (R, +, .) is called .... if in R multiplication is commutative, *i.e.*, $\forall a, b \in R, a{\cdot}b = b \cdot a$.
(A) Commutative ring
(B) Boolean ring
(C) Ring with unity
(D) None of these

273. The ring (R, +, .) is said to .... if there exist a multiplicative identity I $\in$ R, *i.e.*, $\forall a \in R, \exists 1 \in R : a \cdot 1 = a = 1{\cdot}a$.
(A) Commutative ring
(B) Ring with unity
(C) Boolean ring
(D) None of these

274. A ring whose every element a is idempotent *i.e.*, $a^2 = a$ is known as ....
(A) Commutative ring
(B) Boolean ring
(C) Ring with unity
(D) None of these

275. If R is a commutative ring, then $a \neq 0 \in$ R is called a .... If there exist an element $b \neq 0$ $\in$ R such that $ab = 0$—
(A) Zero divisor (B) Prime divisor
(C) Single divisor (D) None of these

276. If in a ring R there exists non-zero elements $a \neq 0$, $b \neq 0 \in$ R such that $ab = 0$, then R is said to be a ring with—
(A) Zero divisor (B) Prime divisor
(C) Single divisor (D) None of these

277. If R be a ring, an element $a \neq 0 \in$ R called .... If $ab = 0$ for some non zero $b \neq 0 \in$ R.
(A) Left zero divisor
(B) Right zero divisor
(C) Both (A) and (B)
(D) None of these

278. If R be a ring, an element $a \neq 0 \in$ R is called ....if $ba = 0$ for some non zero $b \neq 0 \in$ R.
(A) Left zero divisor
(B) Right zero divisor
(C) Both (A) and (B)
(D) None of these

279. A ring R is said to be a ring .... if ab = 0, either $a = 0$ or $b = 0$, $a, b \in$ R—
(A) Without zero divisors
(B) With zero divisor
(C) Both (A) and (B)
(D) None of these

280. Let (R, +, .) be any ring and S a subring of R, then S is said to be a .... of R if $Sr \in$ S. $\forall r \in$ S, $\forall r \in$ R, S $\in$ S S and .... if rS $\in$ S, $\forall r \in$ R, S $\in$ S.
(A) Right ideal, left ideal
(B) Left ideal, left ideal
(C) Right ideal, right ideal
(D) Left ideal, right ideal

281. A non-empty subset S of R is said to be ideal of R, if—
(A) S is a subgroup of R under addition
(B) Both A and B
(C) Sr, rS $\in$ S, $\forall r \in$ R, S $\in$ S
(D) None of these

282. Let R be a commutative ring. An ideal P of ring R is said to be a .... if for $p, q \in$ R, $pq \in$ P $\Rightarrow$ P $\in$ P or $q \in$ P.
(A) Prime ideal of R
(B) Integral Domain
(C) Maximal ideal
(D) None of these

283. An ideal S $\neq$ R in a ring R is said to be a .... if whenever A is an ideal of R such that S $\subseteq$ A $\subseteq$ R, then either R = A or S = A.
(A) Prime ideal of R
(B) Integral domain
(C) Maximal ideal
(D) None of these

284. A commutative ring with unity is said to be an ...., if it is without zero divisors.
(A) Prime ideal of R
(B) Integral domain
(C) Maximal ideal
(D) None of these

285. An integral domain (D, +, .) is called an ordered integral domain, if D contains a subset D+ such that—
(A) D+ is closed with respect to addition and multiplication as defined on D
(B) If $a \in$ D, one and only one of the following is true : $a = 0$ or $a \in$ D+ or $-a \in$ D+
(C) Both A and B
(D) None of these

286. A commutative ring with unity is called a .... if its every non zero element possesses a multiplicative inverse.
(A) Field (B) Integral domain
(C) Group (D) None of these

287. A ring with unity is said to be .... or division ring if each non-zero element possesses multiplicative inverse.
(A) Skew field (B) Group
(C) Field (D) Integral domain

288. 1. Every field is also a division ring but every division ring is not a field.
 2. The multiplicative inverse of a non-zero element of a field is unique.
 3. A field is necessarily an integral domain.
 (A) Only 1 is true (B) Only 3 is true
 (C) One 2 is true (D) All 1, 2, 3 are true

289. 1. Every finite integral domain is a field.
 2. An infinite integral domain need not be a field.
 (A) 1 is true, but 2 is false
 (B) Both 1 and 2 are true
 (C) 1 is false, but 2 is true
 (D) Both 1 and 2 are false

290. A necessary and sufficient conditions for a non-empty subset K of a field F to be a subfield are—
 (A) $a, b \in K \Rightarrow a - b \in K$
 (B) $a, b \neq 0 \in K \Rightarrow ab^{-1} \in K$
 (C) Both A and B
 (D) None of these

291. Prime field is—
 (A) A field which does not contains any proper subfield
 (B) A field when it is ordered as integral domain
 (C) Both (A) and (B)
 (D) None of these

292. Ordered field is—
 (A) A field which contains a subset of positive elements, satisfying the additive and multicative closure and trichatomy
 (B) A field which does not contains any proper subfield
 (C) Both (A) and (B)
 (D) None of these

Answers with Explanation

1. (D) Since $(z, +, .)$ is a commutative ring with unity and it has no zero divisors. Hence $(z, +, .)$ is also an integral domain. Thus, (A), (B) and (C) all are correct but (D) is not correct. Hence the correct answer of this question is (D).
2. (D)
3. (A) Since the set of natural number does not have any additive identity. Thus (N, +, .) is not a ring. Hence (N, +, .) will not be an integral domain.
4. (A) 5. (A) 6. (A)
7. (A) Division ring is Abelian group under '+' and group under multiplication hence two elements *viz* identity under '+' and identity under multiplication '.' is must. Hence at least two elements in division ring is must.
8. (A) $x \in R \Rightarrow x + x \in R$

Now` $(x + x)^2 = (x + x)$ (given)

$\Rightarrow (x + x)(x + x) = (x + x)$

$\Rightarrow (x + x) \cdot x + (x + x) \cdot x = x + x$ (Left distributive law)

$\Rightarrow (x^2 + x^2) + (x^2 + x^2) = x + x$ (Right distributive law)

$\Rightarrow (x + x) + (x + x) = x + x$ $[\because x^2 = x]$

$\Rightarrow (x + x) + (x + x) = (x + x) + 0$ $[\because x + 0 = x]$

$\Rightarrow x + x = 0$ ...(1)

[By left concellation law for addition in R]

Now $x + y = 0$

$\Rightarrow x + y = x + x$ [from (1)]

$\Rightarrow y = x$

[By left concellation law for addition in R]

9. (B)
10. (B) Since the ring of integers does not have multiplicative inverse so the ring of integers can not be field and division ring. The ring of integers is commutative without zero divisor. Hence it is an integral domain.
11. (C)
12. (D) Since all (A), (B) and (C) are correct. Thus (D) is false. Hence (D) is correct option.
13. (B) Since the set of integers is a commutative ring with unity and $14 \in R$ (ring of integers). Thus, the ideal $\{14r : r \in I\}$ is the principle ideal generated by 14.
14. (B)
15. (A) In the commutative ring of integer z the ideal $\{pr : r \in z\}$ is a prime ideal if p is prime because if $ab \in p$, then $p/ab \Rightarrow p/a$ or p/b.

Hence either $a \in p$ or $b \in p \; \forall a, b \in z$.

16. (A) Suppose m is a prime and let $[a], [b], \in z$.
Then $[a]\,[b] = 0 \Rightarrow \; ab = 0 \pmod{m}$
$\Rightarrow ab$ is divisible by m
$\Rightarrow$ either m/a or m/b, thus, $ab = 0 \Rightarrow$ either $a = 0 \pmod{m}$
or, $\quad b \equiv 0 \pmod{m}$
$\therefore$ Set of residue classes mod m is a ring without zero divisors again suppose m is not prime, then let.
$m = rs$ where $1 < r < m$, $1 < s < n$
Therefore, $[r]\,[s] = [rs] = [0] \pmod{m}$
while $[r] \neq [0]$ and $[s] \neq [0]$
$\therefore$ The set of residue classes (mod m) is a ring with zero divisors.
$\therefore$ The correct answer is (A).

17. (B)

18. (D) Complex number are not ordered as, if $x_1 + iy_1, x_2 + iy_2 \in c$ we can not say $(x_1 + iy_1) <$ or $> x_2 + iy_2$.

19. (C)

20. (C) Statement (A) is not correct as a ring may have zero divisors, statement (B) is also not correct always.
Statement (D) is not correct as natural numbers set N has no additive identity hence N is not a ring
(C) is correct it is a will known theorem.

21. (B) $[R = \{0, 1, 2, 3\}, +_4, \times_4]$ is a ring. For check the postulates by following tables.

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	3

Since, $2 \times_4 2 = 0 \Rightarrow 2$ is a zero divisor

22. (A) Similar to 26, R is not a field as '6' is not prime. It is ring with zero divisors as—$2 \cdot 3 = 0$

23. (A) Since, P is prime, hence option (A) is correct.

24. (D) Since, P is prime hence I_P is field so (A), (B), (C) are all correct. Hence option (D) is correct.

25. (D)

26. (C) Let $L_1 = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}, L_2 = \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}$

be any low elements of, then

$$L_1 - L_2 = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} - \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} a-c & 0 \\ b-d & 0 \end{bmatrix} \in L$$

$\therefore$ N is a subgroup of M under addition

Now, Let $U = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$ be any element of M

$$\therefore \quad UL_1 = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} wa + xb & 0 \\ ya + zb & 0 \end{bmatrix} \in L$$

$\therefore$ L is a left ideal of R.

Since $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \notin L$

$\therefore$ L is not right ideal.
Similarly we can prove that k is right ideal but not left ideal on M.

27. (B) 28. (A) 29. (B)

30. (A) Since, (R, +, .) is a ring and non-zero element of the set of real numbers possesses multiplicative inverse in R.
Hence, (R, +, .) is a field.

31. (A)

32. (C) When $x \in \left[\frac{1}{2}, 2\right], f(x) \in \left[-\frac{3}{2}, \frac{3}{2}\right]$
Hence, supremum of $f(x)$ is 3/2.

33. (D) To form a ring, we required at least one element which is 0.
$\therefore$ R {0} form a ring with composition.
$$0 + 0 = 0, \; 0 \cdot 0 = 0$$
This ring known as zero ring

34. (B) We know that set of integers modulo P form a ring with unity to prove that z_p is a field we will have to show that every non zero element of z_p is invertible.

Let $r \in z_p$ and $r \neq 0$,

Now, $r \pm 0 \Rightarrow r \neq 0 \pmod p$

$\Rightarrow$ r is not divisible by p

$\Rightarrow$ r and p are relatively prime

i.e., there exist integers x, y such that

$$rx + py = 1$$

$\Rightarrow$ $rx \in 1 \pmod p$ as $py = 0 \pmod p$

Thus, x is inverse of r in z_p.

$\therefore$ z_p is a field if p is prime.

35. (C) Let $3 + \sqrt{2}$ and $3 - \sqrt{2}$ are two irrational numbers and their product $= (3 + \sqrt{2})(3 - \sqrt{2}) = 1$ which is a rational number

$\therefore$ The statement (C) is false.

36. (D)	37. (B)	38. (C)	39. (A)	40. (A)
41. (B)	42. (A)	43. (B)	44. (A)	45. (A)
46. (A)	47. (A)	48. (A)	49. (A)	50. (A)
51. (A)	52. (A)	53. (A)	54. (A)	55. (A)
56. (A)	57. (A)	58. (A)	59. (A)	60. (A)

61. (A) Here a is one generator of the group

The order of G is 6.

ap is a generator of G of order n iff p is prime to n, *i.e.*,

$$(n, p) = 1$$

Hence $\quad (6, 5) = 1$

$\therefore$ $p = 5$ and a^5 is also a generator.

62. (D) Here 1 is a identity as $(-i)^4 = 1$

$\Rightarrow \quad 0(-i) = 4 \quad [\because\ 0(a) = p]$

$\Rightarrow p$ is the least positive integer : $a^p = e$, an identity.

63. (D) $\quad 5(2) = 2 + 5^2 + 5^2 + 5^2 + 5^2 = 0$

so $\quad 0(2) = 5$

64. (A) $a, b \in G \Rightarrow ab \in G, \forall, a, b \in G$

$$\begin{aligned}
\Rightarrow \quad (ab)^2 &= e \\
\Rightarrow \quad (ab)(ab) &= e \\
\Rightarrow \quad (ab)^{-1} &= ab \\
\Rightarrow \quad b^{-1}a^{-1} &= ab \qquad \ldots(1) \\
\text{But} \quad a^2 &= e \\
\Rightarrow \quad a \cdot a &= e \\
\Rightarrow \quad a^{-1} &= a \qquad \ldots(2) \\
\text{and} \quad b^2 &= e \\
\Rightarrow \quad b^{-1} &= b \qquad \ldots(3)
\end{aligned}$$

$\therefore$ By (2), (3) and (1), we have

$$ab = ba$$

$\therefore$ G is Abelian group.

65. (A)	66. (A)	67. (B)	68. (C)	69. (B)
70. (D)	71. (C)	72. (A)	73. (A)	74. (A)
75. (A)	76. (A)			

77. (C) The inverse of matrix exist only when it is non-singular (M, .) satisfies closure, associative and existence of identity so M is a monoid.

78. (A) We have

$$\begin{aligned}
a * b &= a * b * e \\
&= a * b * (b^{-1} * a^{-1} * b * a) \\
&\qquad [\because\ b^{-1} * a^{-1} * b * a = e] \\
&= a * (b * b^{-1}) * a^{-1} * b * a \\
&= a * e * a^{-1} * b * a \\
&= (a * a^{-1}) * b * a \\
&= e * b * a = b * a
\end{aligned}$$

79. (A) $\forall a \in G, a^2 = e \Rightarrow a \cdot a = a \cdot a^{-1} = e$

$\Rightarrow \quad a = a^{-1}$

$\therefore \quad \forall a \cdot b \in G$

$$\begin{aligned}
ab &= (ab)^{-1} \qquad \ldots(1) \\
\Rightarrow \quad a \cdot b &= b^{-1}a^{-1} \\
&= b \cdot a \quad [\because\ a = a^{-1}, b = b^{-1}]
\end{aligned}$$

$\therefore$ Abelian group.

80. (A)
$$\begin{aligned}
fg &= \begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,3\,2\,4\,5 \end{pmatrix}\begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,2\,3\,4\,5 \end{pmatrix} \\
&= \begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,3\,2\,4\,5 \end{pmatrix}\begin{pmatrix} 1\,3\,2\,4\,5 \\ 1\,3\,2\,5\,4 \end{pmatrix} \\
&= \begin{pmatrix} 1\,2\,3\,4\,5 \\ 1\,3\,2\,5\,4 \end{pmatrix}
\end{aligned}$$

81. (B)	82. (A)	83. (C)	84. (A)	85. (A)

86. (A) $(1\ 2\ 3) \cdot (2\ 4\ 3) \cdot (1\ 3\ 4)$

$$\begin{aligned}
&= \begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,1\,4 \end{pmatrix}\begin{pmatrix} 1\,2\,4\,3 \\ 1\,4\,3\,2 \end{pmatrix}\begin{pmatrix} 1\,2\,3\,4 \\ 3\,2\,4\,1 \end{pmatrix} \\
&= \begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,1\,4 \end{pmatrix}\begin{pmatrix} 2\,3\,1\,4 \\ 4\,2\,1\,3 \end{pmatrix}\begin{pmatrix} 1\,2\,3\,4 \\ 3\,2\,4\,1 \end{pmatrix} \\
&= \begin{pmatrix} 1\,2\,3\,4 \\ 4\,2\,1\,3 \end{pmatrix}\begin{pmatrix} 4\,2\,1\,3 \\ 1\,2\,3\,4 \end{pmatrix} \\
&= \begin{pmatrix} 1\,2\,3\,4 \\ 1\,2\,3\,4 \end{pmatrix}
\end{aligned}$$

87. (A) $\begin{pmatrix} 1\,2\,5\,3\,4 \\ 3\,4\,1\,5\,2 \end{pmatrix} = \begin{pmatrix} 1\,3\,5\,2\,4 \\ 3\,5\,1\,4\,2 \end{pmatrix}$

$$= (1\,3\,5)\,(2\,4)$$
$$= (1\,5)\,(1\,3)\,(2\,4)$$

88. (B) $c = (1\,2\,3\,4\,5\,6\,7)$

c^3 moves every symbol three places a long,

i.e., $c^3 = (1\,4\,7\,3\,6\,5)$

89. (A)
$$c^2 = c \cdot c$$
$$= \begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,4\,1 \end{pmatrix} \begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,4\,1 \end{pmatrix}$$
$$= \begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,4\,1 \end{pmatrix} \begin{pmatrix} 2\,3\,4\,1 \\ 3\,4\,1\,2 \end{pmatrix}$$
$$= \begin{pmatrix} 1\,2\,3\,4 \\ 3\,4\,1\,2 \end{pmatrix}$$
$$= (1\,3)\,(2\,4)$$

90. (D) 91. (A) 92. (A) 93. (A) 94. (B)
95. (A) 96. (B) 97. (D) 98. (A) 99. (B)
100. (A) 101. (A) 102. (B) 103. (C) 104. (A)
105. (B) 106. (C) 107. (A) 108. (A)

109. (A) $a \cdot b = b \cdot a \;\forall\, a, b \in$ E, so (E, +, .) is commutative ring there exist no element $1 \in$ E such that
$$a \cdot 1 = 1 \cdot a = a, \forall a \in \text{E}$$
$\therefore$ (E, +, .) is commutative ring, but not an integral domain.

110. (B) $a \odot b = a + b + ab = b + a + ba = b \odot a$

As addition and multiplication of integers are commutative

$\therefore \odot$ is commutative in I.

$\therefore$ I, $\oplus, \odot$ is a commutative ring.

If b is a unit element, $a \cdot b = a$

$\Rightarrow a + b + ba = a, \Rightarrow b + ba = 0, \Rightarrow b\,(1 + a) = 0, \Rightarrow b = 0$

$\therefore 0 \in$ I is a unit element

$\therefore$ {I, $\oplus$, $\odot$ } is an integral domain.

111. (A) If $e_1 \in$R, then the right identity gives
$$e_1 e_2 = e_1 \qquad \ldots(1)$$
If $e_2 \in$R, then the left identity gives
$$e_1 e_2 = e_2 \qquad \ldots(2)$$
By equation (1) and (2)
$$e_1 e_2 = e_1 = e_2$$
i.e., two identities are equal.

112. (C) If $1 \in$ R is an multiplicative identity then $1 \cdot a = a$, and $1 = 0$

$\Rightarrow a \cdot 1 = 0$, which is the contradiction

$\therefore \quad 1 \neq 0$

113. (A) $e_1 \in$ R is unity then
$$e_1 \cdot a = a \cdot e_1 = a, \forall a \in \text{R}$$
$e_2 \in$ R is unity then
$$e_2 \cdot a = a e_2 = a, \forall a \in \text{R}$$
$$\Rightarrow \quad e_1 \cdot a = e_2 \cdot a$$
$$\Rightarrow \quad e_1 = e_2$$

114. (B) 115. (C) 116. (A) 117. (A) 118. (A)
119. (C) 120. (A) 121. (A) 122. (A) 123. (A)
124. (B) 125. (A) 126. (B)

127. (B) $xy \in$ R
$$\Rightarrow (xy)\,(xy) = xyxy$$
$$= xxyy \quad \text{(if R is commutative)}$$
$$= x^2y^2$$

128. (C) Let a is nilpotent element of the ring R for some positive integer n,

$a^n = 0 \Rightarrow a^{n-1} \cdot a = 0$, but $a^{n-1} \neq 0$ and also $a \neq 0$.

$\therefore$ a is a divisor of zero, *i.e.*, zero divisor

129. (A) A and B are zero divisors if AB = 0 $\Rightarrow$ A $\neq 0$ and B $\neq 0$

Here $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

130. (B) Let 1 be the unity of the field F.

$a \cdot a_1^{-1} = 1$ and $a \cdot a_2^{-1} = 1$
$$\Rightarrow \quad a \cdot a_1^{-1} = a \cdot a_2^{-1}$$
$$\Rightarrow \quad a_1^{-1} = a_2^{-1}$$

131. (B) 132. (D) 133. (D) 134. (B) 135. (C)

136. (C) A is one of the generator

ap is generator of G of order n iff $(n, p) = 1$

$1 < p < n$

Here $n = 4$ and $(4, 3) = 1$

$\therefore$ a^3 is also a generator of G.

137. (B) Here 1 is an identity element

also $\quad (-1)^2 = 1$

$\therefore \quad 0\,(-1) = 2$

$\because \quad 0\,(a) = p$

$\Rightarrow p$ is the least positive integer

$a^p = e$, an identity

138. (B) $c = \{x + yi : x, y$ are real numbers, $i = \sqrt{-1}\}$

c is a commutative ring.

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$
$$= (x_2 + iy_2)(x_1 + iy_1)$$

The zero element is $0 + 0i$ and unit element is $1 + 0i$

Also this ring has no zero divisors because the product of non-zero complex numbers cannot be zero.

Hence c is an integral domain.

139. (C) I is an integral domain $\forall a, b \in$ I, $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$

$\therefore a^2 = a \cdot a = 0 \Rightarrow a = 0$ which is the contradiction

$\therefore \quad a^2 \neq 0$

140. (D)	141. (A)	142. (A)	143. (A)	144. (A)
145. (A)	146. (D)	147. (B)	148. (C)	149. (C)
150. (A)	151. (D)	152. (A)	153. (A)	154. (A)
155. (A)	156. (A)	157. (A)	158. (A)	159. (A)
160. (B)	161. (A)	162. (A)	163. (C)	164. (A)
165. (A)				

166. (A) An integral domain is a field if it possess an multiplicative inverse.

167. (D) $a > b \Rightarrow a - b > 0, \Rightarrow (a + c) - (b + c) = a - b > 0$

$\Rightarrow \quad a + c > b + c$

168. (A) R is commutative

$$(ab)^4 = ab \cdot ab \cdot (ab)^2$$
$$= aabb \cdot ab \cdot a \cdot b$$
$$= a^2b^2 \cdot ba \cdot a \cdot b$$
$$= a^2b^3a \cdot ba$$
$$= a^2b^3 \cdot ba \cdot a$$
$$= a^2b^4 \cdot a \cdot a \quad (\because b^4 = b, \forall b \in R)$$
$$= a^2b \cdot a \cdot a$$
$$= a^2 \cdot a \cdot b \cdot a = a^3ba$$
$$= a^3 \cdot a \cdot b = a^4 \cdot b$$
$$= a \cdot b$$

169. (A) R is commutative

$$(ab)^2 = ab \cdot ab$$
$$= a \cdot a \cdot b \cdot b$$
$$= a^2b^2$$

170. (B)	171. (B)	172. (B)	173. (A)	174. (A)
175. (B)	176. (D)	177. (A)	178. (A)	179. (A)
180. (B)	181. (B)	182. (A)	183. (D)	184. (A)
185. (C)	186. (C)	187. (A)	188. (A)	189. (B)
190. (A)	191. (A)	192. (C)	193. (A)	194. (A)
195. (C)	196. (B)	197. (A)	198. (A)	199. (A)
200. (B)	201. (A)	202. (A)	203. (C)	204. (D)
205. (A)	206. (A)	207. (A)	208. (B)	209. (B)
210. (A)	211. (A)	212. (B)	213. (B)	214. (A)
215. (B)	216. (B)	217. (B)	218. (A)	219. (B)
220. (B)	221. (A)	222. (A)	223. (B)	224. (A)
225. (C)	226. (D)	227. (A)	228. (D)	229. (B)
230. (B)	231. (D)	232. (D)	233. (B)	234. (B)
235. (D)	236. (A)	237. (B)	238. (B)	239. (A)
240. (C)	241. (B)	242. (B)	243. (B)	244. (A)
245. (C)	246. (C)	247. (A)	248. (A)	249. (A)
250. (A)	251. (A)	252. (A)	253. (A)	254. (A)
255. (A)	256. (A)	257. (B)	258. (A)	259. (A)
260. (A)	261. (A)	262. (A)	263. (B)	264. (B)
265. (A)	266. (A)	267. (A)	268. (B)	269. (B)
270. (A)	271. (D)	272. (A)	273. (C)	274. (B)
275. (A)	276. (A)	277. (A)	278. (B)	279. (A)
280. (A)	281. (B)	282. (A)	283. (C)	284. (C)
285. (C)	286. (B)	287. (A)	288. (D)	289. (B)
290. (B)	291. (A)	292. (C)		

Functional Analysis

Banach Spaces

Vector space—A vector space over the field F, consists a set X, a mapping $(x, y) \rightarrow x + y$ of X. X into X and a mapping $(a, x) \rightarrow ax$ of F. X into X, such that :

(a) X is an Abelian group with group operation $(x, y) \rightarrow x + y$

(b) $$a\,(bx) = (ab)\,x,$$

$x \in$ X, $a, b \in$ F (associativity)

(c) $$(a + b)\,x = ax + bx$$
$$a\,(x + y) = ax + ay,$$

$x, y \in$ X, $a, b \in$ F (distributivity)

(d) $1 \cdot x = x$ (1, and identity with respect to multiplication)

Direct product—Let X_α, $\alpha \in$ A be a set of vector spaces. The direct product $X = \pi\,(X\alpha : \alpha \in A]$ has an elements, the functions $[x_\alpha]$, where $x_\alpha \in$ X for each $\alpha \in$ A, and their scalar multiplication and addition are given by a $[x_\alpha] = [ax_\alpha]$ and $[x_\alpha + y_\alpha] = [x_\alpha] + [y_\alpha]$ respectively.

Direct sum—Let X_α, $\alpha \in$ A be a set of vector spaces whose identity elements are all designated as 0. The direct sum of vector spaces X_α, $\alpha \in$ A is $X = \Sigma\,[X_\alpha : \alpha \in A]$, consists of those $[x_\alpha] \in \pi\,[x_\alpha : \alpha \in A]$ for which $x_\alpha = 0$ for all but finite number of the $\alpha \in$ A.

1. $\Sigma\,[X_\alpha : \alpha \in A]$ is a vector subspace of $\pi\,[X_\alpha : \alpha \in A]$
2. Two spaces are identical iff A is finite.

Linear transformation—A linear transformation f is a mapping $f : X \rightarrow Y$ on X into Y, such that $f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$ for all $x_1, x_2 \in \times$ and $\alpha, \beta \in$ R.

L (X, Y)—The set of linear transformations from vector space X into Y and it is also a vector space.

L (X, R)—The set of linear transformations from vector space X into R, the elements of it are called linear functional. It is also called Algebraic dual denoted by X*.

Sub Spaces

Sub space—Let X be a vector space. A subset $Y \subset X$ is a subspace of X, if $x_1, x_2 \in$ X and $\alpha, \beta \in$ R the elements $\alpha x_1 + \beta x_2 \in$ Y.

Improper subspace—X and null vector space {0} of X.

Proper subspace—$Y \subset X$ and $Y \neq X$ and $Y = \{0\}$.

Intersection of sub spaces : $Y = \cap\,\{Y_\alpha : \alpha \in A\}$ where Y_α, $\alpha \in$ A is a set of subspaces of X.

Subspace generated by S, (S) : $S \subset X$, the sub space (S) generated by S is a smallest subspace of X that contains S.

Disjoint subspaces—Subspaces Y, Z of X are disjoint if $Y \cap Z = \{0\}$.

Hahn Banach Theorem

Hahn Banach Theorem—If S is a convex set which contains an interior point and $0 \notin$ S, there is a hyperplane H, passing through 0, such that all of S is on the same side of H.

1. If X is a vector space and S is a convex set in X, which contains an interior point, and if Y is a subspace of X which has no points in common with S, then there is a hyperplane H such that $Y \subset H$ and all points of S are on the same side of H.
2. If X is a vector space and S is a convex set in X and Y is a translation of a subspace of X which has no points in common with S, then there is a hyper plane H (not necessarily a sub space) containing Y, such that all points in S are on the same side of H.

Extension form : Let X be a vector space, Y is a subspace of X and P a real function on X such that

$$P\,(x) \geq 0,\; P\,(x + y) \leq P\,(x) + P\,(y)$$
and $$P\,(ax) = |\,a\,|\,P\,(x)$$

If f is a linear functional on Y such that $|f(x)| \leq P(x)$ for every $x \in Y$, there is a linear F on X such that $F(x) = f(x)$ on Y and $|F(x)| \leq P(x)$ on X.

Extension form—Let X be a (real) vector space, P a real valued function on X such that

$$P(x+y) \leq P(x) + P(y)$$

and $$P(ax) = aP(x)$$

for $a \geq 0$ and Ya sub space of X.

If f is linear on Y and $f(x) \leq P(x)$ for every $x \in Y$, there is a linear F on X such that

$$F(x) = f(x) \text{ on } Y$$

and $$F(x) \leq P(x) \text{ on } X.$$

Banach Limits Let m be the vector space of all bounded sequences of real numbers. C the sub-space of all convergent sequences. If $x \in C$, then $l(x) = \lim_n x_n$ is defined and l is a linear functional on C.

Banach limit—A Banach limit is any linear functional L defined on m such that :

(a) $L(x) \geq 0$ if $x_n \geq 0$ for all n

(b) $L(x) = L(\sigma x)$, where $\sigma x = \sigma(x_1, x_2, \ldots\ldots) = (x_2, x_3, \ldots\ldots)$

(c) $L(x) = 1$ if $x = (1, 1, \ldots\ldots)$

Almost convergent and F-limit—$x \in m$ is called almost convergent and the number S is called F-limit of x if $L(x) = s$ for all Banach limits L.

P (x) and P′ (x) : $P(x) = \inf \overline{\lim_{j \to \infty}} \frac{1}{k} \sum_{i=1}^{k} x_{n_i} + j$

and $P'(x) = -P(-x), x \in m$.

Theorems

1. If L is a Banach limit, then $\underline{\lim}\, x_n \leq L(x) \leq \overline{\lim}\, x_n$ for all $x \in m$.
2. For any $x \in m$, $\lim x_n \leq P'(x) \leq P(x) \leq \lim x_n$. In particular, $P(x) = l(x)$ for $x \in c$.
3. For any Banach limit L and $x \in m$, $P'(x) \leq L(x) \leq P(x)$.
4. $P(x)$ is such that $P(x+y) \leq P(x) + P(y)$ and $P(ax) = aP(x)$ if $a \geq 0$.
5. x is almost convergent iff $P'(x) = P(x)$.
6. x is almost convergent and F-limit of x is x iff $\lim_{P \to \infty} \frac{1}{P}(x_n + x_{n+1} + \ldots\ldots + x_{n+P-1}) = S$ holds uniformly in n.

Banach Spaces Dual Space

Norm—Let X be a vector space and $x \in X$. A real valued function on a vector space X, $\|x\|$ is a norm on X if,

(a) $x \neq 0 \Rightarrow \|x\| > 0$

(b) $\|\alpha x\| = \alpha \|x\|$

(c) $\|x + y\| \leq \|x\| + \|y\|$

Semi-norm—Let X be a vector space, Y is a sub space of X and P a real valued function on X :

(a) $x \neq 0 \Rightarrow P(x) > 0$

(b) $P(x+y) \leq P(x) + P(y)$

(c) $P(\alpha x) = |\alpha| P(x)$

Banach space—A complete normed vector space is Banach space.

Bounded linear transform—A linear transform T is bounded if there is an integer M such that

$$\|T_x\| \leq M\|x\| \text{ for every } x \in X.$$

Norm (bound) or T—Let T be a bounded linear transform and

$$\|T\| = \text{Inf}\{M : \|T_n\| \leq M\|x\|, x \in X\}$$

Then $\|T\|$ is norm (bound) of T.

Here, $\|T_x\| \leq \|T\|\,\|x\|$ for every $x \in X$.

Dual of X—If X is a normed vector space, then space X′ of continuous linear functionals on X, with norm.

$$\begin{aligned} \|x'\| &= \sup\{|x'(x)| : \|x\| = 1\} \\ &= \inf\{M : |x'(x)| \\ &\leq M\|x\|, x \in X \end{aligned}$$

is called dual of X.

Theorems

1. **Hahn Banach theorem**—If X is a normed vector space, Y is a sub space of X and f is a bounded linear functional on Y with bound $\|f\|$, relative to Y, then f has a continuous linear extension to an $x' \in X'$ with $\|x'\| = \|f\|$.
2. For any $x \neq 0$ in X there is an $x' \in X'$, such that $x'(x) = \|x\|$ and $\|x'\| = 1$.
3. If X is a complete normed vector space and f is a continuous linear functional on Y, a sub space of X, then f can be extended to a linear functional F on X such that

$$\|F\| = \|f\|$$

4. For every normed vector space X there is a set A such that X is isomorphic with a sub space of the Banach space of bounded functions f on A with

$$\| f \| = \sup \{ | f(t) | : t \in A \}$$

Uniform Boundedness Principle

1. Let X be a Banach space $\{f_n\}$ is a sequence of continuous linear functional on X and for every $x \in X$, the sequence $\{| f_n(x) |\}$ is bounded, then the sequence of $\{\| f_n \|\}$ of norms is bounded.

   If X is a Banach space, Y is a normed vector space and $\{T_n\}$ a sequence of continuous linear transforms on X into Y such that for every $x \in X$, then sequence $\{\| T_n (x) \|\}$ is bounded, then the sequence $\{\| T_n \|\}$ of norm is bounded.
2. Let X be a Banach space and Y a normed vector space for every $m = 1, 2, \ldots\ldots$, Let $\{T_n (x, m)\}$ be a sequence of continuous linear operators on X with values in Y such that for every m there is an $x_m \in X$ for which

   $\overline{\lim\limits_{n}} \; \| T_n (x_m, m) \| = +\infty$. Then there is an $x \in X$ for which $\overline{\lim\limits_{n}} \; \| T (x, m) \| = +\infty$, $m = 1, 2 \ldots\ldots$
3. Let X be a Banach space and Y a normed vector space. For every $m = 1, 2, \ldots\ldots$ let $\{T_n (x, m)\}$ be a sequence of continuous linear operators on X with values in Y such that for every m, there is an $x_m \in X$ for which $\{T_n (x_m, m)\}$ does not converge. Then there is an $x \in X$ for which $\{T_n (x, m)\}$ does not converge, for any $m = 1, 2, \ldots\ldots$
4. If X is a Banach space and Y is a subspace of X which is a Borel set, then Y is either of first category in X or is identical with X.

Lemma of Riesz

1. If Y is a closed proper subspace of X and $\in > 0$, there is an $x_{\in}$ on the unit sphere such that

$$\text{Inf} \{ \| x_0 - x_{\in} \| : x_0 \in Y \} > -\in$$

2. A normed vector space is finite dimensional iff the closed bounded sets are compact.
3. There is a uniform convergent sequence of functions of finite Baire type whose limit is not of finite Baire type.

Compact Transformation

Compact transformation—Let X and Y be Banach spaces. A linear transformation T on X into Y is called compact if every bounded set in X is taken by T into a set whose closure is compact.

B (X, Y)—Let X and Y be Banach spaces then B (X, Y) is the set of linear transformation from X into Y. It is also a Banach space.

Adjoint of transformation—Let X and Y be Banach spaces. Let X, ∞' and Y' be the dual of X and Y respectively. Then X' and Y' are Banach spaces.

Let $T \in B(X, Y)$, then adjoint of T, T^* is a transformation $T^* \in B(Y,' X')$ defined as : for $y' \in X'$ for each $x \in X$ has the value.

$$(T^* y')(x) = y'(Tx)$$

Characteristic number—Given the equation $x - \lambda Tx = 0$.

The number $\lambda_0 \in R$ is called characteristic number of this equation if $x - \lambda_0 Tx = 0$ has a non zero solution.

Theorems

1. The subset $C(X, Y) \subset B(X, Y)$ of compact transformation is a Banach space.
2. If $T \in C(X, X)$, X a Banach space, then vector space of solution of $x - Tx = 0$ is finite dimensional.
3. If $T \in C(X, X)$, then set of characteristic number of $x - \lambda Tx = 0$ is an isolated set.

Weak Convergence

Weak convergence—If X is a normed vector space and $x'_n \in X'$, $n = 1, 2, \ldots\ldots$ we say x'_n convergence weakly to $x' \in X'$ if for each $x \in X$, $\lim\limits_{n \to \infty} x'_n(x) = x'(x)$.

Total set—If X is a normed vector space, $Y \subset X$ is total if its closed linear space is X.

A-limit—A sequence $x = (x_1, x_2, \ldots\ldots)$ is said to have A-limit, A (x) if each $i = 1, 2, 3 \ldots\ldots$ the sum $A_i(x) = \sum\limits_{k=1}^{\infty} a_{ik} \; x_k$ is convergent and lim $A_i(x) = A(x)$.

A-regular—A is regular if for each convergent sequence x, the A-limit exists and equals the ordinary limit of the sequence x.

A-almost regular—A is called almost regular if for every convergent sequence $x = (x_1,$

x_2, ……) there is an N such that for every $m > N$, $\sum_{n=1}^{\infty} a_{mn} x_n$ convergence and

$$\lim_{m \to \infty} \sum_{n=1}^{\infty} a_{mn} x_n = \lim_{n \to \infty} x_n$$

Theorems

1. If X is a Banach space, then (x'_n) converges weakly to $x' \in X'$ iff the sequence $\{\| x'_n \|\}$ of norms is bounded and $\lim_n x'_n (x) = x' (x)$ for every x in some total subset of X.
2. A is regular iff
   (a) $\sum_{k=1}^{\infty} | a_{ik} | \leq M, i = 1, 2, ……$for some M.
   (b) $\lim_i a_{ik} = 0, k = 1, 2, ……$ and
   (c) $\lim_i \sum_{k=1}^{\infty} a_{ik} = 1$.
3. A is almost regular iff there is a fixed m_0 such that :
   (a) $\sum_{m=1}^{\infty} | a_{mn} | < M$ for fixed M and every $m > m_0$, (b) $\lim_n a_{mn} = 0$ for every n, and
   (c) $\lim_m \sum_{n=1}^{\infty} a_{mn} = 1$.
4. If $\{S_n^{(m)}\}$ is a sequence of sequences such that $\sum_{n=1}^{\infty} | S_n^{(m)} | = \infty$. $m = 1, 2, ……$, there is a sequence $\{t_n\}$ converging to zero, such that all the series $\sum_{n=1}^{\infty} S_n^{(m)} t_n, m = 1, 2, …$ diverge.

Dual of l_P—If $l \leq P \leq \infty$, and q is conjugate to P, then $(l_p) = l_q$.

Second dual space—Let X be normed vector space and X′ be the dual of X.

For every $x_0 \neq 0 \in X$ consider the functional x_0 on X′ defined as $x_0 (x') = x' (x_0)$.

Dual of C (*a, b*) and Riesz Representation Theorem

BV (*a, b*)—Class of all functions of bounded variation on $[a, b]$ *i.e.*, for all f which the total variation $V (f) = \sup \sum_{i=1}^{\infty} | f (x_i) - f (x_{i-1}) |$ is finite, where the supremum is taken over all partition $a = x_0 < x_1 < …… < x_n = b$.

BVN [*a, b*]—BV $[a, b]$ which satisfies normalizing condition C $[a, b]$: metric space of all continuous functions.

Riesz representation theorem—If $F \in C'$ $[a, b]$, there exist $g \in$ BVN $[a, b]$ such that

$$F (f) = \int_t^b f dg,$$

where $f \in C [a, b]$ and $\| g \| = V (g) = \| F \|$.

Open Mapping and Closed Graph Theorem

Closed linear transformation—A linear transformation T from Banach space X into a Banach space Y with dom (T) $\subset$ X is closed if $x_n \in$ dom (T),

$$\lim_n x_n = x_n$$

and $\lim_n Tx_n = y \Rightarrow x \in$ dom (T)

and $y = Tx$

Graph—A graph G (T) of mapping T : X $\to$ Y is the set of points $(x, Tx) \in$ X. Y, with $x \in$ dom T.

Theorems

1. If dom (T) is closed in X and T is bounded then T is closed.
2. T is a closed linear transformation iff G (T) is a closed vector subspace of X, Y.

Closed graph theorem—If X and Y are Banach spaces and T is a linear transformation from X and Y, then dom (T) is closed and graph G (T) closed $\Rightarrow$ T is bounded.

Open mapping theorem—If X and Y are Banach spaces and T is bounded linear transformation which maps X on to all of Y, then T is an open mapping (*i.e.*, the image of open set in X under T, is open set in Y)

1. Let X and Y are complete metric space and F is continuous mapping of X onto Y such that for every $r > 0$, there is $k > 0$ such that for every $x \in X$, the closure of the image of the sphere, $\sigma (x, r)$ in X, of centre x and radius r, contains the sphere $\sigma [F (x), k]$ in Y, then for every $\rho > r$ the image of $\sigma (x, \rho)$ contains $\sigma [F (x), k]$.

Hilbert Spaces

Inner product—Let X be a vector space over the field F (real or complex). A mapping of X. X into F, which takes each ordered pair $(x, y) \in$ X. X

into the number $(x, y) \in F$ is called inner product in X if,

(i) $(x, y) = \overline{(y, x)}$

(ii) $(\alpha x_1 + \beta x_2, y) = \alpha (x_1, y) + \beta (x_2, y)$ (linearity)

(iii) $(x, x) \geq 0$ and $(x, x) = 0$ iff $x = 0$

1. If X is a real vector space, then $(x, y) = (y, x)$

2. $(x, \alpha y) = \bar{\alpha} (x, y)$

Norm—Let $x \in X$, then norm in the space X is $\| X \| = \sqrt{(x, x)} = (x, x)^{1/2}$.

Inner product space—A vector space X, together with an inner product in X, is called **inner product space.**

Hilbert space—If X is complete under the norm obtained from its inner product, then X is called **Hilbert space.**

Theorems

Cauchy-Schwarz inequality—For evey

$$x, y \in X, |(x, y)| \leq \| x \| \| y \|$$

The Cauchy-Schwarz inequality states that inner product space is a continuous function on X. X into F, where the norm of $\{x, y\} \in X \cdot X$ is $\| x \| + \| y \|$.

i.e., $\lim x_n = x, \lim y_n = y$

$\Rightarrow \lim (x_n, y_n) = (x, y)$

Pythagorean theorem—

$$x \perp y$$

$$\Rightarrow \| x \|^2 + \| y \|^2 = \| x + y \|^2$$

is valid in inner product space.

Parallelogram law—

$$\| x + y \|^2 + \| x - y \|^2 = 2 \{\| x \|^2 + \| y \|^2\}$$

holds in inner product space.

Banach space is Hilbert space iff the parallelogram law holds.

Projection theorem and dual—If k is a closed convex set in X and $x_0 \in X \sim k$, then there is a unique $y_0 \in k$, such that

$$\| x_0 - y_0 \| = \inf \{\| x_0 - y \| : y \in k\}$$

Projection theorem—If M is a closed subspace of a Hilbert space X, then every $x \in X$ has a unique decomposition $x = y + z$, $y \in M$, $z \in M^{\perp}$, where $M^{\perp} = \{z \in X : z \perp y \text{ for all } y \in M\}$.

The operator $x \to P_x = y$ is a bounded linear idempotent (*a*-projection) such that $M = PX$ and $M^{\perp} = (1 - P) X$ are mutually orthogonal closed subspaces with $X = M + M^{\perp}$.

For each $x \in X$, P_x is the unique element of M which minimizes the distance from x to M.

If x' is a bounded linear functional on a Hilbert space X, then there is a unique $y_0 \in X$ such that

$$x'(x) = (x, y_0)\ x \in X$$

1. Let V be a bounded linear operator on a Hilbert space X with

$$\| V \| \leq 1$$

and $f \in X$

Put $\phi_n = \frac{1}{n} [f + Vf + \ldots + V^{n-1} f]$, $n = 1, 2\ldots$

Then $\phi \in X$, is such that

$$\lim \phi_n = \phi \text{ in } X$$

Orthonormal Sets and Fourier Expansion

Orthogonal set—A set S of elements of an inner product space X is an orthogonal set if $x \perp y$ whenever $x, y \in S$ and $x \neq y$.

Orthonormal set—An orthonormal set S such that for each $x \in S$,

$$(x, y) = \begin{cases} 0, & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases} \quad x, y \in S$$

Complete set—A set S is complete in X if no non-zero $x \in X$ is orthogonal to S.

Fourier coefficients—Let $x \in X$ and $\{x_\alpha\}$ an orthonormal family in X, possibly uncountable.

Then the fourier coefficients of x with respect to $\{x_\alpha\}$ are the numbers.

$$(x, x_\beta) = \sum a_\alpha (x_\alpha, x_\beta) = a_\beta$$

Orthonormal basis—Let X be a Hilbert space and $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set in X, then S is a orthonormal basis orthonormal closed set) in X if the fourier series representation $x = \sum_{\alpha \in A} (x, x_\alpha) x_\alpha$ is valid for each $x \in X$.

Theorems—If $\{x_1, x_2, \ldots\ldots x_n\}$ is an orthonormal set, then for any $x \in X$,

$$\sum_{i=1}^{n} | x, x_i |^2 \leq \| x \|^2$$

Bessel's inquality—If X is a Hilbert space $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set in X and $x \in X$, then

$$\sum_{\alpha \in A} | (x, x_\alpha) |^2 \leq \| x \|^2$$

Let X be an Hilbert space and $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set in X, then S is a basis iff it is complete in X.

Parseval's formula—Let X be an Hilbert space, $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set. Then S is an basis iff equality holds in Bessel's inequality.

$$\| x \|^2 = \sum_{\alpha \in A} |(x, x_\alpha)|^2, x \in X$$

or, $$(x, y) = \sum_{\alpha \in A} (x, x_\alpha) \overline{(y, x_\alpha)},$$

where $x, y \in X$.

Parseval's formula (for real trigonometric system)—

$$a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \int_0^{2\pi} |f(t)|^2 \, dt,$$

where $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt = b_n \sin nt)$ is the fourier series of $f \in L_2 (0, 2\pi)$.

Isoperimetric theorem—Among all single, closed piecewise smooth curves of length L in the plane the circle encloses the maximum area.

Muntz Theorem

Gramm determinant—Let $x_1, x_2, \ldots\ldots x_n \in L_2 [0{\cdot}1]$.

The gramm determinant of $x_1, x_2, \ldots\ldots x_n$ is

$$G = G(x_1, x_2, \ldots\ldots x_n)$$

$$= \begin{vmatrix} (x_1, x_1) & (x_1, x_2) & \ldots\ldots & (x_1, x_n) \\ (x_2, x_1) & (x_2, x_2) & \ldots\ldots & (x_2, x_n) \\ \vdots & & & \\ (x_n, x_1) & (x_n, x_2) & \ldots\ldots & (x_n, x_n) \end{vmatrix}$$

Theorems—A necessary and sufficient condition for $(x_1, x_2, \ldots\ldots x_n)$ to be linearly independent is that $G \neq 0$.

Muntz theorem—A necessary and sufficient condition for the set t^n 1.

t^n 2, ..., $1 \leq n_1 < n_2 < \ldots\ldots$ to be complete in L_2 is that $\sum_{i=1}^{\infty} \frac{1}{n_i} = \infty$.

Dimension and Riesz-Fischer Theorem

Dimension—Orthogonal dimension of a Hilbert space is the cardinality of its bases.

Theorems

1. The cardinality is the same for each basis of a given Hilbert space.
2. The Hilbert space X is separable iff there is a countable basis for X.
3. Two Hilbert spaces are isomorphic if they have the same dimension.
4. **Riesz-fischer theorem**—If $\sum_{n=\infty}^{\infty} |C_n|^2 < \infty$, then there is $f \in L_2 (0, 2\pi)$ such that $\sum_{-\infty}^{\infty} C_n e^{inx}$ is the fourier series of f and coverges to f in the sense of L_2.

Reproducing Kernel

Proper functional space—A vector space X of functions on a set S, together with a norm in X, is called a proper functional space if for every $s \in S$, the evaluation functional at S is continuous, *i.e.*, there exist M such that $|x(S)| \leq M_S \| x \|, x \in X$.

Proper functional completion—A proper functional completion of X is a proper functional space $\overline{X}$ on the same basic set S such that $\overline{X}$ is complete and X is a dense subspace of $\overline{X}$.

Reproducing Kernel—A Hilbert space of functions on a set S is said to have a reproducing Kernel if there is a function $k(S, t)$ on S.

S has the propeties :

(i) $k(\cdot, t) \in X, t \in S$

(ii) $x(t) = [x, k(\cdot, t)], x \in X, t \in S$ (reproducing property).

Positive matrix—A function $k(S, t)$ on S. S is a positive matrix if for each $n = 1, 2, \ldots\ldots$ and each choice of points $t_1, t_2, \ldots\ldots t_n$ the quardratic form $\sum_{i,j=1}^{n} k(t_i, t_j) \xi_i \overline{\xi}_j$ in $\xi_1, \xi_2, \ldots\ldots \xi_n$ is non negative.

Theorems

1. A proper functional space X has a proper functional completion iff for each Cauchy sequence $\{x_n\}$ in X, $\lim_n x_n(S) = 0$ for all $s \in S$

$$\Rightarrow \quad \lim_n \| x \| = 0$$

2. If X has a reproducing Kernel, then $|x(t)| = |(x, k(\cdot, t)| \leq \| k(\cdot, t) \| \| x \|, x \in X$, $t \in S$ and X is proper functional space.

3. Each reproducing Kernel is a positive matrix.
4. If X is a proper functional Hilbert space with $k(S, t)$ as reproducing Kernel and if $[x_\alpha : \alpha \in A]$ is a complete orthonormal set in X, then

$$k(\cdot, t) = \sum_{\alpha \in A} (k(\cdot, t), x_\alpha) x_\alpha$$

$$= \sum_{\alpha \in A} x_\alpha, t \in S$$

or, $$k(S, t) = \sum_{\alpha \in A} x_\alpha(S)\, \overline{x_\alpha(t)}$$

5. The mapping function f can be expressed in terms of k by the formula

$$f(z) = \sqrt{\frac{\pi}{k(\xi, \xi)} k(z, \xi)}$$

Adjoint Operator

Adjoint—Let T be a bounded linear operator of Hilbert space H, then there exist an unique operator on H called adjoint of T, T^* such that $(T_x, y) = (x, T^*y)$ $x, y \in$ H and also

$$\| T^* \| = \| T \|$$

1. $(T + S)^* = T^* + S^*$
2. $(\alpha T)^* = \bar{\alpha} T^*$
3. $(ST)^* = T^* S^*$
4. $T^{**} = T$
5. $\| T^* T \| = \| T \|^2$

Bounded Operator

Bounded linear operator—Let T be a bounded linear operator on Hilbert space H, then

(a) T is normal if $T^* T = TT^*$

(b) T is self adjoint if $T^* = T$

(c) T is unitary if $T^* T = 1 = TT^*$, 1 being and identity operator on H

(d) T is a projection if $T^2 = T$.

Theorems

1. If T is a bounded linear operator on Hilbert space H, then

   (a) T is normal iff $\| T_x \| = \| T^*_x \|$ for every $x \in$ H.

   (b) If T is normal and $Tx = \alpha x$ for $x \in$ H, $\alpha \in c$, then

   $$T^* x = \bar{\alpha} x$$

   (c) If T is normal and if α, β are distinct eigen values of T, then the corresponding eigen spaces are orthogonal to each other.

2. If U is an unitary operator on H then $(U_x, U_y) = (x, y)$ and $\| Ux \| = \| x \|$ for every $x \in$ H.
3. If P is a projection on H, then
   (a) P is self adjoint
   (b) P is normal
   (c) $(Px, x) = \| Px \|^2$ for every $x \in$ H.
4. If M, N, T are bounded linear operators. M, N are normal operators and MT = NT, then

$$M^* T = TN^*$$

Some Solved Examples

Example 1. If x and y are any two vectors in a Hilbert space, then prove that

(i) $\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$

(ii) $4(x, y) = \| x + y \|^2 - \| x - y \|^2 + i \| x + iy \|^2 - i \| x - iy \|^2$.

Solution : (i) We have

$$\| x + y \|^2 = (x + y, x + y)$$
$$= (x, x + y) + (y, x + y)$$
$$= (x, x) + (x, y) + (y, x) + (y, y) \quad \ldots(1)$$

Also

$$\| x - y \|^2 = (x - y, x - y)$$
$$= (x, x - y) - (y, x - y)$$
$$= (x, x) - (x, y) - (y, x) + (y, y)$$
$$= \| x \|^2 - (x, y) - (y, x) + \| y \|^2 \quad \ldots(2)$$

Adding (1) and (2), we get

$$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$$

(ii) Subtracting (2) from (1), we get

$$\| x + y \|^2 - \| x - y \|^2 = 2(x, y) + 2(y, x) \quad \ldots(3)$$

Replacing by iy in (3), we get

$$\| x + iy \|^2 - \| x - iy \|^2 = 2(x, iy) + 2(iy, x)$$
$$= 2i(x, y) + 2i(y, x)$$
$$= -2i(x, y) + 2i(y, x) \quad \ldots(4)$$

Multiplying both sides of (4) by i, we get

$$i \| x + iy \|^2 - i \| x - iy \|^2 = -2i^2(x, y) + 2i^2(y, x)$$
$$= 2(x, y) - 2(y, x) \quad \ldots(5)$$

Adding (3) and (5), we get

$$\| x + y \|^2 - \| x - y \|^2 + i \| x + iy \|^2 - i \| x - iy \|^2 = 4(x, y)$$

Example 2. For the special Hilbert space l_2^n, by using Cauchy inequality prove Schwarz's inequality.

Solution : Let $x = (x_1, x_2, \ldots\ldots x_n)$, $y = (y_1, y_2, \ldots\ldots y_n$ be any two members of the Hilbert space L_2^n by Cauchy's inequality.

We have

$$\sum_{i=1}^{n} |x_i\, y_i| \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2} \left[\sum_{i=1}^{n} |y_i|^2\right]^{1/2}$$

For inner product on l_2^n, we have

$$(x, y) = \sum_{i=1}^{n} x_i\, y_i$$

$$\begin{aligned}
\therefore\ |(x \cdot y)| &= \sum_{i=1}^{n} |x_i\, y_i| \\
&= |x_1\, y_1 + x_2\, y_2 + \ldots\ldots + x_n\, y_n| \\
&= |x_1 y_1| + |x_2\, y_2| + \ldots\ldots + |x_n\, y_n| \\
&= |x_1\, y_1| + |x_2\, y_2| + \ldots\ldots + |x_n\, y_n| \\
&= \sum_{i=1}^{n} |x_i\, y_i| \\
&\leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2} \left[\sum_{i=1}^{n} |y_i|^2\right]^{1/2} && \text{[by Cauchy's inequality]} \\
&= \sqrt{(x \cdot y)}\,\sqrt{(y \cdot y)} && \text{[by (1)]} \\
&= \|x\|\,\|y\| && [\because (x, x) = \|x\|^2]
\end{aligned}$$

Which is the required Schwarz's inequality.

Example 3. If P and Q are the projections on closed linear subspaces M and N of Hilbert space H. Prove that the following statements are all equivalent to each other.

(i) $P \leq Q$,

(ii) $\|P_x\| \leq \|Q_x\|$ for every x,

(iii) $M \subset N$,

(iv) $PQ = P$,

(v) $QP = P$

Solution : In order to prove that the five given statements are all equivalent to one another. We shall prove that (i) $\to$ (ii) $\to$ (iii), (iii) $\to$ (v), (v) $\to$ (iv), (iv) $\to$ (i)

It should be recalled that if P is any projection on H, then

$$(Px, x) = \|Px\|^2\ \forall\, x \in H$$

(i) $\Rightarrow$ (ii), we have

$$P \leq Q$$

$$\begin{aligned}
&\Rightarrow && (Px, x) \leq (Qx \cdot x)\ \forall\, x \in H \\
&\Rightarrow && \|Px\|^2 \leq \|Qx\|^2\ \forall\, x \in H \\
&\Rightarrow && \|Px\| \leq \|Qx\|\ \forall\, x \in H
\end{aligned}$$

(iii) $\to$ (iii). It is given that

$$\|Px\| \leq \|Qx\|\ \forall\, x \in H$$

We are to prove that

$$M \subset N$$

Let $x \in M$ since M is the range of of P, therefore

$$x \in M \to Px = x \to \|Px\| = \|x\|$$

$$\begin{aligned}
&\Rightarrow && \|x\| \leq \|Qx\| && [\because \|Px\| \leq \|Qx\| \text{ by hypothesis}] \\
&\Rightarrow && \|x\| = \|Qx\| && [\because \|Qx\| \leq \|x\|\ \forall\, x \in H] \\
&\Rightarrow && Qx = x
\end{aligned}$$

$\Rightarrow x \in N$ *i.e.*, the range of Q.

Thus $x \in M \to x \in N$. Therefore $M \subset N$.

Example 4. Prove that if T is an arbitrary operator on a Hilbert space H, and if α and β are Scalars such that

$$|\alpha| = |\beta| \text{ then } \alpha T + \beta T^*$$

is normal.

Solution : We know that

$$\begin{aligned}
(\alpha T + \beta\, T^*)^* &= (\alpha T)^* + (\beta T^*)^* \\
&= \bar{\alpha} T^* + \bar{\beta} T^{**} = \bar{\alpha} T^* + \bar{\beta} T
\end{aligned}$$

Now $(\alpha T + \beta T^*)\,(\alpha T + \beta T^*)^*$

$$\begin{aligned}
&= (\alpha T + \beta T^*)\,(\alpha T^* + \beta T) \\
&= \alpha\alpha TT^*\ \alpha\beta T^2 + \beta\alpha\ T^{*2} + \beta\beta\ T^*\, T \\
&= |\alpha|^2\, TT^* + \alpha\beta\, T^2 + \beta\, \alpha T^{**} \\
&\qquad + |\beta|^2\, T^*\, T \quad \ldots(1)
\end{aligned}$$

Also $(\alpha T + \beta T)^*\,(\alpha T + \beta T^*)$

$$\begin{aligned}
&= (\alpha T^* + \beta T)\,(\alpha T + \beta T^*) \\
&= (\alpha T^* + \beta T)\,(\alpha T + \beta T^*) \\
&= \alpha\alpha T^*\, T + \alpha\beta T^{*2} + \beta\alpha T^2 + \beta\beta\, TT^* \\
&= |\alpha|^2\, T^*\, T + \alpha\beta\, T^{*2} + \beta\bar{\alpha}\, T^2 \\
&\qquad + |\beta|^2\, TT^* \quad \ldots(2)
\end{aligned}$$

Since $|\alpha| = |\beta|$ therefore, we see that the right hand sides of (1) and (2) are equal. Hence, the left hand sides (1) and (2) are also equal therefore, we have

$$\begin{aligned}
&(\alpha T + \beta\, T^*)\,(\alpha T + \beta\, T^*)^* \\
&\qquad = (\alpha T + \beta\, T^*)^*\,(\alpha T + \beta\, T^*)
\end{aligned}$$

Hence $\alpha T + \beta\, T^*$ is normal.

Example 5. An operator T on a Hilbert space H is normal iff $\| T^* x \| = \| Tx \|$ for every x.

Solution : We have T is normal

$$\Rightarrow \quad TT^* = T^* T$$
$$\Rightarrow \quad TT^* - T^* T = 0$$
$$\Rightarrow \quad [(TT^* - T^* T) x, x) = 0 \ \forall x$$
$$\Rightarrow \quad (TT^* x, x) = T^* Tx, x) \ \forall x$$
$$\Rightarrow \quad (T^* x, T^* x) = (Tx, T^{**} x) \ \forall x$$
$$\Rightarrow \quad (T^* x, T^* x) = (Tx, Tx) \ \forall x$$
$$\Rightarrow \quad \| T^* x \|^2 = \| Tx \|^2 \ \forall x$$
$$\Rightarrow \quad \| T^* x \| = \| Tx \| \ \forall x$$

Example 6. Prove that if T is an operator on H, then the following are equivalent :

(1) $T^* T = I$,

(2) $(Tf, Tg) = (f, g)$, for all $f, g \in H$,

(3) $\| Tf \| = \| f \|$, for all $f \in H$.

Solution : Here, (1) $\Rightarrow$ (2), if $T^* T = I$, then

$$(Tf, Tg) = (f, T^* Tg)$$
$$= (f, g), \text{ for all } f, g \in H$$

(2) $\Rightarrow$ (3) if (Tf, Tg)

$$= (f, g), \text{ for all } f, g \in H,$$

then in particular

$$(Tf, Tf) = (f, f)$$

i.e., $\| Tf \| = \| f \|$, for all $f \in H$

(3) $\Rightarrow$ (1), if $\| Tf \|$

$$= \| f \|, \text{ for all } f \in H, \text{ then}$$

i.e., $\| Tf \|^2 = \| f \|^2$

$$(Tf, Tf) = (f, f)$$

i.e., $(T^* Tf, f) - (f, f) = 0$

Which is equivalent to $(T^* T - I)(f, f) = 0$ for all $f \in H$. Therefore,

$$T^* T = I$$

Example 7. Prove that in a Hilbert space the inner product is jointly continuous, *i.e.*, $x_n \to x$, $y_n \to y \Rightarrow (x_n, y_n) \to (x, y)$.

Solution : We have

$| (x_n, y_n) - (x, y) |$

$$= | (x_n, y_n) - (x_n, y) + (x_n, y) - (x, y) |$$
$$= | (x_n, y_n - y) + (x_n - x, y) |$$

[by linearity property of inner product]

$$\le | (x_n, y_n - y) | + | (x_n - x, y) |$$

$[\because | \alpha + \beta | \le | \alpha | + | \beta |]$

$$\le \| x_n \| \| y_n - y \| + | (x_n - x \| \| y \|$$

[by Schwarz inequality]

Now $x_n \to x$ and $y_n \to y$ for $n \to \infty$

Therefore, $\| y_n - y \| \to 0$ and $\| x_n - x \| \to 0$, for $n \to \infty$

Therefore, $| (x_n, y_n) - (x, y) | \to 0$ for $n \to \infty$

Hence, $(x_n, y_n) \to (x, y)$

Example 8. Prove that an operator T on a Hilbert space H is unitary iff it is an isometric isomorphism of H onto itself.

Solution : Suppose T is a unitary operator on H, then T is invertible and, therefore, T is onto also $T^* T = I$. Therefore we have

$$\| Tx \| = \| x \| \ \forall \ x \in H$$

Thus T preserves norms and so T is an isometric isomorphism of H onto itself.

Conversely suppose that T is an isometric isomorphism of H onto itself. Then T is one-one and onto. Therefore T – 1 exists. Also T is an isometric isomorphism

$$\Rightarrow \quad \| Tx \| = \| x \| \ \forall x, \Rightarrow T^* T = I$$
$$\Rightarrow \quad (T^* T) T^{-1} = IT^{-1}$$
$$\Rightarrow \quad T^* (TT^{-1}) = T^{-1}$$
$$\Rightarrow \quad T^* I = T^{-1} \Rightarrow T^* = T^{-1}$$

$\therefore$ $TT^* = I = T^* T$ and so T is unitary.

Example 9. Prove that if $P_1, P_2, \ldots\ldots P_n$ are projections on closed linear sub spaces $M_1, M_2, \ldots\ldots, M_n$ of Hilbert space H, then $P = P_1 + P_2 + \ldots\ldots + P_n$ is a projection if and only if the P_i S are pairwise orthogonal, and in this case P is the projection on $M = M_1 + M_2 + \ldots\ldots + M_n$.

Solution : Suppose that $P_1, P_2, P_3, \ldots\ldots, P_n$ are the perpendicular projections and P_i S are pairwise orthogonal. Then we have (a) P is self-adjoint since each P_i is a projection.

$P_j^* = P_j = P_j^2$, for each $j = 1, 2, \ldots\ldots, n$

Therefore,

$$P^* = (P_1 + P_2 + \ldots\ldots + P_n)^*$$
$$= P_1^* + P_2^* + \ldots\ldots + P_n^*$$
$$= P_1 + P_2 + \ldots\ldots + P_n = P$$

(b) P is idempotent. If P_i's are pairwise orthogonal, then

$$P^2 = (P_1 + P_2 + \ldots\ldots + P_n)^2$$
$$= \sum_{j=1}^{n} P_i^2 + 2 \sum_{j=1} P_i P_f = \sum_{i=1}^{n} P_i = P$$

Thus P is idempotent.

(c) If P is idempotent, then P_i's are pairwise orthogonal, if $f \in$ range of P_i, then $P_i (f) = f$ and

$$\| f \|^2 = \| P_i f \|^2 \le \sum_{i=1}^{n} \| P_j f \|^2$$

$$= \sum_{j=1}^{n} (P_j f, P_j f)$$
$$= \sum_{j=1}^{n} (P_j, f, P_j^* f)$$
$$= \sum_{j=1}^{n} (P_j f, f) = (Pf, f) = (P^2 f, f)$$
$$= (Pf, Pf) = \| Pf \|^2 \le \| f \|^2$$

Hence, $\sum_{j=1}^{n} \| P_j f \|^2 = \| P_i f \|^2$. Therefore, $\| P_j f \| = 0$, for $j \neq i$, and hence $P_j f = 0$, for $j \neq i$, thus $f \in$ null space of P_j, for $j \neq i$ and hence range of $P_i \subseteq$ null space of P_j for $j \neq i$.

i.e., $M_i \subseteq M_j^{\perp}$, for $j \neq i$.

Consequently, $M_j + M_j$, for $j \neq i$ and so P_i's are pairwise orthogonal *i.e.*, $P_i P_j = 0$, for $i \neq j$.

(d) P is a projection on M. Since $\| P f \| = \| f \|$, for all $f \in M_i$, each M_i is contained in the range of P.

Therefore, $M = \sum_{j=1}^{n} M_j$ is also contained in the range of P. Further if $f \in$ range of P, then

$$f = Pf = (P_1 + P_2 + \ldots\ldots + P_n) f$$
$$= P_1 f + P_2 f + \ldots\ldots + P_n f$$

i.e., $f \in M = \sum_{i=1}^{n} M_i$, since $P_i f \in M_i$

Example 10. Let f be a bounded linear functional on the Hilbert space H which is separable, prove that there is a unique $y \in H$ such that $f(x) = (x, y)$ for all x, moreover, $\| f \| = \| y \|$.

Solution : Let $< \phi_v >$ be a complete orthonormal system for H, and set $b_v = f(\phi_v)$. Then for each n, we have

$$\sum_{v=1}^{n} b_v^2 = f\left(\sum_{v=1}^{n} b_v \phi_v \right)$$
$$= \| f \| \cdot \| \sum_{v=1}^{n} b_v \phi_v \|$$
$$= \| f \| \left[\sum_{v=1}^{n} b_v^2 \right]^{1/2}$$

Thus $\sum_{v=1}^{n} b_v^2 \le \| f \|^2$ and so $\sum_{v=1}^{n} b_v^2 \le \| f \|^2 < \infty$.

Hence, there is an element $y = \sum_{v=1}^{n} b_v \phi_v$. we have $\| y \| \le \| f \|$. Let x be any element of H. Then $\sum_{v=1}^{n} a_v \phi_v \to x$ and so

$$f(x) = \lim f\left(\sum_{v=1}^{n} a_v \phi_v \right)$$
$$= \lim \sum_{v=1}^{n} a_v b_v$$
$$= \sum_{v=1}^{\infty} a_v b_v = (x, y)$$

By the Schwarz inequality $\| f \| \le \| y \|$.

Example 11. Prove that the set of all normal operators on H is closed under scalar multiplication.

Solution : If T is normal operator and a is any scalar, then

$$(aT)(aT)^* = (aT)(\bar{a}T^*) = a\bar{a}(TT^*)$$
$$= a\bar{a}(T^*T) = (\bar{a}T^*)(aT)$$
$$= (aT)^*(aT)$$

Therefore aT is normal.

Example 12. Let H be a Hilbert space and let $S = (l_1, l_2, \ldots\ldots, l_n, \ldots\ldots\}$ be a countably infinite orthonormal set in H. Prove that a series of the form $\sum_{n=1}^{\infty} \alpha_n l_n$ i s convergent if and only if $\sum_{n=1}^{\infty} | \alpha_n |^2 < \infty$. Further if $\sum_{n=1}^{\infty} \alpha_n l_n$ converges to x, then $\alpha_n = (x, l_n)$.

Solution : Consider the partial sum

$$S_n = \sum_{j=1}^{n} \alpha_i l_i$$

For $m > n$, we have

$$\| S_m - S_n \|^2 = \| \sum_{j=n+1}^{m} \alpha_i l_i \|^2$$
$$= \| \sum_{i=n+1}^{m} | \alpha_i |^2$$

If the series $\sum_{n=1}^{\infty} \alpha_n l_n$ is convergent, then the sequence $< S_n >$ of partial sums is convergent and every convergent sequence is a Cauchy sequence.

Therefore as $m, n \to \infty$, we have

$$\| S_m - S_n \|^2 \to 0$$
$$\Rightarrow \quad \sum_{j=n+1}^{\infty} | \alpha_i |^2 \to 0$$

The series $\sum_{n=1}^{\infty} \Sigma | \alpha_n |^2$ is convergent *i.e.*, conversely suppose that the series $\sum_{n=1}^{\infty} | \alpha_n |^2$ is convergent, *i.e.*,

$$\sum_{n=1}^{\infty} | \alpha_n |^2 < \infty$$

Then as $m, n \to \infty$, we have

$$\sum_{i=n+1}^{m} |\alpha_i|^2 \to 0$$

$$\| S_m - S_n \|^2 \to 0$$

Therefore, the sequence $< S_n >$ is a Cauchy sequence in H. But H is complete. So $< S_n >$ is a convergent sequence in H and thus the series $\sum_{n=1}^{\infty} \alpha_n l_n$ is convergent.

Now suppose that the series $\sum_{n=1}^{\infty} \alpha_n L \pm 1 l_n$ is convergent and let $x = \sum_{n=1}^{\infty} \alpha_n l_n$. If $S_n = \sum_{i=1}^{\infty} \alpha_i l_i$ then for $n > j$, we have

$$(S_n, l_j) = \left(\sum_{n=1}^{\infty} \alpha_i, l_i, l_j \right) = \alpha_i$$

Since this relationship is true for any $n > j$ therefore, it must also be true in the limit. So we have $\alpha_j = \lim (S_n, l_j) = (\lim S_n, l_j)$, by continuity of inner product $= (x, l_j)$. Thus $\alpha_j = (x, l_j)$ for each j.

Example 13. Prove that B (N, N′) is a normal linear space.

Solution : Define a function

$$\| \ \| : B(N, N') \to R \text{ by}$$

$$\| T \| = \operatorname{Sup} \| T(f) \| \, | \, \| f \| \le 1,$$

for all B (N, N′), since $\| \ \|$ satisfies.

(i) $\| T \| = \sup \| T(f) \|$,

which is non-negative

$\| f | \le \|$

(ii) $\| T \| = 0$

$\Rightarrow$ $\sup. \| T \| (f) = 0$

$\Rightarrow$ $\sup_{j \ne 0} \left[\frac{\| T(f) \|}{\| f \|} \right] = 0$

$\Rightarrow$ $\frac{\| T(f) \|}{\| f \|} = 0$, for all $f \in N$ for which $f \ne 0$

$\Rightarrow \| T(f) \| = 0$, where $f \ne 0$ and $f \in N$,

$\Rightarrow T(f) = 0$, even when $f = 0$,

$\Rightarrow T = 0$ (Null operator),

(iii) $\sup_{\| f \| \le 1} \| (at)(f) \| = \sup_{\| f \| \le 1} \| a T(f) \|$

$= | a | \sup_{\| f \| \le 1} \| T(f) \|$

$= | a | \| T \|,$

for all scalars a.

(iv) $\| T_1 + T_2 \| = \sup_{\| f \| \le 1} \| (T_1 + T_2)(f) \|$

$\le \sup_{\| f \| \le 1} \| T_1(f) \| + \sup_{\| f \| \le 1} \| T_2(f) \|$

$\le \| T_1 \| + \| T_2 \|$

Thus B (N, N′) is a normed linear space.

Example 14. Prove that if T_1 and T_2 are self-adjoint operators on H, then their product T_1T_2 is self-adjoint if and only if.

$$T_1T_2 = T_2T_1$$

Solution : Let T_1 and T_2 be self-adjoint operators on H,

then, $T_1^* = T_1$ and $T_2^* = T_2$

Suppose that

$$T_1T_2 = T_2T_1$$

Then $(T_1T_2)^*$

$= T_2^* T_1^*$

$= T_2T_1$

$= T_1T_2$

Thus T_1T_2 is self-adjoint

Let T_1T_2 be self-adjoint

Then $T_1T_2 = (T_1T_2)^*$

$= T_2^*T_1^*$

$= T_2T_1$.

Example 15. State and prove that Cauchy-Schwarz inequality.

Solution : Cauchy-Schwarz inequality : For any two vectors f and g of vector space X.

$|f, g| \le \| f \| \cdot \| g \|$

Proof : When $g = 0$,

In this laye both sides are zero and euqality holds.,

when $g \ne 0$,

since $g \ne 0 \Rightarrow \| g | > 0$,

we can write $0 \le \| \, \| g \|^2 f - (f, g) g \, \|^2$

$= (\| g \|^2 f - (f, g) g \, \| g \|^2 \; f - (f, g) g)$

$= \| g \|^4 (f, f) - (f, g) \| g \|^2 (g, f) - \| g \|^2 (\| f \|^2 \| g \|^2 - | (f, g) |^2 - | (f, g) |^2 + | (f, g) |^2)$

$= \| g \|^2 (\| f \|^2 \| g \|^2 - | (f, g) |^2)$

Dividing by $\| g \|^2$ so, we have

$$0 \leq (\| f \|^2 \| g \|^2 - | (f, g) |^2)$$

$$\Rightarrow \quad | (f, g) \leq \| f \| \| g \|.$$

Example 16. Prove that if x and y are any two vectors in a Hibert space then :

(i) $\| x + y \|^2 - \| x - y \|^2 = 4 \text{ Re } (x, y)$

(ii) $(x, y) = \text{Re } (x, y) + i \text{ Re } (x, iy)$

Solution : If $\alpha = \beta + i\gamma$ is a complex number where β, γ are real numbers, then we write

$\beta = \text{Re } \alpha$ and $\gamma = \text{I}_m \alpha$.

(i) We have

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2 + (x, y) + (y, x) \quad \ldots(1)$$

and $\| x - y \|^2$

$$= \| x \|^2 + \| y \|^2 - (x, y) - (y, x) \quad \ldots(2)$$

subtracting (2) from (1), we get

$\| x + y \|^2 - \| x - y \|^2$

$$= 2(x, y) + 2(y, x)$$

$$= 2[(x, y) + \overline{(x, y)}]$$

$$= 2[2 \text{ Re } (x, y)]$$

$[\because \alpha + \overline{\alpha} = 2 \text{ Re } \alpha$, if α is a complex number]

$$= 4 \text{ Re } (x, y)$$

(ii) We have $(x, y) = \text{Re } (x, y) + i \text{ I}_m (x, y)$

If $\alpha = \beta = iy$ is a complex number, then

$$y = \text{I}_m \alpha$$

$$= \text{R}_e \{ -i (\beta + iy) \}$$

$$= \text{R}_e (-i\alpha)$$

$\therefore \text{I}_m (x, y)$

$$= \text{R}_e \{ -i (x, y) \}$$

$$= \text{R}_e (x, iy)$$

$[\because (x, iy) = -i (x, y)]$

$$\therefore \quad (x, y) = \text{R}_e (x, y) + i\text{R}_e (x, iy)$$

Example 17. Prove that an operator T on H is normal if and only if $\| T^* f \| = \| T_f \|$ for all $f \in$ H.

Solution : We see that

$$\| T^* f \| = \| T f \|$$

$$\Leftrightarrow \quad \| T^* f \|^2 = \| T f \|^2$$

$$\Leftrightarrow \quad (T^* f, T^* f) = (Tf, Tf)$$

$$\Leftrightarrow \quad (TT^* f, f) = (T^* Tf, f)$$

$$\Leftrightarrow ((TT^* - T^*T) f, f) = 0$$

for all $f \in$ H,

$$\Rightarrow (TT^* - T^*T) f = 0$$

or $\quad TT^* = T^* T,$

Thus T is normal.

Example 18. Prove that if T_1 and T_2 are normal operators on H with the property that either commutes with the adjoint of the other, than $T_1 + T_2$ and $T_1 T_2$ are normal.

Solution : (a) $T_1 + T_2$ is normal

Since T_1 and T_2 commutes with the adjoint of others

$\therefore T_1T_2^* = T_2^*T_1$ and $T_1^*T_2 = T_2T_1^*$.

Therefore, $\quad (T_1 + T_2) (T_1 + T_2)^*$

$$= (T_1 + T_2) (T_1^* + T_2^*)$$

$$= T_1T_1^* + T_1T_2^* + T_2T_1^* + T_2T_2^*$$

$$= T_1^*T_1 + T_2^*T_1 + T_1^*T_2 + T_2^*T_2$$

$$= (T_1^* + T_2^*) (T_1 + T_2)$$

$$= (T_1 + T_2)^* (T_1 + T_2)$$

Thus $T_1 + T_2$ is normal.

(b) T_1T_2 is normal

Since T_1 and T_2 commutes with the adjoint of others.

$\therefore (T_1T_2) (T_1T_2)^*$

$$= (T_1T_2) (T_2^*T_1^*)$$

$$= T_1 (T_2T_2^*) T_1^*$$

$$= (T_1T_2^*) (T_2T_1^*)$$

$$= (T_2^*T_1) (T_1^*T_2)$$

$$= T_2^*(T_2T_1^*)T_2$$

$$= T_2^*(T_1^*T_1)T_2$$

$$= (T_2^*T_1^*) (T_1T_2)$$

$$= (T_1T_2)^* (T_1T_2)$$

Thus T_1T_2 is normal.

Example 19. Prove that if the real and imaginary parts commute, then T is normal.

Solution : Let T_1 and T_2 be the real and imaginary parts of T, then

$$T = T_1 + iT_2$$

and $\quad T^* = T_1 - iT_2$

Further, assume that

$$T_1T_2 = T_2T_1,$$

then $\quad TT^* = (T_1 + iT_2) (T_1 - iT_2)$

$$= T_1^2 + T_2^2 + i (T_2T_1 - T_1T_2)$$

and $\quad T^*T = (T_1 - iT_2)(T_1 + iT_2)$

$$= T_1^2 + T_2^2 + i(T_1T_2 - T_2T_1)$$

Since T_1 and T_2 commute, we obtain

$TT^* = T^*T$, *i.e.*, T is normal conversely, suppose that T is normal, then we have.

$$TT^* = T^*T$$

$$\Rightarrow T_2T_1 - T_1T_2 = T_1T_2 - T_2T_1,$$

$$\Rightarrow \quad 2T_1T_2 = 2T_2T_1$$

$$\Rightarrow \quad T_1T_2 = T_2T_1$$

i.e., T_1 and T_2 commute.

Example 20. If T is an arbitrary operator on a Hilbert space H, then T is a zero operator iff $(T_x, y) = 0$ for all x and y.

Solution : Suppose $T = 0$ (*i.e.*, zero operator), than for all x and y

We have $\quad (Tx, y) = (ox, y)$

$$= (o, y)$$

$$= 0$$

Conversely $\quad (Tx, y) = 0 \ \forall x, y \in H$

$$\Rightarrow \quad (T_x, T_x) = 0 \ \forall_x \in H$$

(taking $y = Tx$)

$$\Rightarrow \quad Tx = 0 \ \forall_x \in H$$

$$\Rightarrow \quad T = 0, \text{ i.e., zero operator.}$$

Example 21. If A_1 and A_2 are self-adjoint operators on Hilbert space H, then their product A_1A_2 is self adjoint iff $A_1 A_2 = A_2 A_1$.

Solution : Let A_1 and A_2 be two self-adjoint operators on a Hilbert space H. Then $A_1^* = A_1$ and $A_2^* = A_2$.

Suppose A_1 and A_2 commute, *i.e.*, $A_1A_2 = A_2A_1$. Then to prove that A_1A_2 is self-adjoint, we have.

$$(A_1A_2)^* = A_2^* A_1^*$$

$$= A_2A_1$$

$$= A_1A_2$$

Threefore A_1A_2 is self-adjoint.

Conversely, suppose that A_1A_2 is self-adjoint then,

$$(A_1A_2)^* = A_1A_2$$

$$\Rightarrow \quad A_2^*A_1^* = A_1A_2$$

$$\Rightarrow \quad A_2A_1 = A_1A_2$$

$\Rightarrow$ A, and A_2 commute.

Example 22. Let N be a normal linear space and let $x, y \in N$. Then

prove that $\quad | \, \|x\| - \|y\| \, | \leq \|x - y\|$

Solution : Since N is a normal linear space

$$\therefore \quad \|x\| = \|(x - y) + y\|$$

$$\leq \|x - y\| + \|y\|$$

or $\quad \|x\| - \|y\| \leq \|x - y\| \quad \ldots(1)$

Similarly $\quad \|y\| - \|x\| \leq \|y - x\| \quad \ldots(2)$

But $\quad \|y - x\| = \|(-1)(x - y)\|$

$$= |-1| \, |x - y\|$$

$$= \|x - y\| \quad \ldots(3)$$

From (2) and (3), we have

$$\|y\| - \|x\| \leq \|x - y\| \quad \ldots(4)$$

Then from (1) and (4), we obtain

$$| \, \|x\| - \|y\| \, | \leq \|x - y\|$$

Example 23. Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H. Then prove that the following conditions are equivalent to one another.

(1) $\{e_i\}$ is complete.

(2) $f \perp \{e_i\} \Rightarrow f = 0$

(3) if f is an arbitrary vector in H, then $f = \in (f, e) \, e_i$,

(4) if f is an arbitrary vector in H. then $\|f\|^2$. $\Sigma \, |(f_i e_i|^2$

Solution : (1) $\Rightarrow$ (2). If (2) is not true, then we can obtain a vector $f \neq 0$ such that $f \perp \{e_i\}$ taking $e = \dfrac{f}{\|f\|}$, we observe that $\{e, e_i\}$ is an orthonormal set which properly contains $\{e_i\}$. This controdicts the completeness of $\{e_i\}$. Thus (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). As $f - \Sigma \, (f, e_i) \, e_i$ is orthogonal to $\{e_i\}$, so it follows from (2) that $f - \Sigma \, (f, e_i) \, e_i = 0$ or equivalently, that

$$f = \Sigma \, (f, e_i) \, e_i$$

(3) $\Rightarrow$ (4) As the inner product is jointly continuous, we have

$$\|f\|^2 = (f, f) = (\Sigma \, (f, e_i) \, e_i \, \Sigma \, (f, e_i) \, e_i)$$

$$= \Sigma \, (f, e_i) \left(\overline{fe_i}\right)$$

$$= \Sigma \, |(f, e_i)|^2$$

(4) $\Rightarrow$ (1). If $\{e_i\}$ is not complete, then it is a proper subset of an orthonormal set $\{e, e_i\}$. Since e is orthogonal to all the $e_i s$ (4) gives $\|e\|^2$

$= \Sigma \mid (e, e_i) \mid^2 = 0$. This contradicts the fact that e in a unit vector.

Example 24. Prove that a bounded linear operator is uniformly continuous. If a linear operator is continuous at one point, it is bounded.

Solution : Suppose A is bounded linear operator. Then

$\| Ax_1 - Ax_2 \| \leq \| A \| \cdot \| x_1 - x_2 \| < \varepsilon$. for all x_1 and x_2 in X with $\| x_1 - x_2 \| < \varepsilon \| A \|$. Thus is uniformly continous.

Suppose now that A is a linear operator that is contiuous at x_0. Then there is a $\delta \to 0$ such that $\| Ax - Ax_0 \| < 1$ for all x such that $\| x - x_0 \| < \delta$. For any z in X with $Z \neq \theta$, set $\omega = \eta z' \| z \|$. Where $0 < \eta < \delta$. Then

$$\frac{\eta}{\| z \|} Az = A\omega$$
$$= A(\omega + x_0) - A(x_0)$$

and
$$\frac{\eta}{\| z \|} \| Az \| = \| A(\omega + x_0) - A(x_0) \| < 1$$

Since

$$\| w + x_0 - x_0 \| = \| \omega \|$$
$$= \eta < \delta$$

Consequently,

$$\| Az \| \leq \eta^{-1} \| z \|,$$

and A is bounded.

Example 25. Prove that if T is normal operator on H, then

$$\| T^2 \| = \| T \|^2.$$

Solution : Since T is normal, then

$$\| T^* f \| = \| Tf \|,$$

for every $f \in$ H.

Therefore

$$\| T^* \| = \text{Sup} \{\| T^2 f \| : \| f \| \leq 1\}$$
$$= \text{Sup} \{\| T(Tf) \| : \| f \| \leq 1\}$$
$$= \text{Sup} \{\| T^*(Tf) \| : \| f \| \leq 1\}$$
$$= \text{Sup} \{\| T^* T f \| : \| f \| \leq 1\}$$
$$= \| T^* T \|$$
$$= \| T \|^2.$$

OBJECTIVE TYPE QUESTIONS

1. Let T be a bounded linear operator on Hilbert space H, then T is projection, if—
   (A) T is unitary if $T^* T = 1 = TT^*$, I being an identity operator on H.
   (B) $T^2 = T$
   (C) $T^* T = T^* T^*$
   (D) None of these

2. Let T be a bounded linear operator on Hilbert space H, then T is unitary if—
   (A) T is unitary if $T^* T = I = TT^*$, 1 being an identity operator on H. (A)
   (B) $T^* = T$
   (C) $T^* T = T^* T^*$
   (D) None of these

3. Let T be a bounded linear operator on Hilbert space H, then T is self adjoint if—
   (A) $T^* = T$
   (B) $T^* T = T^* T^*$
   (C) $T^* T = T T^*$
   (D) None of these

4. Let T be a bounded linear operator on Hilbert space H, then T is normal if—
   (A) $T^* T = TT$
   (B) $T^* T = T^* T^*$
   (C) $T^* T = T T^*$
   (D) None of these

5. For inner product—
   (A) $(x, \alpha y) = \bar{\alpha} (x, y)$
   (B) $(x, \alpha y) = (x, y)$
   (C) $(x, \alpha y) = \alpha (x, y)$
   (D) None of these

6. For adjoint operator—
   (A) $\| T^* T \| = \| T \|$
   (B) $\| T^* T \| = \| T^* \|$
   (C) $\| T^* T \| = \| T \|^2$
   (D) None of these

7. For adjoint operator—
   (A) $T^{**} = T$
   (B) $T^{**} = T^*$
   (C) $T^{**} = T + T$
   (D) $T^{**} = T - T$

8. For adjoint operator—
   (A) $(ST)^* = T^* S^*$

(B) $(ST)^* = TS$
(C) $(ST)^* = T^* + S^*$
(D) $(ST)^* = T^* - S^*$

9. For adjoint opearator—
(A) $(\alpha T)^* = \bar{\alpha} T$
(B) $(\alpha T)^* = \bar{\alpha} T^*$
(C) $(\alpha T)^* = \bar{\alpha}$
(D) None of these

10. For adjoint operator—
(A) $(T + S)^* = T^* + S^*$
(B) $(T + S)^* = T + S$
(C) $(T + S)^* = TS$
(D) None of these

11. Let T be a bounded linear operator of Hilbert space H, then there exist an unique operator on H called adjoint of T, T* such that—
(A) $(Tx, y) = (x, T^*y)\, x, y \in H$
(B) $\| T^* \| = \| T \|$
(C) Both (A) and (B)
(D) None of these

12. If x has a reproducing Kernel, then—
(A) $| x(t) | = | (x, k(;t) | \leq \| k(;t) \| \| x \|, x \in X, t \in S$
(B) X is a proper functional space
(C) Both (A) and (B) above
(D) None of these

13. Let x be a Hilbert space and $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set in X, then S is a orthonormal basis (orthonormal closed set) in X if—
(A) The fourier series representation $x = \sum_{\alpha \in A} (x, x_\alpha) x_\alpha$ is valid for each $x \in X$.
(B) The fourier series representation $x = \sum_{\alpha \in A} (x, x_\alpha) x_\alpha$ is not valid for each $x \in X$.
(C) The fourier series representation $x = \sum_{\alpha \in A} (x, x_\alpha) x_\alpha$ is valid for some $x \in X$
(D) None of these

14. A set S is complete in X—
(A) If no non-zero $x \in X$ is orthogonal to S
(B) If non-zero $x \in X$ is orothogonal to S
(C) If no non-zero $x \in X$ is orthogonal to S
(D) None of these

15. Orthonormal set is—
(A) An orthonormal set S such that $(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases}, x, y \in S$
(B) An orthonormal set S such that $(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}, x, y \in S$
(C) An orthonormal set S such that $(x, y) = 0$, $x, y \in S$
(D) None of these

16. If x' is a bounded linear functional on a Hilbert space X, then—
(A) There is unique $y_0 \in X$ such that $x'(x) = (y_0), x \in X$
(B) There is unique $y_0 \in x$ such that $x'(x) = (y_0, x), x \in X$
(C) There is a unique $y_0 \in X$ such that $x'(x) = (x, y_0), x \in X$
(D) None of these

17. If k is a closed convex set in X and $x_0 \in X \sim k$, then there is a unique $y_0 \in k$, such that—
(A) $\| x_0 - y_0 \| = \sup \{\| x_0 - y \| : y \in k\}$
(B) $\| x_0 - y_0 \| = \inf \{\| x_0 - y \| : y \in k\}$
(C) $\| x_0 - y_0 \| = \{\| x_0 - y \| : y \in k\}$
(D) None of these

18. A set S of elements of an inner product space. X is an orthogonal set—
(A) if $x \perp y$ whenever $x, y \in S$ and $x \neq y$
(B) If $x \perp y$ whenever $x, y \in S$ and $x = y$
(C) If $x \perp y$ whenever $x, y \in S$
(D) None of these

19. If X is a real vector space, then—
(A) Inner product $(x, y) = (y, x)$
(B) Inner product $(x, y) \neq (y, x)$
(C) Inner product $(x, y) > (y, x)$
(D) None of these

20. For inner product—
(A) $(x, x) = 0$ iff $x = 0$
(B) $(x, x) > 0$ iff $x = 0$
(C) $(x, x) < 0$ iff $x = 0$
(D) None of these

21. For inner product—
(A) $(x, x) \geq 0$ (B) $(x, x) < 0$
(C) $(x, x) = 0$ (D) None of these

22. For inner product—
(A) $(x, y) = \overline{(y, x)}$
(B) $(x, y) = (y, y)$
(C) $(x, y) = (y, x)$
(D) None of these

23. If M, N, T are bounded linear operators, M, N are normal operators and MT = NT, then—
(A) $M^* T = T N^*$ (B) $M^* T = T^* N^*$
(C) $MT = T^* N^*$ (D) $M^* T^* = T^* N^*$

24. If p is a projection on H, then (i) p is self adjoint (ii) p is normal (iii) $(px, x) = \| px \|^2$ for every $x \in$ H—
(A) (i) is true
(B) (ii) is true
(C) (iii) is true
(D) (i), (ii) and (iii) are true

25. If U is an unitary operator on H, then—
(A) $(Ux, Uy) = (x, y)$
(B) $\| Ux \| = \| x \|$ for every $x \in$ H
(C) Both A and B are true
(D) None of these

26. If T is a bounded linear operator on Hilbert space H, then—
(A) T is normal iff $\| Tx \| = \| T^* x \|$ for every $x \in$ H
(B) T is normal iff $\| Tx \| > \| T^*x \|$ for every $x \in$ H
(C) T is normal iff $\| Tx \| < \| T^*x \|$ for every $x \in$ H
(D) None of these

27. If T is a bounded linear operator on Hilbert space H, then—
(A) T is normal iff $\| Tx \| = \| T^* x \|$ for every $x \in$ H
(B) If T is normal and $Tx = \alpha x$ for $x \in$ H, $\alpha \in$ C, then $T^*x = \bar{\alpha} x$
(C) If T is normal and if α, β are distinct eigen values of T, then the corresponding eigen spaces are orthogonal to each other
(D) All the above

28. If $\{S_n^{(m)}\}$ is a sequence of sequences such that $\sum_{n=1}^{\infty} | S_n^{(m)} | = \infty$. $m = 1, 2, \ldots\ldots$, then—
(A) There is a sequence $\{t_n\}$ diverge, such that all the series $\sum_{n=1}^{\infty} S_n^{(m)} t_n, m = 1, 2, \ldots\ldots$ converge
(B) There is a sequence $\{t_n\}$ converging to zero, such that all the series $\sum_{n=1}^{\infty} S_n^{(m)} t_n, m = 1, 2, \ldots\ldots$ converge
(C) Three is a sequence $\{t_n\}$ converging to zero, such that all the series $\sum_{n=1}^{\infty} S_n^{(m)} t_n, m = 1, 2, \ldots\ldots$ diverge
(D) There is a sequence $\{t_n\}$ diverging, such that all the series $\sum_{n=1}^{\infty} S_n^{(m)} t_n, m = 1, 2, \ldots\ldots$ diverge

29. A is called almost regular if, for every convergent sequence $x = (x_1, x_2 \ldots\ldots)$—
(A) There is an N such that for every $m >$ N, $\sum_{n=1}^{\infty} a_{mn} x_n$ diverges
(B) There is an N for every $m >$ N, $\sum_{n=1}^{\infty} a_{mn} x_n$ converges and $\lim_{m \to \infty} \sum_{n=1}^{\infty} a_{mn} x_n = \lim_{n \to \infty} x_n$
(C) There is an N such that for every $m <$ N, $\sum_{n=1}^{\infty} a_{mn} x_n$ converges
(D) None of these

30. If dom (T) is closed in Banach space X and linear transformation T is bounded, then—
(A) T is closed
(B) T is open
(C) T is constant
(D) T is null

31. Given the equation $x - \lambda Tx = 0$. The number $\lambda_0 \in$ R is called characteristic number of this equation if—
(A) $x - \lambda_0 Tx = 0$ has a non-zero solution
(B) $x - \lambda_0 Tx = 0$ has a no solution
(C) $x - \lambda_0 Tx = 0$ has a finite solution
(D) None of these

32. Projection theorem is—
   (A) If X and Y are Banach Spaces and T is a linear trasnformation from X and Y, then dom (T) is closed and graph G(T) closed $\Rightarrow$ T is bounded.
   (B) If X and Y are Banach spaces and T is bounded linear transformation which maps X on to all of Y, then T is an open mapping (*i.e.,* the image of open set in X under T, is open set in Y).
   (C) If M is a closed subspace of a Hilbert space X, then every $x \in$ X, has a unique decomposition $x = y + z$, $y \in$ M, Z $\in$ $M^{\perp}$, where $M^{\perp} = \{z \in X : z \perp y \text{ for all } y \in M\}$.
   (D) None of these
33. Open mapping theorem is—
   (A) If X and Y are Banach spaces and T is a linear transformation from X and Y, then dom (T) is closed and graph G(T) closed $\Rightarrow$ T is bounded
   (B) If X and Y are Banach spaces and T is bounded linear transformation which maps X on to all of Y, then T is an open mapping (*i.e.,* the image of open set in X under T, is open set in Y)
   (C) If M is a closed subspace of a Hilbert space X, then every $x \in$ X has a unique decomposition $x = y + z$, $y \in$ M, $z \in M^{\perp}$ where $M^{\perp} = \{z \in X : z \perp y \text{ for all } Y \in M\}$
   (D) None of these
34. Closed graph theorem is—
   (A) If X and Y are Banach spaces and T is a linear transformation from X and Y, then dom (T) is closed and graph G(T) closed $\Rightarrow$ T is bounded
   (B) If X and Y are Banach spaces and T is bounded linear transformation which maps X on to all of Y, then T is an open mapping (*i.e.,* the image of open set in X under T, is open set in Y)
   (C) If M is a closed subspace of a Hilbert space X, then every $x \in$ X has a unique decomposition $x = y + z$, $y \in$ M, $z \in M^{\perp}$, where $M^{\perp} = \{z \in X : z \perp Y \text{ for all } y \in M\}$
   (D) None of these
35. T is closed linear transformation if graph G (T) is a—
   (A) Closed vector subspace of Banach spaces $X \cdot Y$
   (B) Open vector subspace of Banach spaces $X \cdot Y$
   (C) Null vector subspace of Banach spaces $X \cdot Y$
   (D) None of these
36. A graph G(T) of a mapping T : X $\rightarrow$ Y is the set of points—
   (A) $(x, Tx) \in X \cdot Y$, with $Tx \in$ dom T
   (B) $(x, Tx) \in X \cdot Y$ with $x \in$ dom T
   (C) $(x, Tx) \in X \cdot X$ with $x \in$ dom T
   (D) None of these
37. A linear transformation T from a Banach space X into a Banach space Y with dom (T) $\subset$ X is closed if—
   (A) $x_n \in$ dom (T)
   (B) $\lim\limits_{n} x_n = x_n$
   (C) $\lim\limits_{n} Tx_n = y \Rightarrow x \in$ dom (T) and $y = Tx$
   (D) All are true
38. If X is a Banach space, Y is a normed vector space and $\{T_n\}$ a sequence of continuous linear transformation on X into Y such that for every $x \in$ X, the sequence $\{\| T_n (x) \|\}$ is bounded, then—
   (A) The sequence $\{\| T_n \|\}$ of norm is bounded
   (B) The sequence $\{\| T_n \|$ of norm is unbounded
   (C) The sequence $\{\| T_n \|\}$ of norm is open
   (D) None of these
39. Let X be a Banach space, $\{f_n\}$ is a sequence of continuous linear functional on X and for every $x \in$ X, the sequence $\{| f_n (x)|\}$ is bounded, then—
   (A) The sequence of $\{\| f_n \|\}$ of norms is bounded
   (B) The sequence of $\{\| f_n \|\}$ of norms is unbounded
   (C) The sequence of $\{\| f_n \|\}$ of norms is closed
   (D) None of these

40. For every normed vector space X, these is a set A such that X is isomorphic with a subspace of the Banach space of bounded functions f on A with—
   (A) $\|f\| = \inf\{|f(t)| : t \in A\}$
   (B) $\|f\| = \{|f(t)| : t \in A\}$
   (C) $\|f\| = \sup\{|f(f)| : t \in A\}$*
   (D) None of these

41. If X is a complete normed vector space and f is a continuous linear functional on Y, a subspace of X, then—
   (A) f can be extended to a linear functional F on X such that $\|F\| = \|f\|$
   (B) f can not be extended to a linear functional F on X such that $\|F\| = \|f\|$
   (C) f can be extended to a linear functional F on X such that $F = f$
   (D) None of these

42. If X is a normed vector space, Y is a subspace of X and f is a bounded linear functonal on Y with bound $\|f\|$ relative to Y then f has a continuous linear extension to an $x' \in X'$ with $\|x'\| = \|f\|$—
   (A) Hahn Banach theorem
   (B) Uniform Boundedness Principle
   (C) Muntz theorem
   (D) None of these

43. Each reproducing Kernel is—
   (A) A positive matrtix
   (B) A negative matrix
   (C) A null matrix
   (D) A identity matrix

44. Muntz theorem states necessary and sufficient condition for the set $t^{n^1}, t^{n^2}, \ldots, 1 \le n_1 < n_2 <$ …… to be complete in L_2 is that—
   (A) $\sum_{i=1}^{\infty} \frac{1}{n_i} > \infty$
   (B) $\sum_{i=1}^{\infty} \frac{1}{n_i} = \infty$
   (C) $\sum_{i=1}^{\infty} \frac{1}{n_i} = 0$
   (D) $\sum_{i=1}^{\infty} \frac{1}{n_i} < \infty$

45. A function K(s, t) on S · S is a positive matrix if for each $n = 1, 2$…… and each choice of points $t_1, t_2 \ldots, t_n$ the quardratic form $\sum_{i,j=1}^{n} kc\, t_i, t_j$ $\xi_i \bar{\xi}_j$ in $\xi_1, \xi_2, \ldots \xi_n$—
   (A) is non-negative
   (B) is non-positive
   (C) is negative
   (D) is positive

46. A Hilbert space of functions on a set S is said to have a reproducing Kernel if there is a function K(s, t) on S · S is such that—
   (1) $k(;t) \in X, t \in S$
   (2) $x(t) = (x, K(;t), x \in X, t \in S$
   (A) (1) is true only
   (B) (2) is true only
   (C) (1) and (2) both are true
   (D) None is true

47. A proper functional completion of X is a proper functional space $\overline{X}$ on the same basic set S such that—
   (A) $\overline{X}$ is complete
   (B) X is a dense subspace of $\overline{X}$
   (C) Both (A) and (B)
   (D) None of these

48. A vector space X of functions a set S, together with a norm in X, is called a proper functional space if for every $s \in S$—
   (A) There exist Ms such that $|x(S)| = Ms\|x\|, x \in X$
   (B) There exist Ms such that $|x(S)| > Ms\|x\|, x \in X$
   (C) There exist Ms such that $|x(S| \le Ms\|x\|, x \in X$
   (D) None of these

49. A vector space X of functions on a set S, together with a norm in X, is called a proper functional space—
   (A) If for evrey $s \in$ S, the evaluation functional at s is discontinuous
   (B) If for every $s \in$ S, the evaluation functional at s is constant
   (C) If for every $s \in$ S, the evaluation functional at s is continuous*
   (D) None of these

50. Riesz-Fischer theorem states that if $\sum_{n-\infty}^{\infty} |C_n|^2 < \infty$, then there is $f \in L_2(0, 2\pi)$ such that $\sum_{-\infty}^{\infty} C_n e^{inx}$ is—
    (A) the fourier series of f
    (B) converges to f in the sense of L_2
    (C) both (A) and (B) above
    (D) None of these

51. If there is a countable basis for X, then—
    (A) The Hilbert space X is separable
    (B) The Hilbert space X is not separable
    (C) The Hilbert space X is null
    (D) None of these

52. Two Hilbert spaces are isomorphic if they have—
    (A) Same dimension
    (B) Different dimension
    (C) Almost different dimension
    (D) None of these

53. If Hilbert space X is separable, then there is a—
    (A) Countable basis for X
    (B) Non countable basis for X
    (C) Null basis for X
    (D) None of these

54. The Cardinality is……for each basis of a given Hilbert space—
    (A) Same
    (B) Different
    (C) Almost different
    (D) None of these

55. A necessary and sufficient condition for the set $t^{n^1}, t^{n^2}, \ldots\ldots 1 \le n_1 < n_2 < \ldots$ to be complete in L_2 is that $\sum_{i=1}^{\infty} \frac{1}{n_1} = \infty$. This is—
    (A) Muntz theorem
    (B) Isoperimetric theorem
    (C) Bassel's inequality
    (D) Parseval's formula

56. A necessary and sufficient condition for $(x_1, x_2 \ldots x_n)$ to be linearly independent is that—
    (A) Gramm determinant $G \neq 0$
    (B) Gramm determinant $G > 0$
    (C) Gramm determinant $G < 0$
    (D) Gramm determinant $G = 0$

57. Among all simple, closed, piecewise smooth curves of length L in the place the circle encloses the maximum area. This is—
    (A) Isoperimetric theorem
    (B) Bassel's inequality
    (C) Cauchy-Schwarz inequality
    (D) Parseval's formula

58. Let X be an Hilbert space, $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set. Then S is basis iff equality holds in Bassel's inequality this is—
    (A) Bassel's inequality
    (B) Cauchy-Schwarz inequality
    (C) Parseval's formula
    (D) None of these

59. Let X be an Hilbert space and $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set in X, then S is a basis—
    (A) If it is complete in X
    (B) If it is not complete in X
    (C) If it is compact in X
    (D) None of these

60. If X is a Hilbert space, $S = \{x_\alpha : \alpha \in A\}$ is an orthonormal set in X and $x \in X$, then $\sum_{\alpha \in A} |(x, x_\alpha)|^2 \le \|x\|^2$. This is—
    (A) Bassel's inequality
    (B) Cauchy-Schwarz inequality
    (C) Parseval's formula
    (D) None of these

61. If $\{x_1, x_2, \ldots\ldots, x_n\}$ is an orthonormal set, then for any $x \in X$—
    (A) $\sum_{i=1}^{n} |x, x_i)|^2 \le \|x\|^2$
    (B) $\sum_{i=1}^{n} |x, x_i|^2 \le |x|^2$
    (C) $\sum_{i=1}^{n} |x, x_i|^2 \le x^2$
    (D) None of these

62. Parallelogram law states that—
    (A) $x \perp y \Rightarrow \|x\|^2 + \|y\|^2 = \|x + y\|^2$ is valid in inner product space

(B) $\| x+y \|^2 + \| x-y \|^2 = 2 \{\| x \|^2 + \| y \|^2\}$ holds in inner products space

(C) $x, y = \begin{cases} 0, & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases}$,$x, y \in$ S

(D) None of these

63. Pythagorean theorem states that—

(A) $x \perp y \Rightarrow \| x \|^2 + \| y \|^2 = \| x+y \|^2$ is valid in inner product space*

(B) $\| x+y \|^2 + \| x-y \|^2 = 2 \{\| x \|^2 + \| y \|^2\}$ holds in inner product space

(C) $x, y = \begin{cases} 0, & \text{if } x \neq y \\ 1, & \text{if } x = y \end{cases}$, $x, y \in$ S

(D) None of these

64. Banach space is Hilbert space if—

(A) Parallelogram law holds

(B) Pythagorean theorem holds

(C) Projection theorem holds

(D) None of these

65. $\| x+y \|^2 + \| x-y \|^2 = 2\{\| x \|^2 + \| y \|^2\}$ holds in inner product space this is—

(A) Pythagorean theorem

(B) Parallelogram law

(C) Cauchy-Schwarz inequality

(D) None of these

66. $\perp y \Rightarrow \| x \|^2 + \| y \|^2 = \| x+y \|^2$ is valid in inner product space. This is—

(A) Pythagorean theorem

(B) Parallelogram law

(C) Cauchy-Schwarz inequality

(D) None of these

67. The Cauchy-Schwarz inequality states that inner product space is—

(A) Continuous function on X. X into F, where the norm of $(x, y) \in$ X · X is $\| x \| + \| y \|$

(B) Constant function on X. X into F, where the norm of $(x, y) \in$ X · X is $\| x \| + \| y \|$

(C) Discontinuous function on X · X into F. where norm of $(x, y) \in$ X, X is $\| x \| + \| y \|$

(D) None of these

68. For a fixed m_0 such that—

(1) $\sum_{m=1}^{\infty} | a_{mn} | <$ M for fixed M and every $m > m_0$

(2) $\lim_m a_{mn} = 0$ for every n, and $\lim_m \sum_{n=1}^{\infty} a_{mn} = 1$, then

(A) A is regular (B) A is almost regular

(C) A not regular (D) None of these

69. If

(1) $\sum_{k=1}^{\infty} | a_{ik} | \leq$ M, $i = 1, 2, \ldots$ for some M

(2) $\lim_i a_{ik} = 0, k = 1, 2, \ldots$ and

(3) $\lim_i \sum_{k=1}^{\infty} a_{ik} = 1$, then—

(A) A is regular

(B) A is almost regular

(C) A not regular

(D) None of these

70. If parallelogram law holds, then—

(A) Banach space is Hilbert space

(B) Hilbert space is Banach space

(C) Banach space is vector space

(D) None of these

71. Cauchy-Schwarz inequality states—

(A) For every $x, y \in$ X, $| (x, y) | \leq \|x\| \|y\|$

(B) For every $x, y \in$ X, $| (x, y) | > \|x\| \|y\|$

(C) For every $x, y \in$ X, $| (x, y) | < \|x\| \|y\|$

(D) For every $x, y \in$ X, $| (x, y) | = \|x\| \|y\|$

72. X is called Hilbert space, then—

(A) If X is complete under the norm obtained from its inner product*

(B) If X is complete space

(C) If X is not complete under theorem obtained from its inner product

(D) None of these

73. A vector space X, together with an inner product in X, is called—

(A) Inner product space

(B) Outer product space

(C) Closed space

(D) None of these

74. Let $x \in$ X, then norm in the space X is—

(A) $\| x \| = \sqrt{(x, x)} = (x, x)^{1/2}$

(B) $\| x \| = (x, x)^2$

(C) $\| x \| = 1$

(D) None of these

75. If $l \leq p \leq \infty$, and q is conjugate to p, then—
(A) $(l_p)' = l_q$ (B) $(l_p)' > l_q'$
(C) $(l_p)' < l_q$ (D) None of these

76. A is almost regular iff there is a fixed m_0 such that—
(A) $\sum_{m=1}^{\infty} |a_{mn}| < M$ for fixed M and every $m > m_0$
(B) $\lim_{m} a_{mn} = 0$ for every n
(C) $\lim_{m} \sum_{n=1}^{\infty} a_{mn} = 1$
(D) All the above

77. A is regular iff—
(A) $\sum_{k=1}^{\infty} |a_{ik}| \leq M$, $i = 1, 2, \ldots$ for some M
(B) $\lim_{i} a_{ik} = 0$, $k = 1, 2, \ldots\ldots$
(C) $\lim_{i} \sum_{k=1}^{\infty} a_{ik} = 1$
(D) All the above

78. A is regular if for each convergent sequence x—
(A) the A-limit exists
(B) A-limit equals the ordinary limit of the sequence x
(C) Both (A) and (B)
(D) None of these

79. If x is a normed vector space Y $\subset$ X is total if—
(A) its closed linear span is X
(B) its open linear span is X
(C) its closed linear span is not X
(D) None of these

80. If X is a normed vector space and $x'_n \in X'$, (n = 1, 2……) we say x'_n converges weakly to $x' \in X'$ if—
(A) for each $x \in X$, $\lim_{n \to \infty} x'_n(x) = x'(x)$
(B) for each $x \in X$, $x'_n(x) = x'(x)$
(C) for each $x \in X$, $\lim_{n \to \infty} x'_n(x) = 0$
(D) None of these

81. A normed vector space is finite dimensional if—
(A) the closed bounded sets are compact
(B) the closed bounded sets are not compact
(C) the open bounded sets are compact
(D) the open bounded sets are not compact

82. If X is a Banach space and Y is a subspace of X, which is a Borel set, then—
(A) Y is of first category in X and is identical with X
(B) Y is of second category in X and is identical with X
(C) Y is either of first category in X or is identical with X
(D) Y is either of second category in X or is identical with X

83. If $P'(x) = P(x)$, then—
(A) x is almost convergent
(B) x is divergent
(C) x is constant
(D) None of these

84. The closed bounded sets are compact if—
(A) A normed vector space is finite dimensional
(B) A vector space is finite dimensional
(C) A normed vector space is inifinite
(D) None of these
(E) $P'(x) = P(x)$

85. A linear transformation T is bounded if—
(A) T is continuous
(B) T is discrete
(C) T is dis continuous
(D) None of these

86. If X is normed vector space. The space X of continuous linear functional on X, is referred as dual of X if—
(A) $\| x' \| = \{M : |x'(x)| \leq M \|x\|, x \in X\}$
(B) $\| x' \| = \text{Sup } \{M : |x'(x)| \leq M \|x\|. x \in X\}$
(C) $\| x' \| = \inf \{M : |x'(x)| \leq M \|x\|, x \in X\}$
(D) None of these

87. A linear transformation T of X into Y is continuous, then—
(A) T is bounded
(B) T is not bounded
(C) T is constant
(D) None of these

88. If X is a normed vector space, the space X′ of continuous linear functional on X′ is refered as dual of X if—
(A) $\| x' \| = \inf \{ |x'(x)| : \| x \| = 1 \}$
(B) $\| x' \| = \text{Sup} \{ |x'(x)| : \| x \| = 1 \}$
(C) $\| x' \| = \{ | x'(x) | : \| x \| = 1 \}$
(D) None of these

89. If $\| T \|$ is norm of T, then—
(A) $\| T_x \| \leq \| T \| \| x \|$ for every $x \in X$
(B) $\| T_x \| < \| T \| \| x \|$ for every $x \in X$
(C) $\| T_x \| > \| T \| \| x \|$ for every $x \in X$
(D) $\| T_x \| = \| T \| \| x \|$ for every $x \in X$

90. If $\| T \|$ is norm of T if—
(A) T is a linear transform
(B) $\| T \| \inf \{ M : \| T_x \| \leq M \| x \|, x \in X \}$
(C) T is bounded
(D) All the above

91. A linear transform T is bounded if there is an integer M such that—
(A) $\| T_x \| < M \| x \|$ for every $x \in X$
(B) $\| T_x \| = M \| x \|$ for every $x \in X$
(C) $\| T_x \| < M \| x \|$ for every $x \in X$
(D) None of these

92. Banach space is a—
(A) Complete normed vector space
(B) Normed vector space
(C) Complete vector space
(D) None of these

93. Let x be a vector space and p a real valued function on X is semi norm if—
(A) $x \neq 0 \Rightarrow p(x) > 0$
(B) $p(x + y) \leq p(x) + p(y)$
(C) $p(\alpha x) = | \alpha | p(x)$
(D) All the above

94. Let X be a vector space, and p a real valued function on X is semi norm then—
(A) $p(\alpha x) = \alpha \, p(x)$
(B) $p(\alpha x) = | \alpha | p(x)$
(C) $p(\alpha x) = \alpha / |p(x)|$
(D) $p(\alpha x) = \alpha + p(x)$

95. Let X be a vector space, and p a real valued function on X is semi norm then—
(A) $p(x + y) \leq p(x) + p(y)$
(B) $p(x + y) = p(x) + p(y)$
(C) $p(x + y) > p(x) + p(y)$
(D) $p(x + y) < p(x) + p(y)$

96. Let X be a vector space, and p a real valued function on X is semi norm then—
(A) $x \neq 0 \Rightarrow p(x) = 0$
(B) $x \neq 0 \Rightarrow p(x) < 0$
(C) $x \neq 0 \Rightarrow p(x) > 0$
(D) None of these

97. Let X be a vector space and $x \in X$. A real valued function on a vector space X, $\| x \|$ is a norm on X, then—
(A) $\| \alpha x \| = \alpha / \| x \|$
(B) $\| \alpha x \| = \| x \|$
(C) $\| \alpha x \| = \alpha \| x \|$
(D) None of these

98. Let X be a vector space and $x \in X$. A real valued function an a vector space X, $\| x \|$ is a norm on X, then—
(A) $\| x + y \| < \| x \| + \| y \|$
(B) $\| x + y \| > \| x \| + \| y \|$
(C) $\| x + y \| = \| x \| + \| y \|$
(D) $\| x + y \| \leq \| x \| + \| y \|$

99. Let x be a vector space and $x \in X$, A real valued function on a vector space X, $\| x \|$ is a norm on X, then—
(A) $x \neq 0 \Rightarrow \| x \| > 0$
(B) $x \neq 0 \Rightarrow \| x \| < 0$
(C) $x \neq 0 \Rightarrow \| x \| = 0$
(D) None on these

100. x is almost convergent if—
(A) $p'(x) = p(x)$ (B) $p'(x) > p(x)$
(C) $p'(x) < p(x)$ (D) None of these

101. If L is a Banach limit then—
(A) $\underline{\lim}\, x_n \leq L(x) \leq \overline{\lim}\, x_n$ for all $x \in m$
(B) $L(x) \leq \overline{\lim}\, x_n \leq \underline{\lim}\, x_n$ for all $x \in m$
(C) $\overline{\lim}\, x_n \leq \underline{\lim}\, x_n \leq L(x)$ for all $x \in m$
(D) None of these

102. The subspaces Y, Z of vector space X are disjoint if—
(A) $Y \cap Z = \{0\}$ (B) $Y \cap Z = X$
(C) $Y \cap Z = Y$ (D) $Y \cap Z = Z$

103. The subspace generated by S of vector space X is a—
(A) Smallest subspace of X that contains S
(B) Largest subspace of X that contains S
(C) Largest subspace of X that does not contains S
(D) Largest subspace of X that does not contains S

104. $Y = \cap \{Y_\alpha : \alpha \in A\}$ where $Y_\alpha, a \in A$ is a set of subspace of X, is refered as—
(A) Intersection of subspace
(B) Union of subspace
(C) Disjoint intersection of subspace
(D) Disjoint union of subspace

105. Proper subspace of vector space X is—
(A) $Y \subset X$ and $Y \neq X$ and $Y \neq \{0\}$
(B) $Y \subset X$ and $Y = X$ or $Y = \{0\}$
(C) $Y \subset X$ and $Y \neq X$ but $Y = \{0\}$
(D) None of these

106. Improper subspace of vector space X are—
(A) X only
(B) Null vector space $\{0\}$ of X only
(C) X and null vector space $\{0\}$ of X
(D) None of these

107. Let m be the vector space of all bounded sequences of real numbers, $x \in m$ is called almost convergent and the number's is called F-limit of x if—
(A) $L(x) = s$ for all Banach limits L
(B) $L(x) > s$ for all Banach limits L
(C) $L(x) < s$ for all Banach limits L
(D) None of these

108. A Banach limit is any linear functional L defined on m, m. be the vector space of all bounded sequences of real numbers—
(A) $L(x) = L(\sigma x)$, where $\sigma x = \sigma(x_1, x_2, \dots) = x_2, x_3, \dots\dots)$
(B) $L(x) = 1$ if $x = (1, 1, \dots\dots)$
(C) Both (A) and (B)
(D) None of these

109. A Banach limit is any linear functional L defined on $m \cdot m$ be the vector space of all bounded sequences of real numbers, then—
(A) $L(x) \geq 0$ if $x_n \geq 0$ for all n
(B) $L(x) = 0$ if $x_n > 0$ for all n
(C) $L(x) > 0$ if $x_n = 0$ for all n
(D) None of these

110. A sequence $x = (x_1, x_2, \dots)$ is said to have A-limit, A(x), if for each $i = 1, 2, 3 \dots\dots$ the sum $A_i(x) = \sum_{k=1}^{\infty} a_{ik} x_k$—
(A) is convergent and $\lim A_i(x) = A(x)$
(B) is divergent and $\lim A_i(x) = A(x)$
(C) is oscillating
(D) None of these

111. A dual of normed vector space is—
(A) Banach space
(B) Not a Banach space
(C) Bounded space
(D) None of these

Answer

1. (C)	2. (C)	3. (A)	4. (A)	5. (A)
6. (B)	7. (B)	8. (A)	9. (B)	10. (A)
11. (C)	12. (C)	13. (B)	14. (C)	15. (A)
16. (C)	17. (B)	18. (A)	19. (A)	20. (A)
21. (A)	22. (A)	23. (A)	24. (D)	25. (A)
26. (A)	27. (D)	28. (C)	29. (B)	30. (A)
31. (A)	32. (C)	33. (B)	34. (A)	35. (D)
36. (B)	37. (D)	38. (A)	39. (A)	40. (C)
41. (A)	42. (A)	43. (A)	44. (B)	45. (A)
46. (C)	47. (C)	48. (C)	49. (C)	50. (A)
51. (D)	52. (A)	53. (A)	54. (A)	55. (A)
56. (A)	57. (A)	58. (C)	59. (A)	60. (A)
61. (A)	62. (B)	63. (A)	64. (A)	65. (B)
66. (A)	67. (A)	68. (B)	69. (A)	70. (A)
71. (A)	72. (A)	73. (A)	74. (A)	75. (A)
76. (A)	77. (B)	78. (C)	79. (A)	80. (A)
81. (A)	82. (C)	83. (A)	84. (C)	85. (A)
86. (C)	87. (A)	88. (B)	89. (A)	90. (D)
91. (C)	92. (A)	93. (D)	94. (B)	95. (A)
96. (C)	97. (C)	98. (D)	99. (A)	100. (A)
101. (A)	102. (A)	103. (A)	104. (A)	105. (A)
106. (C)	107. (A)	108. (B)	109. (A)	110. (A)
111. (A)				

Numerical Analysis

Error

Error—Error in numerical calculation occurs due to—

(i) Error in input data/experimental error.

(ii) **Round off error**—Occurs due to rounding of digits (use of finite number of digits).

(iii) **Truncation error**—Occurs due to approximation.

(iv) Errors that occur due to mistakes in numerical computation.

Absolute error—$\in = a - \tilde{a}$, where $\tilde{a}$ is an approximation of exact value a.

$$\text{True Value} = \text{Approximation} + \text{Error}$$

Relative Error—

$$\in_r = \frac{\in}{a} = \frac{a - \tilde{a}}{a} = \frac{\text{Error}}{\text{True Value}}$$

Error bound β—For $|\in| \le \beta$, $|a - \tilde{a}| \le \beta$

Error Propagation

1. In addition and subtraction, an error bound for the results is given by the sum of error bounds for the terms.
2. In multiplication and division, a bound for the relative error of the results is given (approximately) by the sum of the bounds for the relative errors of the given numbers.

Finite Difference Operators

(a) **Forward Operator—**

Let h be the finite difference.

then

$$\Delta f(x) = f(x+h) - f(x)$$
$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$
$$\Delta^3 f(x) = f(x+3h) - 3f(x+2h) + 2f(x+h) - f(x)$$
$$\Delta^n f(x) \sum_{r=0}^{n} (-1)^{n-r}\, {}^nC_r\, f(x+rh)$$

(b) **Shift Operator—**

Let h be the finite difference.

Then
$$E\, f(x) = f(x+h)$$
$$E^n\, f(x) = f(x+nh)$$

(c) **Backward differences—**

Let h be the finite difference.

Then
$$\nabla f(x) = f(x) - f(x-h)$$
$$\nabla^2 f(x) = f(x-2h) - 2f(x-h) + f(x)$$

....

....

....

$$\nabla^n f(x) = \sum_{r=0}^{n} (-1)^{n-r}\, {}^nC_r\, f(x-rh)$$

(d) **Factorial Notation—**

Let h be the finite difference.

Then
$$x^{(n)} = x(x-h)(x-2h)\ldots\left(x - \overline{n-1}\right)h$$
$$x^{(n)} = \frac{x!}{(x-n)!},\ (n < x)$$
$$\Delta x^{(n)} = nhx^{(n-1)}$$
$$\Delta^n x^n = n!\, h^n$$

(e) **Central difference operator δ—**

$$\delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$$
$$\delta^n f(x) = \Delta^n f\left(x - \frac{1}{2}nh\right)$$

(f) **Averaging operator μ—**

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{1}{2}h\right) + f\left(x - \frac{1}{2}h\right)\right]$$

Relation Between Different Finite Operators

1. Relation between Δ and E

$$E \equiv 1 + \Delta \text{ and } \Delta \equiv E - 1$$
$$E^n \equiv (1+\Delta)^n \text{ and } \Delta^n \equiv (E-1)^n$$

$$E^n f(x) = f(x + nh)$$
$$= \sum_{r=0}^{n} {}^nC_r \, \Delta^r f(x)$$

Theorem—1. If $f(x)$ is a rational integral function (polynomial) of degree n in x the nth difference of this polynomial is constant and $(n + 1)$th and higher difference are zero.

2. Relation between ∇, Δ and E

- $\nabla E \equiv E \nabla \equiv \Delta$
- $E^{-1} \equiv 1 - \nabla$
- $E \equiv (1 - \nabla)^{-1}$

3. Central difference

- $\delta = E^{1/2} - E^{-1/2}$
- $\delta \equiv \Delta E^{-1/2} \equiv E^{1/2} \Delta$
- $\delta \equiv \Delta E^{1/2} \equiv E^{1/2} \nabla$
- $\delta^n f(x) = \Delta^n f\left(x - \frac{1}{2}nh\right)$
- $\delta^n f(x) = \nabla^n f\left(x + \frac{1}{2}nh\right)$
- $\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$
- $\mu \equiv \frac{1}{2}[E^{1/2} + E^{-1/2}]$

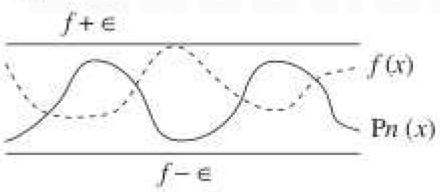
Interpolation

Polynomial of degree n (Algebraic polynomial)—

$p_n f(x) = a_n x^n + a_n x^{n-1} + \ldots\ldots a_1 x + a_n$ where n is non-negative integer and a_n, a_{n-1} …… a_1, a_0 are real constants.

Weierstrass approximation theorem—

For any continuous function $f(x)$ on an interval J : $a \le x \le b$ and error bound $\beta > 0$, there is a polynomial $p_n(x)$ of sufficiently high degree n such that $| f(x) - p_n(x) | < \beta$ for all $x \in$ J. *i.e.*, $p_n(x) \approx (x)$



Taylor's polynomial approximation—In Taylor's polynomial all the information ued in the aproximation is concentrated at the signle point x_0. This limits Taylor polynomial approximation to the situation in which approximations are needed only at points close to x_0.

Interpolation—It means to find (approximate) value of function $f(x)$ for an x between different x values $x_0, x, \ldots\ldots, x_n$ at which the value of $f(x)$ are given *i.e.*, $f(x_i) = f_i$ $(i = 0, 1 \ldots\ldots n)$.

If $x_0, x_1, \ldots\ldots x_n$ are $(n + 1)$ distinct values of real valued function $f(x)$. One has a polynomial $p_n(x_i) \approx f(x)$ of degree n or less *i.e.*, there is at most one polynomial of degree $\le n$ which interpolates $f(x)$ at $(n + 1)$ distinct points $x_0, x_1, \ldots\ldots x_n$.

Lagrange Interpolation polynomial—

If $x_0, x_1, \ldots\ldots x_n$ are $(n + 1)$ distinct numbers and f is a function whose values are given at these numbers, then there exists a unique polynomial $p(x)$ of degree at most n with property

$$f(x_k) \approx p(x_k),\ k = 0, 1 \ldots\ldots n.$$

This polynomial (Lagrange interpolation polynomial of degree n) is given by

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} L_k(x) f_k$$
$$= \sum_{k=0}^{n} \frac{L_k(x)}{L_{k(xk)}} f_k$$

where
$$L_k(x) = \prod_{\substack{j=0 \\ j \ne k}}^{n} \frac{(x - x_j)}{(x_k - x_j)}$$

and
$$L_k(x) = \prod_{\substack{j=0 \\ j \ne k}}^{n} (x_k - x_j)$$

Remainder term (error)—

$$\text{Remainder} = f(x) - p_n(x)$$
$$= \left(\prod_{j=1}^{n} (x - x_j)\right) \frac{f^{n+1}(\xi)}{(n+1)!},$$

where $\xi \in (a, b)$ if $x_1 \ldots\ldots x_n \in [a, b]$

Interpolation with unequal interval first divided difference—

$$\underset{x_1}{\Delta}\, y_0 = \frac{y_1 - y_0}{x_1 - x_0}$$
$$= \frac{y_0 - y_1}{x_0 - x_1}$$
$$= \underset{x_0}{\Delta}\, y_1$$

$$\underset{x_{n-1}}{\Delta}\, y_n = \frac{y_{n-1} - y_n}{x_{n-1} - x_n}$$
$$= \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$
$$= \underset{x_n}{\Delta}\, y_{n-1}$$

The divided difference are symmetrical in all their arguments *i.e.,* the value of any difference is independent of the other of their arguments.

i.e., $\underset{x_{n-1}}{\Delta} y_n = \underset{x_n}{\Delta} y_{n-1}$

n–th divided difference

$$\underset{x_1 \ldots x_n}{\Delta^n} y_0 = \sum_{i=0}^{n} \frac{y_1}{\prod_{\substack{j=0\\ j\neq i}}^{n} (x_i - x_j)}$$

$$= f(x_0, x_1 \ldots x_n)$$

Then-divided difference of a polynomial of the nth degree are constant.

Newton's divided difference formula—

$$f(x) = y_0 + (x - x_0) f(x_0, x_1)$$

$$+ (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \ldots\ldots + (x - x_0) \ldots\ldots (x - x_{n-1}) f(x_0, x_1 \ldots\ldots x_n) + R_n$$

Relation between divided difference and ordinary difference—

$$\underset{x_1}{\Delta} y_0 = \frac{y_0 - y_1}{x_0 - x_1}$$

$$= \frac{1}{h} \Delta f(0)$$

$$= \frac{\Delta y_0}{h}$$

$$\underset{x_1 \ldots x_n}{\Delta^n} y_0 = f(x_0, x_1 \ldots x_n)$$

$$= \frac{\Delta y_0}{n!\, h^n}$$

Interpolation with Equal Interval

Newton's Binomial expansion method (missing terms)—

Suppose $y = f(x)$ has values $y_0, y_1, \ldots\ldots\ldots y_n$ $(n + 1)$ values corresponding to the arguments

$$a, a + h, a + 2h \ldots\ldots a + nh$$

1. **If one value y_1 is missing—**

Then $\Delta^n y_x = 0$

or, $(E - 1)^n y_0 = y_n - {}^n c_1\, y_{n-1} + {}^n c_2\, y_{n-2} \ldots\ldots (-1)^n y_0$

$= 0$

$f(x)$ can be represented by a polynomial of degree $(n - 1)$ since n–values of $f(x)$ is known.

2. **If two values are missing—**

$f(x)$ can be represented, as a polynomial of degree $n - 2$ since $n - 1$ value of $f(x)$ is known.

Newton-Gregory forward difference interpolation formula—

$$f(x + nh) = \sum_{r=0}^{n} {}^n c_r\, \Delta^r f(x)$$

Newton–Gregory advance difference formula—

$$f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \Delta^r f(0)$$

$$= f(0) + x\,\Delta f(0) + \frac{x(x-1)}{2!} \Delta^2 f(0) + \frac{x(x-1)(x-2)}{3!} \Delta^3 f(0) \ldots\ldots + \frac{x^{(n)}}{n!} \Delta^n f(0)$$

Newton backward difference formula—

$$f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \Delta^r f(0)$$

$$= f(0) + x\,\nabla f(0) + \frac{x(x+1)}{2!} \nabla^2 f(0) + \ldots\ldots + \frac{x(x+1)\ldots\ldots(x+n-1)}{n!} \nabla^n f(0)$$

Central Difference Interpolation Formula

Gauss's forward formula for equal intervals.

$$y_u = y_0 + {}^u c_1\, \Delta y_0 + {}^u c_2\, \Delta^2\, y - 1 + {}^{u+1} c_3\, \Delta^3\, y_{-1} + {}^{u+1} c_4\, \Delta^4\, y_{-2} + \ldots\ldots$$

Gauss's backward formula for equal intervals

$$y_u = y_0 + {}^u c_1\, \Delta y_{-1} + {}^{u+1} c_2\, \Delta^2 y_{-1} + {}^{u+1} c_3\, \Delta^3\, y_{-2} + {}^{u+1} c_4\, \Delta^4\, y_{-2} + \ldots\ldots$$

Sterling's formula—The arithmetic mean of Gauss's forward and backward formula is known as Sterling's formula.

$$y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta_{y-1}}{2}\right) + \frac{u^2}{2!} \Delta^2\, y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) \frac{u^2(u^2-1)}{4!} \Delta^4\, y_{-2} + \ldots\ldots$$

where, $u = \dfrac{x - x_0}{h}$

Bessel's formula—Shifting the origin in Gauss's backward formula one have Bessel's formula (Gauss's third formula)

$$y_u = y_0 + u\Delta y_0 + \frac{p(p-1)}{2\,!}\left(\frac{\Delta^2{}_{y-1} + \Delta^2 y_0}{2}\right) + \frac{p(p-1)\left(p-\frac{1}{2}\right)}{3\,!}\Delta^3 y - 1 + \frac{(p+1)\,p\,(p-1)\,(p-2)}{4\,!}\left(\frac{\Delta^4{}_{y-2} + \Delta^4{}_{y-1}}{2}\right) + \ldots\ldots$$

Everett's formula—

$$y_u = vy_0 + \frac{v(v^2-1^2)}{3\,!}\Delta^2{}_{y-1} + \frac{v(v^2-1^2)\,(v^2-2^2)}{5\,!}\Delta^4 y - 2 + \ldots + uy_1 + \frac{u(u^2-1^2)}{3\,!}\Delta^2 y_0 + \frac{u(u^2-1^2)\,(u^2-2^2)}{5\,!}\Delta^4 y - 1 + \ldots\ldots$$

where $v = 1 - u$.

Everett's formula truncated after second difference is equivalent to Bessel's formula truncated after third differences.

Numerical Soulution of Algebraic Equations

Given an equation $f(x) = 0$

A solution of $f(x) = 0$,

is a number $x = s : f(s) = 0$.

Iteration method—To solve $f(x) = 0$, when there is no formula for the exact solution one can use approximation method, an iteration method, in it one start from an initial guess x_0 (which may be poor) and compute step by step (*i.e.*, searching better) approximation x_0, x_1, x_2, …… of an unknown solution of $f(x) = 0$.

(A) Fixed point iteration—

$$f(x) = 0$$

$$\Leftrightarrow \quad x = g(x)$$

choose x_0 and compute $(x_{n+1}) = g(x_n)$

The solution of $x = g(x)$ is called a fixed point of g.

Convergence of fixed-point iteration—

Let $x = s$ be a solution of $x = g(x)$ and suppose that g has a continuous derivative in some interval J containing s. Then if $|\,g'(x)\,| \le k < 1$ in J, the iteration process defined by $x_{n+1} = g(x_n)$ converges for any x_0 in J.

Existence of solution—

If g is continuous in closed interval J, its range lies in J then $x = g(x)$ has at least one solution in J.

Fixed point—

A fixed point for a given function g is a number p for which $g(p) = p$.

Theorems—

(1) If g is continuous on interval $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$. then g has a fixed point in $[a, b]$.

(2) If in addition $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exist with $|\,g'(x)\,| \le k$ or $g'(x) \le k$ for all $x \in (a, b)$, then the fixed point in $[a, b]$ is unique.

(3) **Fixed-point theorem**—Let g is a continuous function on $[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose, in addition that g' exist on (a, b) and positive constant $k < 1$ exist with $|\,g'(x)$ $\le k$, for all $x \in (a, b)$. Then for any number P in $[a, b]$, the sequence defined by $p_n = g(p_{n-1})$, $n \ge 1$, converges to the unique fixed point P in $[a, b]$.

Corollary—If g satisfies the hypothesis of fixed point theorem, bounds for the error invalved in using p_n to approximate p are given by—

$|\,p_n - p\,| \le k^n \max.\{p_0 - a, b - p_0\}$ and $|\,p_n - p|$ $\le \frac{k_n}{1-k}|\,p_1 - p_0\,|$ for all $n \ge 1$

(B) **Newton-Rapson Method—**

$$f(x) = 0$$

where f is assumed to have a continuous derivative

$$f'\,x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The convergence is at least quardratic for simple root if $f''(x)$ is continuous.

Second order quadratic convergence of Newton's method—

If $f(x)$ is three times differentiable and f', f'' are not zero at a solution s of $f(x) = 0$, then for x_0 sufficiently close to s. Newton's method is of second order.

Theorem—Let f is continuous in $[a, b]$ in XY plane. If $p \in [a, b]$ is such that $f(p) = 0$ and $f'(p) \ne 0$, then there exists a $\delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any intial approximation $p_0 \in [p - \delta, p + \delta]$.

(C) Second Method (Regula Falsi)—Replacing the derivative $f'(x_n)$ by the difference quotient in Newton's method.

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

$$x_{n+1} = x_n - f(x_n)\frac{x_n\ x_{n-1}}{f(x_n) - f(x_{n-1})}$$

(D) Bisection (Binary–Search) Method—Let f is continuous function defined on the interval $[a, b]$, with $f(a)$ and $f(b)$ of opposite sign. By the intermediate value theorem, there exists a number p in (a, b) with $f(p) = 0$.

[**Note**—This procedure will work for the case when $f(a)$ and $f(b)$ have opposite signs and there is more than one root in the interval (a, b)].

'The bisection method calls for a repeated halving of sub-intervals of $[a, b]$ and locating the half-containing p.

To find a solution to $f(x) = 0$ given the continuous function f on the interval $[a, b]$ where $f(a)$ and $f(b)$ have opposite signs.

Set $a_1 = a$

and $b_1 = b$

and let $p_1 = \frac{1}{2}(a_1 + b_1)$

If $f(p_1) = 0,$

then $p = p_1$

If not, then $(f_{p_1})f(a_1) < 0$

or $f(p_1)f(b_1) > 0$

opposite sign

If $f(p_1)f(a_1) < 0$

then $p \in (a_1, p_1)$

and set $a_2 = a_1,$

$b_2 = p_1$

Some sign

If $f(p_1)f(a_1) > 0$

Then $p \in (p_1, b_1)$

and set $a_2 = p_1,$

$b_2 = b_1$

Repeat the procedure.

Theorem—Let f is continuous function on $[a, b]$ and $f(a)\ f(b) < 0$, then bisection method generater a sequence $\{p_n\}$ approximating a zero p of f with—

$$|p_n - p| \leq \frac{b-a}{2^n},$$

$$n \geq 1$$

Solution of linear system of equations :

(A) **Gauss elimination method**—

Given system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2\ y + c_2z = d_2$$
$$a_3x + b_3\ y + c_3z = d_3 \qquad \ldots(1)$$

Step 1. Eliminate x from second and third equation, here $r_1 = \frac{a_2}{a_1}$, multiply first equation by r_1 and then subtract it by second equations.

Here $r_2 = \frac{a_3}{a_1},$

multiply first equation by r_2 and then subtract it by third equation.

The resulting system is,

$$a_1x + b_1y + c_1z = d_1$$
$$b'_2y + c_2'\ z = d'_2$$
$$b'_3\ y + c_3'\ z = d_3' \qquad \ldots(2)$$

The first equation is pivotal equation and a_1 is the first pivot.

Step 2. Eliminate y from third equation in (2).

Here $r_2 = \frac{b_3'}{b_2'}$

multiply second equation in (2) by r_3 and then subtract it by third equation.

The resulting equation is,

$$a_1x + b_1y + c_1z = d_1$$
$$b_2'y + c_2'\ z = d_2'$$
$$c''_3z = d'_3$$

The second equation is pivotal equation and b'_2 is the new pivot.

Step 3. The value of x, y, z can be found from (3) by back substitution.

(B) Cruot's Triangularization Method—

Given the system of equations :

$$a_{11}\,x_1 + a_{12}\,x_2 + a_{13}\,x_3 = b_1$$
$$a_{21}\,x_1 + a_{22}\,x_2 + a_{23}\,x_3 = b_2$$
$$a_{31}\,x_1 + a_{32}\,x_2 + a_{33}\,x_3 = b_3$$

which is equivalent to

$$AX = B \qquad \ldots(1)$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and $$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let $$A = LU \qquad \ldots(2)$$

where, the lower triangular matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

and the upper triangular matrix

$$U = \begin{bmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ 0 & \mu_{22} & \mu_{23} \\ 0 & 0 & \mu_{33} \end{bmatrix}$$

Then (1) becomes

writting $$LUX = B, \qquad \ldots(3)$$

$$UX = V \qquad \ldots(4)$$

Equation (3) becomes $LV = B$ which is equivalent to the equations :

$$v_1 = b_1$$

$$l_{21}\, v_1 + v_2 = b_2$$

$$l_{31}\, v_1 + l_{32}\, v_2 + v_3 = b_3$$

Solving these for v_1, v_2, v_3 we get V. Then (4) becomes

$$u_{11}\, x_1 + u_{12}\, x_2 + u_{13}\, x_3 = v_1$$

$$u_{22}\, x_2 + u_{23}\, x_3 = v_2$$

and $$u_{33}\, x_3 = v_3$$

from which x_3, x_2 and x_i can be found by back-substitution. To compute the matrices L and U write (2) as,

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and equating corresponding elements from both sides, we obtain.

(i) $$u_{11} = a_{11},$$

$$u_{12} = a_{12}$$

and $$u_{13} = a_{13}$$

(ii) $$l_{21}\, u_{11} = a_{21}$$

or, $$l_{21} = a_{21} / a_{11}$$

and $$l_{31}\, u_{11} = a_{31}$$

or, $$l_{31} = a_{31} / a_{11}$$

(iii) $$l_{21}\, u_{12} + u_{22} = a_{22}$$

or, $$u_{22} = a_{22} - \frac{a_{21}}{a_{11}}\, a_{13}$$

and $$l_{31}\, u_{13} + u_{23} = a_{23}$$

or, $$u_{23} = a_{23} - \frac{a_{21}}{a_{11}}\, a_{13}$$

(iv) $$l_{31}\, u_{12} + l_{32}\, u_{22} = a_{32}$$

or $$l_{32} = \frac{1}{\mu_{22}}\left[a_{32} - \frac{a_{31}}{a_{11}}\, a_{12}\right]$$

(v) $$l_{31}\, u_{13} + l_{33}\, u_{23} + u_{33} = a_{33}$$ which gives u_{33}.

And we compute the elements of L and U in the following set order—

(i) First row of U,

(ii) First Column of L

(iii) Second row of U

(iv) Second column of L

(v) Third row of U

(C) Matrix Inversion Method

Given
$$\left.\begin{aligned} a_1x + b_1y + c_1\, z &= d_1 \\ a_2x + b_2y + c_2\, z &= d_2 \\ a_3x + b_3y + c_3\, z &= d_3 \end{aligned}\right\} \qquad \ldots(1)$$

Which can be written as—

where $$AX = D \qquad \ldots(2)$$

where $$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and $$D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Multiplying both sides of (2) by the inverse matrix A^{-1}, we get $A^{-1}\, AX = A^{-1}\, D$

$$IX = A^{-1}D$$
$$X = A^{-1}D$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \times \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \ldots(3)$$

Where $A_1, B_1, \ldots\ldots$ are the cofactors of $a_1\ b_1, \ldots\ldots$ in the determinant $|A|$.

Numerical Integration—It is the numerical evaluation of integrals $J = \int_a^b f(x)\,dx$, where a and b are given and f is a function given analytically by a formula or empirically by table of values *i.e.*,

$$J = \int_a^b f(x)\,dx \cong \int_a^b \phi(x)\,dx,$$

where $\phi(x)$ is a polynomial which is equivalent to $f(x)$.

Rectangular rule—Subdivide the interval of integration $a \leq x \leq b$ into n-subintervals of equal length $h = \frac{b-a}{n}$ and in each subinterval approximate f by $f(x^*_j)$ = value of f at the midpoint x_j of the jth subinterval.

$$J = \int_a^a f(x)\,dx = h\,[f(x^*_1) + f(x^*_2) + \ldots\ldots + f(x^*_n)]$$

where $$h = \frac{b-a}{n}$$

Trapezoidal rule—Subdivide the interval $a \leq x \leq b$ in to n-subintervals of equal length

$$h = \frac{b-a}{n},$$

where $x_0 = a$

and $x_0 + nh = b$

$$J = \int_a^h f(x)\,dx$$
$$= \int_{x_0}^{x_0+nh} f(x)dx$$
$$= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})]$$

Given y_0 and y_1 for corresponding values of x_0 and x_1 respectively for formula for function $f(x)$. Let $f(x)$ is a polynomial of degree 1. Then Trapezoidal rule is—

$$J = \frac{1}{2}h(y_0 + y_1)$$

Simpson's one-third rule—Sub-divide the interval $a \leq x \leq b$ into even number of equal interval $n = 2m$ of length

$$h = \frac{b-a}{2m}$$
$$J = \int_a^b f(x)dx,$$
$$= \int_{x_0}^{x_0+nh} f(x)dx$$
$$= \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots\ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots\ldots + y_{n-2})]$$

1. Given y_0, y_1, y_2 corresponding to values x_0, x_1, x_2 for function $y = f(x)$ and let $f(x)$ is polynomial of degree 2, then Simpson's one-third rule is.

$$\boxed{J = \frac{1}{3}h\,[y_0 + 4y_1 + y_2]}$$

Simpson's three-eight rule—Sub-divide the interval $a \leq x \leq b$ into multiple of 3 of equal intervals $n = 3m$ of length $h = \frac{b-a}{3m}$

$$J = \int_a^b f(x)dx,$$
$$= \int_{x_0}^{x_0+nh} f(x)dx$$
$$= \frac{3h}{8}[y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots\ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots\ldots + y_{n-3})]$$

2. Given y_0, y_1, y_2, y_3 corresponding to values x_0, x_1, x_2, x_3 for function $y = f(x)$.

Let $f(x)$ is a polynomial of degree 3, then Simpson's three eight rule is—

$$\boxed{J = \frac{3}{8}h\,[y_0 + 3y_1 + 3y_2 + y_3]}$$

Weddle's Rule :

Subdivide the interval $a \leq x \leq b$ into multiple of 6 of equal intervals $n = 6m$ length $h = \frac{b-a}{6\,m}$

$$J = \int_a^b f(x)dx,$$
$$= \int_{x_0}^{x_0+nh} f(x)dx$$

$$= \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots)$$

Cote's formula :

$$1 = nh\,(c_0{}^n y_0 + c_1{}^n y_1 + c_2{}^n y_2 + \ldots\ldots + c_k{}^n y_n)$$
$$= \int_a^b f(x)dx,$$

Where $c_0{}^n$ is a cote number.

Closed Cote's formula—a and b are nodes ($x_0 = a, x_n = b$)

Newton-cote's formula and different rules of integration—The Trapezoidal and Simpson's rule are special Newton cotes formula.

(i) Trapezoidal rule :

$f(x)$ is interpolated at equally spaced nodes by a polynomial of degree $n = 1$.

(ii) Simpson rule—$f(x)$ is interpolated at equally spaced nodes by a polynomial of degree $n = 2$.

(iii) Three-eight rule—$f(x)$ is interpolated at equally spaced nodes by a polynomial of degree $n = 3$.

(iv) Booles rule—$f(x)$ is interpolated at equally spaced nodes by polynomial of degree $n = 4$.

(v) Weddle's rule—$f(x)$ is interpolated at equally spaced nodes by polynomial of degree, $n = 6$.

Numerical Solution of Ordinary Differential Equation—

(A) Taylor's Series Method :

Given, initial value problem

$$y' = \frac{dy}{dx} = f(x, y),$$
$$\text{with} \quad y(x_0) = y_0 \qquad \ldots(1)$$
$$\text{Then} \quad y'' = \frac{d^2y}{dx^2} = f_x + f_y y' \quad = f_x + f_y f$$
$$y''' = \frac{d^3y}{dx^3} = f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_x f_y + f_y{}^2 f \quad \ldots(2)$$

By Taylor's theorem, the series about a point $x = x_0$, $y = y_0 + (x - x_0)(y')_0 + \frac{(x + x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \ldots\ldots$ (3) From (3) one can find the value y_1 of y for $x = x_1$ and $y', y'', y''' \ldots$ can be found at $x = x_1$ with (1) and (2) and so on.

(B) Picard's Method :

Given initial value problem

$$y' = \frac{dy}{dx} = f(x, y),$$
$$\text{with} \quad y(x_0) = y_0$$

Integrating both sides, we have

$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$

First approximation :

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

Second approximation :

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

nth approximation :

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

Theorem :

If the function $f(x, y)$ is bounded in same region about the point (x_0, y_0) and if $f(x, y)$ satisfies Lipshitz condition $| f(x, y) - f(x, \bar{y}) | \leq k \,| y - \bar{y} |$ (k being a constant), then the sequence $y_1, y_2 \ldots\ldots$ converges to the solution of initital value problem $y' = \frac{dy}{dx} = f(x, y)$, with $y(x_0) = y_0$.

(C) Euler's method :

Given initial value problem

$$y' = \frac{dy}{dx} = f(x, y),$$
$$\text{with} \quad y(x_0) = y_0$$

By Taylor series, for $h \to 0$, $y(x + h) \approx y(x) + hy'(x) = y(x) + hf(x, y)$

This gives, $\quad y_{n+1} = y_n + hf(x_n, y_n)$

$$\text{where} \quad h = \frac{x_n - x_0}{n}$$
$$(i.e., x_n = x_0 + nh)$$

(D) Modified Euler's Method :

Given initial value problem

$$y' = \frac{dy}{dx} = f(x, y),$$

with $\quad y(x_0) = y_0$

Integrating both sides,

$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$

Approximating by means of one have

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$$

And the iteration formula

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1)\, y_1^{(n)}]$$

where $y_1^{(n)}$ is the nth approximation to y_1.

And $\quad y_1^{(0)} = y_0 + hf(x_0, y_0),$

Where $\quad h = \dfrac{x_n - x_0}{n}$

$(i.e., \ x_n = x_0 + nh)$

(D) Runge-Kutta Method :

Given initial value problem

$$y' = \frac{dy}{dx} = f(x, y),$$

where $\ y(x_0) = y_0$

Calculate : $k_1 = hf(x_n, y_n)$

$$k_2 = hf\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}k_2\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

and $\quad y_{n+1} = y_n + \dfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where $\quad h = \dfrac{x_n - x_0}{n}$

$(i.e., \ x_n = x_0 + nh)$

(E) Predictor–Corrector method :

Given initial value problem :

$$y' = \frac{dy}{dx} = f(x, y)$$

where $\ y(x_0) = y_0$

Predicted Value—

$$y^*_{n+1} = y_n + hf(x_n, y_n)$$

Corrected Value—

$$y_{n+1} = y_n + h\,[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]$$

where $\quad h = \dfrac{x_n - x_0}{n} \quad (i.e., \ x_n = x_0 + nh)$

Existence and uniquenece of solutions for initial value problem :

Given $\quad y' = f(x, y)$

and $\quad y(x_0) = y_0$

1. Existence Theorem :

If $f(x, y)$ continuous at all points (x, y) is some rectangle R : $| x - x_0 | < a$, $| y - y_0 | < b$ and bounded in R, $|f(x, y)| \le k$, for all $(x, y) \in$ R, then the initial value problem has at least one solution $y(x)$.

2. Uniqueness Theorem :

If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded $|f| \le k \left|\dfrac{\partial f}{\partial y}\right| \le m$ for all $(x, y) \in$ R, then the initial value problem has at least one solution $y(x)$.

Some Solved Examples

Example 1. Prove that—

$$\mu_4 = \mu_3 + \Delta\mu_2 + \Delta^2\mu_1 + \Delta^3\mu_1$$

Solution : We know that

$$\Delta\mu_x = \mu_{x+h} - \mu_x$$

$$\therefore \quad \mu_4 - \mu_3 = \Delta\mu_3,$$

$$\mu_3 - \mu_2 = \Delta\mu_2,$$

$$\mu_2 - \mu_1 = \Delta\mu_1$$

Hence $\quad \mu_4 = \mu_3 + \Delta\mu_3$

$$= \mu_3 + \Delta(\mu_2 - \Delta\mu_2)$$

$$[\because \ \mu_3 - \mu_2 = \Delta\mu_2]$$

$$= \mu_3 + \Delta\mu_2 + \Delta^2\mu_2$$

$$= \mu_3 + \Delta\mu_2 + \Delta^2(\mu_1 + \Delta\mu_1)$$

$$[\because \ \Delta\mu_1 = \mu_2 - \mu_1]$$

$$= \mu_3 + \Delta\mu_2 + \Delta^2\mu_1 + \Delta^3\mu_1$$

Example 2. Let $x = s$ be a solution of $x = g(x)$ and suppose that g has a continuous derivative in some interval J containing s. Then if $| g'(x) | \le k < 1$ in J, the iteration process defined by $x_n = g(x_{n-1})$ converges any $x_0 \in$ J.

Solution : Since g has a continuous derivative in interval I by mean value theorem.

$$g'(t) = \frac{g(x)-g(s)}{x-s} \ (s \in \text{I})$$

$$\Rightarrow \quad g(x)-g(s) = g'(t)\,(x-\delta)\ (\delta \in \text{I})$$

Here $g(s) = s$

and $x_1 = g(x_0)$,

$x_2 = g(x_1), \ldots\ldots$

we have
$$\begin{aligned}|x_n - s| &= |g(x_{n-1}) - g(s)| \\ &= |g'(t)|\,|x_{n-1}-s| \\ &\le k\,|x_{n-1}-s| \\ &= k\,|g(x_{n-2})-g(s)| \\ &= k\,|g'(t)|\,|x_{n-2}-s| \\ &\le k^2\,|x_{n-2}-s| \\ &\vdots \\ &\le k^n\,|x_0 - s|\end{aligned}$$

$\because$ $k < 1$ we have $k^n \to 0 \Rightarrow |x_n - s| \to 0$ as $n \to \infty$

$\Rightarrow$ The iteration process defined by $x_n = g(x_{n-1})$ converges to $x_0 \in$ I.

Example 3. Find the square root of 18, correct to four decimal places by fixed point iteration formula.

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{18}{x_n}\right]$$

Solution : Given the iteration formula

$$x_1 = \frac{1}{2}\left[x_n + \frac{18}{x_n}\right] \qquad \ldots\ldots(1)$$

Put $n = 0$,

$$x_1 = \frac{1}{2}\left[x_0 + \frac{18}{x_0}\right] \qquad \ldots\ldots(2)$$

Let $x_0 = \sqrt{18}$

$= 4{\cdot}2$ (Approximately)

Then from (2), we get

$$\begin{aligned}x_1 &= \frac{1}{2}\left[4{\cdot}2 + \frac{18}{4{\cdot}2}\right] \\ &= \frac{1}{2}[4{\cdot}2 + 4{\cdot}286] \\ &= 4{\cdot}243 \qquad \ldots\ldots(3)\end{aligned}$$

Putting $n = 1$ in (1), we get

$$\begin{aligned}x_2 &= \frac{1}{2}\left[x_1 + \frac{18}{x_1}\right] \\ &= \frac{1}{2}\left[4{\cdot}243 + \frac{18}{4{\cdot}243}\right], \text{ from (3)} \\ &= \frac{1}{2}[4{\cdot}243 + 4{\cdot}2423] \\ &= 4{\cdot}24265 \qquad \ldots\ldots(4)\end{aligned}$$

Putting $n = 2$ in (1), we get

$$\begin{aligned}x_3 &= \frac{1}{2}\left[x_2 + \frac{18}{x_2}\right] \\ &= \frac{1}{2}\left[4{\cdot}24265 + \frac{18}{4{\cdot}2465}\right] \\ &= \frac{1}{2}[4{\cdot}24265 + 4{\cdot}24263] \\ &= 4{\cdot}24264 \qquad \ldots(5)\end{aligned}$$

$\therefore$ From (4) and (5), the square root of 18, correct to four decimal places is $4{\cdot}2426$.

Example 4. Evaluate the following—

(a) $\Delta\left(\dfrac{x^2}{\cos 2x}\right)$ (b) $\Delta^2 \cos 2x$.

Solution : (a) $\Delta\left(\dfrac{x^2}{\cos 2x}\right)$

$$\begin{aligned}&= \frac{(x+h)^2}{\cos 2(x+h)} - \frac{x^2}{\cos 2x} \\ &= \frac{(x+h)^2 \cos 2x - x^2 \cos 2(x+h)}{\cos 2(x+h)\cos 2x} \\ &= \frac{[(x+h)^2 - x^2]\cos 2x + x^2\,[\cos 2x - \cos 2(x+h)]}{\cos 2(x+h)\cos 2x} \\ &= \frac{(2hx+h^2)\cos 2x + 2x^2 \sin h \sin(2x+h)}{\cos 2(x+h)\cos 2x}\end{aligned}$$

(b) $\Delta^2 \cos 2x$

$$\begin{aligned}&= \Delta\,(\cos 2(x+h) - \cos 2x] \\ &= \Delta \cos 2(x+h) - \Delta \cos 2x \\ &= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x] \\ &= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h \\ &= -2\sin h\,[\sin(2x+3h) - \sin(2x+h)] \\ &= -2\sin h\,[2\cos(2x+2h)\sin h] \\ &= -4\sin^2 h \cos(2x+2h)\end{aligned}$$

Example 5. Evaluate $\Delta^2\left(\dfrac{5x+12}{x^2+5x+6}\right)$, where the interval of differencing is unity.

Solution : $\Delta^2\left(\dfrac{5x+12}{x^2+5x+6}\right)$

$$= \Delta^2 \left\{ \frac{5x+12}{(x+2)(x+3)} \right\}$$

$$= \Delta^2 \left\{ \frac{2}{x+2} + \frac{3}{x+3} \right\}$$

$$= \Delta \left\{ \Delta\left(\frac{2}{x+2}\right) + \Delta\left(\frac{3}{x+3}\right) \right\}$$

$$= \Delta \left\{ 2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right) \right\}$$

$$= -2\Delta \left\{ \frac{1}{(x+2)(x+3)} \right\} - 3\Delta \left\{ \frac{1}{(x+3)(x+4)} \right\}$$

$$= -2 \left\{ \frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)} \right\}$$

$$= -3 \left\{ \frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)} \right\}$$

$$= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)}$$

$$= \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$$

Example 6. Find the real root of the equation $x^4 - x - 10 = 0$. By Newton—Raphson method upto 3 decimal places.

Solution : Here,

$$f(x) = x^4 - x - 10 = 0 \quad \ldots(1)$$

and $f'(x) = 4x^3 - 1$

By Newton-Raphson formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x_i^4 - x_i - 10}{4x_i^3 - 1}$$

$$\Rightarrow \quad x_{i+1} = \frac{4x_i^4 - x_i - x_i^4 + x_i + 10}{4x_1^3 - 1}$$

$$= \frac{3x_i^4 + 10}{4x_i^3 - 1} \quad \ldots(2)$$

Also from (1), we have

$$f(1) = 1 - 1 - 10$$
$$= -10$$
$$= \text{Negative}$$
$$f(2) = 2^4 - 2 - 10$$
$$= 16 - 2 - 10$$
$$= 4$$
$$= \text{Positive}$$

$\therefore$ The root of the given equation lies between 1 and 2.

Also $f(1{\cdot}5) = (1{\cdot}5)^4 - 1{\cdot}5 - 10$
$= -\text{ve}$

$f(1{\cdot}8) = (1{\cdot}8)^4 - 1{\cdot}8 - 10$
$= 10{\cdot}5 - 1{\cdot}8 - 10$
$= -\text{ve}$

$f(1{\cdot}9) = (1{\cdot}9)^4 - 1{\cdot}9 - 10$
$= 13{\cdot}032 - 1{\cdot}9 - 10$
$= +\text{ve}$

$\therefore$ The root of (1) lies between 1·8 and 1·9.

First approximation—Let $i = 0$,

So $x_i = x_0$
$= 1{\cdot}8$ (say)

Then from (2)

$$x_1 = \frac{3x_0^4 + 10}{4x_0^3 - 1}$$

$$= \frac{3(1{\cdot}8)^4 + 10}{4(1{\cdot}8)^3 - 1}$$

$$= \frac{3(10{\cdot}498) + 10}{4(5{\cdot}832) - 1}$$

$$= \frac{41{\cdot}494}{22{\cdot}328}$$

$$= 1{\cdot}858$$

Second approximation—Let $i = 1$,

So $x_i = x_1$
$= 1{\cdot}858$

Then from (2),

$$x_2 = \frac{3(x_1^4) + 10}{4x_1^3 - 1}$$

$$= \frac{3(1{\cdot}858)^4 + 10}{4(1{\cdot}858)^3 - 1}$$

$$\Rightarrow \quad x_2 = \frac{3(11{\cdot}9174) + 10}{4(6{\cdot}414) - 1}$$

$$= \frac{45{\cdot}7522}{24{\cdot}656}$$

$$= 1{\cdot}8556$$

Third approximation—Let

$i = 2$,

So $x_i = x_2$
$= 1{\cdot}8556$

Then from (2),

$$x_3 = \frac{3(x_2)^4 + 10}{4x_2^3 - 1}$$

$$= \frac{(1{\cdot}8556)^4 + 10}{4\,(1{\cdot}8556)^3 - 1}$$

$$\Rightarrow \quad x_3 = \frac{3\,(11{\cdot}856) + 10}{4\,(6{\cdot}3893) - 1}$$

$$= \frac{45{\cdot}568}{24{\cdot}557}$$

$$= 1{\cdot}855602$$

The root of the equation is 1·855.

Example 7. By using Newton–Raphson's method (a) prove that the recurrence formula for solution of $x^n = a$ is x_{i+1}

$$= \frac{(n-1)\,x_i^n + a}{nx_i^{n-1}}$$

and (b) evaluate $x^n = 12$.

Solution : (a) Given

$$x^n = a$$

Let $\qquad f(x) \equiv x^n - a = 0;$

$$f'(x) = n \cdot x^{n-1}$$

By Newton–Raphson formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{x_i^x - a}{nx_i^{n-1}}$$

$$= \frac{nx_i^n - x_i^n + a}{nx_i^{n-1}}$$

$$= \frac{(n-1)\,x_i^n + a}{nx_i^{n-1}} \qquad \ldots(1)$$

(b) To evaluate $\quad x^n = 12$

We know $\quad (8)^{1/3} = (2)^{1/3} = 2$

$\therefore$ Taking $\quad a = 12$

and $\quad n = 3$ in (1)

$$x_{i+1} = \frac{2x_i^3 + 12}{3x_i^2} \qquad \ldots(2)$$

For first approximation—Let

$$x_0 = 2{\cdot}2$$

$$x_1 = \frac{2x_0^3 + 12}{3x_0^2}$$

$$= \frac{2(2{\cdot}2)^3 + 12}{3(2{\cdot}2)^2}$$

$$= \frac{2(10{\cdot}648) + 12}{3(4{\cdot}84)}$$

$$= \frac{33{\cdot}296}{14{\cdot}52}$$

$$= 2{\cdot}293$$

For second approximation—Let $i = 1$,

$$x_i = x_1$$

$$= 2{\cdot}293,$$

then $\qquad x_2 = \dfrac{2x_1^3 + 12}{3x_1^2}$

$$= \frac{2\,(2{\cdot}293)^3 + 12}{3(2{\cdot}293)^2}$$

$$= \frac{2(12{\cdot}0562) + 12}{3(5{\cdot}2578)}$$

$$= \frac{36{\cdot}1125}{15{\cdot}7785}$$

$$= 2{\cdot}2894$$

$\therefore$ The required value of x correct to three places of decimal, is 2·289.

Example 8. Solve $\dfrac{dy}{dx} = \dfrac{1}{x+y}$ for $x = 0{\cdot}5$ by using Runge-Kutta method with $x_0 = 0$, $y(x_0) = 1$, taking $h = 0{\cdot}5$.

Solution : Given $x_0 = 0$, $y_2 = 1$, $h = 0{\cdot}5 \quad \ldots(1)$

and $\qquad f(x, y) = \dfrac{1}{x+y} \qquad \ldots(2)$

$$\therefore \quad f(x_0, y_0) = \frac{1}{(x_0 + y_0)}$$

$$= \frac{1}{(0 + 1)}$$

$$= 1$$

Here $\qquad k_1 = hf(x_0, y_0)$

$$= (0{\cdot}5)\,(1)$$

$$= 0{\cdot}5 \qquad \ldots(3)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\; y_0 + \frac{1}{2}k_1\right)$$

$$= (0{\cdot}5)\,f\,(0 + 2{\cdot}5,\; 1 + 0{\cdot}25)$$

$$= (0{\cdot}5)\,f\,(0{\cdot}25,\; 1{\cdot}25)$$

$$= (0{\cdot}5)\left[\frac{1}{(1{\cdot}50)}\right]$$

$$= 0{\cdot}333 \qquad \ldots(4)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\; y_0 + \frac{1}{2}k_2\right)$$

$$= (0{\cdot}5)\,f\,(0{\cdot}25,\; 1{\cdot}167)$$

$$= (0{\cdot}5)\left[\frac{1}{(0{\cdot}25 + 1{\cdot}167)}\right]$$

$$= (0\cdot5)\left[\frac{1}{1\cdot417}\right]$$

$$= 0\cdot353 \qquad \ldots(5)$$

and $\quad k_4 = hf(x_0 + h, y_0 + k_3)$

$$= (0\cdot5)\, f(0\cdot5,\ 1\cdot353)$$

$$= (0\cdot5)\left[\frac{1}{(0\cdot5 + 1\cdot353)}\right]$$

$$= (0\cdot5)\left[\frac{1}{(1\cdot853)}\right]$$

$$= 0\cdot270 \qquad \ldots(6)$$

$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= \frac{1}{6}[0\cdot5 + 2(0\cdot333) + 2(0\cdot353) + 0\cdot27]$$

From (3), (5), (4), (6)

$$= \frac{1}{6}[0\cdot5 + 0\cdot666 + 0\cdot706 + 0\cdot27]$$

$$= \frac{1}{6}[2\cdot142]$$

$$= 0\cdot357$$

$\therefore \quad y_{n+1} = y_{n+k}$

$\Rightarrow \quad y_1 = y_0 + k$

$\Rightarrow \quad 1 + 0\cdot357 = 1\cdot357$

Thus the required value of y when $x = 0\cdot5$ is $1\cdot357$.

Example 9. Evaluate.

(a) $\frac{\Delta^2}{E}x^2$

(b) $\Delta(\sin x \cos 3x)$.

Solution : (a) $\frac{\Delta^2}{E}x^2$

$$= \Delta^2 E^{-1} x^2$$

$$= \Delta^2 (x-h)^2 \qquad [\because \; E^n y_x = y_{x+nh}]$$

$$= \Delta[\Delta(x-h)^2]$$

$$= \Delta[\{(x+h)-h\}^2 - \{x-h\}^2]$$

$$= \Delta[x^2 - (x-h)^2]$$

$$= \Delta[x^2 - (x^2 - 2xh - h^2)]$$

$$= \Delta(2xh - h^2)$$

$$= [\{2(x+h)h - h^2\} - \{2xh - h^2\}] \qquad [\because \Delta y_n = y_{x+h} - y_x]$$

$$= \{2xh + 2h^2 - h^3\} - (2xh - h^2\}$$

$$= 2h^2$$

(b) $\Delta(\sin x \cos 3x)$

$$= \Delta\left[\frac{1}{2}(2 \sin x \cos 3x)\right]$$

$$= \Delta\left[\frac{1}{2}(\sin 4x - \sin 2x)\right]$$

$$= \frac{1}{2}[\Delta(\sin 4x) - \Delta(\sin 2x)]$$

$$= \frac{1}{2}[\{\sin 4(x+h) - \sin 4x\} - \{\sin 2(x+h) - \sin 2x\}]$$

$$= \frac{1}{2}[2\cos(4x+2h)\sin 2h - 2\cos(2x+h)\sin h]$$

$$= \cos(4x+2h)\sin 2h - \cos(2x+h)\sin h$$

$$= (\sin h)[2\cos(4x+2h)\cos h - \cos(2x+h)]$$

Example 10. Using Picard's method solve $\frac{dy}{dx} = 1 - 2xy$, given $y = 0$ at $x = 0$, upto third approximation.

Solution : Here, $y_0 = 0$

Integrating the given equation between the limits given

$$[dy]_{y=0}^{y} = \int_{x=0}^{x}(1-2xy)\,dx$$

or $\quad y = \int_0^x (1-2xy)\,dx \qquad \ldots(i)$

and $\quad y_n = y_0 + \int_0^x f(x, y_{n-1})\,dx$

First approximation y_1, replace y by 0 in $(1 - 2xy)$, from (i),

$$y_1 = \int_0^\infty (1)\,dx = x \qquad \ldots(ii)$$

Second approximation y_2, replace y by 0 in $(1 - 2xy)$ from (i),

$$y_2 = \int_0^x (1 - 2xy_1)\,dx$$

$$= \int_0^x (1 - 2x^2)\,dx, \qquad \text{from (ii)}$$

$$= \left(x - \frac{2}{3}x^3\right)_0^x$$

$$= x - \frac{2}{3}x^3 \qquad \ldots(iii)$$

Third approximation y_3, replace y by y_2 in $(1 - 2xy)$ from (i),

$$y_3 = \int_0^x (1 - 2xy_2)\, dx$$

$$= \int_0^x \left[1 - 2x\left(x - \frac{2}{3}x^3\right)\right] dx,$$

from (iii)

or
$$y_3 = \int_0^x \left(1 - 2x^2 + \frac{4}{3}x^4\right) dx$$

$$= x - \frac{2}{3}x^3 - \frac{4}{15}x^5$$

Thus the solution of the given equation upto third approximation are

$$y_1 = x$$

$$y_2 = x - \frac{2}{3}x^3,$$

$$y^3 = x - \frac{2}{3}x^3 - \frac{4}{15}x^5$$

Example 11. Solve

$$\begin{aligned} 10x - 7y + 3z + 5u &= 6 \\ -6x + 8y - z - 4u &= 5 \\ 3x + y + 4z + 11u &= 2 \\ 5x - 9y - 2z + 4u &= 7 \end{aligned}$$

by Crout's triangularization method.

Solution : Here,

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

A = LU

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix}$$

So that

(i) R_1 of U : $u_{11} = 10, u_{12} = -7, u_{13} = 3, u_{14} = 5$

(ii) C_1 of L : $l_{21} = -0{\cdot}6,\ l_{31} = 0{\cdot}3,\ l_{41} = 0{\cdot}5$

(iii) R_2 of U : $u_{22} = 3{\cdot}8,\ u_{23} = 0{\cdot}8,\ u_{24} = -1$

(iv) C_2 of L : $l_{32} = 0{\cdot}815,\ l_{42} = 1{\cdot}447$

(v) R_3 of U : $u_{33} = 2{\cdot}447,\ u_{34} = 10{\cdot}315$

(vi) C_3 of L : $l_{43} = -0{\cdot}956$

(vii) R_4 of U : $u_{44} = 9{\cdot}924$

Thus

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0{\cdot}6 & 1 & 0 & 0 \\ 0{\cdot}3 & 0{\cdot}815 & 1 & 0 \\ 0{\cdot}5 & -1{\cdot}447 & -0{\cdot}956 & 1 \end{bmatrix} \begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3{\cdot}8 & 0{\cdot}8 & -1 \\ 0 & 0 & 2{\cdot}447 & 10{\cdot}315 \\ 0 & 0 & 0 & 9{\cdot}924 \end{bmatrix}$$

Writing UX = V, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0{\cdot}6 & 1 & 0 & 0 \\ -0{\cdot}3 & 0{\cdot}815 & 1 & 0 \\ 0{\cdot}5 & -1{\cdot}447 & -0{\cdot}956 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

Solving this system, we have

$v_1 = 6,\ v_2 = 8{\cdot}6,\ v_3 = -6{\cdot}815,\ v_4 = 9{\cdot}924$

Hence the original system becomes.

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3{\cdot}8 & 0{\cdot}8 & -1 \\ 0 & 0 & 2{\cdot}447 & 10{\cdot}315 \\ 0 & 0 & 0 & 9{\cdot}924 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 8{\cdot}6 \\ -6{\cdot}815 \\ 9{\cdot}924 \end{bmatrix}$$

$$\begin{aligned} 10x - 7y + 3z + 5u &= 6, \\ 3{\cdot}8y + 0{\cdot}8z - u &= 8{\cdot}6 \\ 2{\cdot}447z + 10{\cdot}315u &= -6{\cdot}815u = 1 \end{aligned}$$

So by back substitution, we get

$u = 1,\ z = -7,\ y = 4$ and $x = 5$.

Example 12. By using Runge-Kutta method find the value of y when $x = 1{\cdot}1$ given.

$$y(x_0) = 1{\cdot}2,\ x_0 = 1 \text{ and } \frac{dy}{dx} = 3x + y.$$

Solution : Given

$$x_0 = 1,\ y_0 = 1{\cdot}2,\ h = 0{\cdot}1 \qquad \ldots\text{(i)}$$

and $f(x, y) = 3x + y^2$ ...(ii)

So $f(x_0, y_0) = 3x_0 + y_0^2$

$$= 3(1) + (1{\cdot}2)^2$$

$$= 3 + 1{\cdot}44$$

$$= 4{\cdot}44$$

Here $k_1 = hf(x_0, y_0)$

$$= (0{\cdot}1)(4{\cdot}44)$$

$$= 0{\cdot}444 \qquad \ldots\text{(iii)}$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h \cdot y_0 + \frac{1}{2}k_1\right)$$

$$= (0{\cdot}1)\, f(1{\cdot}05,\ 1{\cdot}422)$$
$$= (0{\cdot}1)\,(3{\cdot}15 + 2{\cdot}022)$$
$$= (0{\cdot}1)\,(5{\cdot}172)$$
$$= 0{\cdot}517 \qquad \text{...(iv)}$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right)$$
$$= (0{\cdot}1)\, f(1{\cdot}05,\ 1{\cdot}459)$$
$$= (0{\cdot}1)\,(3{\cdot}15 + 2{\cdot}129)$$
$$= (0{\cdot}1)\,(5{\cdot}279)$$
$$= 0{\cdot}528 \qquad \text{...(v)}$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3)$$
$$= (0{\cdot}1)\, f(1{\cdot}1,\ 1{\cdot}728)$$
$$= (0{\cdot}1)\,(3{\cdot}3 + 2{\cdot}986)$$
$$= (0{\cdot}1)\,(6{\cdot}286)$$
$$= 0{\cdot}629 \qquad \text{...(vi)}$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0{\cdot}444 + 2\,(0{\cdot}517) + 2\,(0{\cdot}528) + 0{\cdot}629]$$
$$= \frac{1}{6}[3{\cdot}163] = 0{\cdot}527$$

$$\therefore \quad y_{n+1} = y_n + k$$
$$\Rightarrow \quad y_1 = y_0 + k$$
$$= 1{\cdot}2 + 0{\cdot}527 = 1{\cdot}727$$

Thus, the required value of y when $x = 1{\cdot}1$ is $1{\cdot}727$.

Example 13. By Gauss elimination method solve the system of equations,

$$x + 4y - z = -5$$
$$x + y - 6z = -12$$
$$3x - y - z = 4.$$

Solution : We have

$$x + 4y - z = -5 \qquad \text{...(i)}$$
$$x + y - 6z = -12 \qquad \text{...(ii)}$$
$$3x - y - z = 4 \qquad \text{...(iii)}$$

Step I. Here $r_1 = 1$ operate (ii)—(i) and $r_2 = 3$. (iii)—3 (i) to eliminate x,

$$-3y - 5z = -7 \qquad \text{...(iv)}$$
$$-13y + 2z = 19 \qquad \text{...(v)}$$

Step II. Here $r_3 = \frac{13}{3}$ operate (v) $- \frac{13}{3}$ (iv) to eliminate y,

$$\frac{71}{3}z = \frac{148}{3} \qquad \text{...(vi)}$$

Step III. By back-substitution, we get

From (vi), $\quad z = \frac{148}{71}$

From (iv), $\quad y = \frac{7}{3} - \frac{5}{3}\left(\frac{148}{71}\right) = -\frac{81}{71}$

From (i), $\quad x = -5 - 4\left(-\frac{81}{71}\right) + \frac{148}{71} = \frac{117}{71}$

$\therefore$ The solution is $x = \frac{117}{71}$, $y = -\frac{81}{71}$ and $z = \frac{148}{71}$

Example 14. Given

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find the value of $\int_0^{80} y\,dx$ by Trapezoidal rule.

Solution : The trapezoidal rule

$$\int_a^b y\,dx = h\left[\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right]$$

Taking $h = 10$, we have

$$\therefore \int_0^{80} y\,dx$$
$$= h\left[\frac{1}{2}y_0 + y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + \frac{1}{2}y_8\right]$$
$$= 10\left[\frac{1}{2}(0) + 4 + 7 + 9 + 12 + 15 + 14 + 8 + \frac{1}{2}(3)\right] \quad \text{(From table)}$$
$$= 10\,[69 + 1{\cdot}5]$$
$$= 10\,(70{\cdot}5)$$
$$= 705$$

The value of $\int_0^{80} y\,dx = 705$.

Example 15. Evaluate $\int_1^2 \log x\,dx$ by Trapezoidal rule.

Solution : Dividing the interval (1, 2) into five equal parts each of width $0{\cdot}2$.

The values of the function $y = \log_e x$ for each point of sub-division is given by

x	$y = \log x$	x	$y = \log x$
1·0	$0{\cdot}00000 = y_0$	1·6	$0{\cdot}20412 = y_3$
1·2	$0{\cdot}07918 = y_1$	1·8	$0{\cdot}25527 = y_4$
1·4	$0{\cdot}14613 = y_2$	2·0	$0{\cdot}30103 = y_5$

By Trapezoidal rule, we have

$$\int_1^2 \log x\, dx = \int_1^{1+5h} y\, dx$$

$$= h\left[\frac{1}{2}y_0 + y_1 + y_2 + y_3 + y_4 + \frac{1}{2}y_5\right]$$

$$= (0{\cdot}2)\left[\frac{1}{2}(0) + 0{\cdot}07918 + 0{\cdot}14613 \quad 0{\cdot}20412 + 0{\cdot}25527 + \frac{1}{2}(0{\cdot}30103)\right]$$

$$= (0{\cdot}2)\,[0{\cdot}07918 + 0{\cdot}14613 + 0{\cdot}20412 + 0{\cdot}25527 + 0{\cdot}15052]$$

$$= (0{\cdot}2)\,[0{\cdot}83522]$$

$$= 0{\cdot}16704$$

Example 16. Using Taylor's series method, find y to five places of decimals when $x = 1{\cdot}02$, given that

$$\frac{dy}{dx} = xy - 1 \text{ and } y = 2, \text{ when } x = 1.$$

Solution : Given

$$(y)_0 = 2 \qquad \ldots(i)$$

and $\quad y' = xy - 1 \qquad \ldots(ii)$

Differentiating (ii) successively,

$$y'' = xy' + y;$$
$$y''' = xy'' + 2y',$$
$$y^{(iv)} = xy''' + 3y'' \qquad \ldots(iii)$$

Put $x = 1$ and using (i), from (ii) and (iii), we have

$$(y')_0 = (y')_{x=1} = 1\,(2) - 1 = 1$$
$$(y'')_0 = (y'')_{x=1} = 1\,(1) + 2 = 3$$
$$(y''')_0 = (y''')_{x=1} = (1)\,(3) + 2\,(1) = 5$$
$$(y^{iv})_0 = (y^{iv})_{x=1} = (1)\,(5) + 3\,(3) = 14$$

By Taylor's series about $x = x_0$

$$y = (y_0) + (x - x_0)\,(y')_0 + \frac{(x-x_0)^2}{2\,!}(y'')_0 + \frac{(x-x_0)^3}{3\,!}(y''')_0 + \ldots\ldots$$

$$\Rightarrow y = 2 + (x-1)\,(1) + \frac{(x-1)^2}{2\,!}(3) + \frac{(x-1)^3}{3\,!}(5) + \frac{(x-1)^4}{4\,!}(14) + \ldots\ldots,$$

Since $x_0 = 1$ (given)

Put $x = 1{\cdot}02$ the required value of y is

$$y = 2 + (0{\cdot}02) + \frac{(0{\cdot}02)^2}{2}(3) + \frac{(0{\cdot}02)^3}{6}(5) + \frac{(0{\cdot}02)^4}{24}(14) + \ldots$$

$$= 2 + 0{\cdot}02 + 0{\cdot}0006 + 0{\cdot}0000067 + \ldots$$
$$= 2{\cdot}020606$$
$$= 2{\cdot}02061$$

Example 17. By Euler's method to initial value problem.

$\frac{dy}{dx} = x + y$, $y = 0$, when $x = 0$ in the range $x = 0$ to $x = 0{\cdot}04$

Taking $h = 0{\cdot}2$, find the value at y_1, y_2, y_3.

Solution : Taking $h = 0{\cdot}2$ split up the interval (0, 1) into sub-intervals of width 0·2 each with the help of $x_n = 0 + nh$ so that the points of division are given by

$x_0 = 0$, $x_1 = 0{\cdot}2$, $x_2 = 0{\cdot}4$, $x_3 = 0{\cdot}6$, $x_4 = 0{\cdot}8$, $x_5 = 1{\cdot}0$

Let us write

$$f(x, y) = x + y \qquad \ldots(i)$$

Here $x_0 = 0$, $y = 0$ and

$$h = 0{\cdot}2 \qquad \ldots(ii)$$

and we have to calculate y_5

$$\therefore \quad f(x_0, y_0) = x_0 + y_0 \text{ from (i)}$$
$$= 0 + 0 = 0$$
$$[\because x_0 = 0 = y_0]$$

Now $\quad y_1 = y_0 + hf(x_0, y_0)$

i.e., $\quad y_1 = 0 + (0{\cdot}2)\,(0) = 0$

$$y_1 = 0 \qquad \ldots(iii)$$

$$\therefore \quad f(x_1, y_1) = x_1 + y_1 \text{ from (i)}$$
$$= 0{\cdot}2 + 0 = 0{\cdot}2 \qquad \ldots(iv)$$

Now $\quad y_2 = y_1 + hf(x_1, y_1)$

$$= 0 + (0{\cdot}2)\,(0{\cdot}2), \text{ from (iv)}$$

or $\quad y_2 = 0{\cdot}04$

$$\therefore \quad f(x_2, y_2) = x_2 + y_2 \text{ from (i)}$$
$$= 0{\cdot}4 + 0{\cdot}04$$
$$= 0{\cdot}44$$

Now
$$\begin{aligned} y_3 &= y_2 + hf(x_2, y_2) \\ &= (0{\cdot}04) + (0{\cdot}02)(0{\cdot}44) \\ &= 0{\cdot}128 \end{aligned}$$

Thus we have $y_1 = 0$, $y_2 = 0{\cdot}04$ and $y_3 = 0{\cdot}128$.

Example 18. Using Taylor's series, find the solution of the differential equation

$xy' = x - y$ ($y(2) = 2$ at $x = 2{\cdot}1$ correct to five decimal places.

Solution : The first few derivatives and their values at $x = 2$, $y = 2$ are

$$y' = 1 - y/x \qquad y'_0 = 0$$

$$y'' = \frac{-y'}{x} + \frac{y}{x^2} \qquad y''_0 = \frac{1}{2}$$

$$y''' = \frac{-y''}{x} + \frac{2y'}{x^2} - \frac{2y}{x^3} \qquad y'''_0 = \frac{-3}{4}$$

$$y^{iv} = \frac{-y'''}{x} + \frac{3y''}{x^2} - \frac{6y'}{x^3} + \frac{6y}{x^4} \qquad y^{iv}_0 = \frac{3}{2}$$

The Taylor series expansion about $x_0 = 2$ is

$$\begin{aligned} y(x) &= y_0 + (x-2)\,y'_0 + \frac{1}{2}(x-2)^2\,y''_0 \\ &\quad + \frac{1}{6}(x-2)^3\,y'''_0 + \frac{1}{24}(x-2)\,y^{iv}_0 + \ldots\ldots \\ &= 2 + (x-2)\,0 + \frac{1}{4}(x-2)^2 \\ &\quad - \frac{1}{8}(x-2)^3 + \frac{1}{16}(x-2)^4 + \ldots\ldots \end{aligned}$$

At $x = 2{\cdot}1$, we obtain

$$\begin{aligned} y(2{\cdot}1) &= 2 + 0{\cdot}0025 - 0{\cdot}000125 \\ &\quad + 0{\cdot}0000062\ldots\ldots \\ &= 2{\cdot}00238 \end{aligned}$$

Example 19. Given

$$\begin{aligned} 3x + y + 2z &= 3 \\ 2x - 3y - z &= -3 \\ x - 2y + z &= 4 \end{aligned}$$

Solve it by matrix inversion method.

Solution : Here

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$\Leftrightarrow \quad AX = D$$

$$\therefore \quad X = \frac{1}{|A|}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix},$$

where $A_1, B_1, \ldots\ldots$ are cofactors of $a_1, b_1, \ldots\ldots$ in the determinant $|A|$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & 11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Thus $x = 1$, $y = 2$ and $z = -1$.

Example 20. $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$.

Solution : $\Delta \log f(x) = \log f(x+h) - \log f(x)$, by definition of Δ

$$\begin{aligned} &= \log\left[\frac{f(x+h)}{f(x)}\right] \\ &= \log\left[\frac{E\,f(x)}{f(x)}\right] \qquad [\because E\,u_x = u_{x+h}] \\ &= \log\left[\frac{(\Delta+1)\,f(x)}{f(x)}\right] \qquad [\because E = \Delta + 1] \\ &= \log\left[\frac{\Delta f(x) + f(x)}{f(x)}\right] \\ &= \log\left[\frac{f(x)}{f(x)} + \frac{\Delta f(x)}{f(x)}\right] \\ &= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right] \end{aligned}$$

Example 21. Prove that

$$\Delta\sqrt{u_x} = \frac{\Delta u_x}{\sqrt{u_x} + \sqrt{u_{x+h}}}.$$

Solution : $\Delta\sqrt{u_x} = \sqrt{u_{x+h}} - \sqrt{u_x}$

By definition of Δ

$$\begin{aligned} &= \frac{(\sqrt{u_{x+h}} - \sqrt{u_x})(\sqrt{u_{x+h}} + \sqrt{u_x})}{(\sqrt{u_{x+h}} + \sqrt{u_x})} \\ &= \frac{u_{x+h} - u_x}{\sqrt{u_x} + \sqrt{u_{x+h}}} \\ &= \frac{\Delta u_x}{\sqrt{u_x} + \sqrt{u_{x+h}}} \end{aligned}$$

OBJECTIVE TYPE QUESTIONS

1. Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, where $y(x_0) = y_0$. In Runge-Kutta method—
   (A) $y_{n+1} = y_n + \frac{1}{6}(k_1 + k_2 + k_3 + k_4)$
   (B) $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
   (C) $y_{n+1} = y_n + \frac{1}{4}(k_1 + 2k_2 + 3k_3 + 4k_4)$
   (D) None of these

2. Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, where $y(x_0) = y_0$. In Runge-Kutta method—
   (A) $k_3 = hf(x_n + h, y_n + k_2)$
   (B) $k_4 = hf(x_n + h, y_n + k_3)$
   (C) $k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$
   (D) None of these

3. Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$. where $y(x_0) = y_0$. In Runge-Kutta method—
   (A) $k_3 = hf(x_n + h, y_n + k_2)$
   (B) $k_3 = hf(x_n, y_n)$
   (C) $k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$
   (D) None of these

4. By Taylor's theorem, the series about a point $x = x_0$ is given by—
   (A) $y = y_0 + x_0 (y')_0 + \frac{x_0^2}{2!}(y'')_0 + \frac{x_0^3}{3!}(y'')_0 + \ldots\ldots$
   (B) $y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \ldots\ldots$
   (C) $y = y_0 + (x + x_0)(y')_0 + \frac{(x + x_0)^2}{2!}(y'')_0 + \frac{(x + x_0)^3}{3!}(y''')_0 + \ldots\ldots$
   (D) None of these

5. Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, where $y(x_0) = y_0$. In Runge-Kutta method—
   (A) $k_1 = hf(x_n)$ (B) $k_1 = hf(x_n, y_n)$
   (C) $k_1 = f(y_n)$ (D) None of these

6. In Newton-Cote's formula, if $f(x)$ is interpolated at equally spaced nodes by a polynomial of degree six, then it represents—
   (A) Trapezoidal rule
   (B) Simpson rule
   (C) Three-eight rule
   (D) Weddles rule

7. In Newton-Cotes formula, if (x) is interpolated at equally spaced nodes by a polynomial of degree one, then it represents—
   (A) Trapezoidal rule
   (B) Simpson rule
   (C) Three-eight rule
   (D) Booles rule

8. Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, where $y(x_0) = y_0$. In Runge-Kutta method—
   (A) $k_2 = hf(x_n + h, y_n + k_1)$
   (B) $k_2 = hf(x_n, y_n)$
   (C) $k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$
   (D) None of these

9. In Newton-Cotes formula, if $f(x)$ is interpolated at equally spaced nodes by a polynomial of degree four, then it represents—
   (A) Trapezoidal rule
   (B) Simpson rule
   (C) Three-eight rule
   (D) Booles rule

10. In Newton-Cotes formula, if $f(x)$ is interpolated at equally spaced nodes by a polynomial of degree three, then it represents—
   (A) Trapezoidal rule (B) Simpson rule
   (C) Three-eight rule (D) Booles rule

11. In Newton-Cotes formula, if $f(x)$ is interpolated at equally spaced nodes by a polynomial of degree two, then it represents—
    (A) Trapezoidal rule (B) Simpson rule
    (C) Three-eight rule (D) Booles rule

12. The n-divided difference of a polynomial of the nth degree are—
    (A) Constant (B) Zero
    (C) Variable (D) None of these

13. The n-th divided difference is defined as—
    (A) $\underset{x_1 \ldots x_n}{\Delta^n} y_0 = \sum_{i=0}^{n} \frac{y_i}{\prod_{\substack{j=0 \\ j \neq 1}}^{n} (x_i - x_j)}$
    (B) $\underset{x_1 \ldots x_n}{\Delta^n} y_0 = \sum_{i=0}^{n} \frac{y_i}{(x_i - x_j)}$
    (C) $\underset{x_1 \ldots x_n}{\Delta^n} y_0 = \sum_{i=0}^{n} \frac{y_1}{\prod_{j=0}^{n} (x_i - x_j)}$
    (D) None of these

14. Let $\tilde{a}$ is an approximation of exact value a and absolute error is $\in$, then exact value is given by—
    (A) $a = \varepsilon\tilde{a}$ (B) $a = \varepsilon + \tilde{a}$
    (C) $a = \frac{\varepsilon}{\tilde{a}}$ (D) None of these

15. Let $\tilde{a}$ is an approximation of axact value a, then absolute error ε, is defined as—
    (A) $\varepsilon = a\tilde{a}$ (B) $\varepsilon = \frac{a}{\tilde{a}}$
    (C) $\varepsilon = a - \tilde{a}$ (D) None of these

16. If $f(x, y)$ continuous at all points (x, y) is some rectangle R : $|x - x_0| < a$, $|y - y_0| < b$ and bounded in R, if $(x, y)| \leq k$, for all $(x, y) \in$ R, then the initial value problem has at least one solution $y(x)$. This is—
    (A) Uniqueness theorem for initial value problem
    (B) Existence theorem for initial value problem
    (C) Green's theorem
    (D) None of these

17. The bisection (Binary-search) method : Let f is continuous function defined on the interval $[a, b]$. with $f(a)$ and $f(b)$ of opposite sign—
    (A) By the intermidiate value theorem, there exists a number p in (a, b) with $f(p)$ equals to zero
    (B) By the intermidiate value theorem, there exists anumber p in (a, b) with $f(p)$ not equals to zero
    (C) By the intermidiate value theorem, there exists a number p in (a, b) with $f(p)$ equals to positive number
    (D) None of these

18. Subdivide the interval $a \leq x \leq b$ into mutiple of 6 of equal intervals $n = 6$m length $h = \frac{b-a}{6m}$ the integral $J = \int_a^b f(x)\, dx = \int_{x_0}^{x_0+nh} f(x)\, dx$ by Weddles rule is given by—
    (A) $\frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots\ldots)$
    (B) $\frac{3h}{10}(y_0 + 2y_1 + 3y_2 + 4y_3 + 5y_4 + 6y_5 + 7y_6 + 8y_7 + 9y_8 + \ldots\ldots)$
    (C) $(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots\ldots)$
    (D) None of these

19. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded, $|f| \leq k \left|\frac{\delta f}{\partial y}\right| \leq$ M for all $(x, y) \in$ R, then the initial value problem has at least one solution $y(x)$. This is—
    (A) Uniquencess theorem for initial value problem
    (B) Existence theorem for initial value problem
    (C) Greens theorem
    (D) None of these

20. Let f is ...... on $[a, b]$ and $f(a) f(b) < 0$. Then bisection method generates a sequence $\{P_n\}$ approximating a zero p of f with $|P_n - P| \leq \frac{b-a}{2^n}$, $n \geq 1$—

(A) Continuous function
(B) Discontinuous function
(C) Constant function
(D) None of these

21. Relation between divided difference and ordinary difference is—

(A) $\underset{x_1}{\Delta}\, y_0 = \dfrac{y_0 - y_1}{x_0 - x_1} = \dfrac{\Delta y_0}{h}$

(B) $\underset{x_1}{\Delta}\, y_0 = \dfrac{y_0 - y_1}{x_0 - x_1} = \Delta y_0$

(C) $\underset{x_1}{\Delta}\, y_0 = \dfrac{y_0 - y_1}{x_0 - x_1} = h\, \Delta y_0$

(D) None of these

22. Newton's divided difference formula is—

(A) $f(x) = y_0 + (x - x_0) f(x_0, x_1) + (x - x_0) (x - x_1) f(x_0, x_1, x_2) + \ldots\ldots + (x - x_0) \ldots\ldots (x - x_{n-1}) f(x_0, x_1 \ldots x_n) + R_n$

(B) $f(x) = y_0 + (x - x_0) + (x - x_0)(x - x_1) + \ldots + (x - x_0) \ldots (x - x_{n-1}) + R_n$

(C) $f(x) = y_0 + f(x_0, x_1) + f(x_0, x_1, x_2) + \ldots + f(x_0, x_1, \ldots x_n) + R_n$

(D) None of these

23. Relation between divided difference and ordinary difference is—

(A) $\underset{x_1 \ldots x_n}{\Delta^n}\, y_0 = \Delta y_0$

(B) $\underset{x_1 \ldots x_n}{\Delta^n}\, y_0 = \dfrac{\Delta y_0}{n\,!}$

(C) $\underset{x_1 \ldots x_n}{\Delta^n}\, y_0 = \dfrac{\Delta y_0}{n\,!\, h^n}$

(D) None of these

24. Given y_0, y_1, y_2, y_3 corresponding to values x_0, x_1, x_2, x_3 for function $y = f(x)$. Let $f(x)$ is a polynomial of degree 3, then by Simpson's three-eight rule, the integral $J = \int_a^b f(x) dx$ is equivalent to—

(A) $J = \dfrac{3}{8} h\, [y_0 + 3y_1 + 3y_2 + y_3]$

(B) $J = \dfrac{1}{3} h\, [y_0 + 4y_1 + y_2]$

(C) $J = \dfrac{1}{3} h\, [2y_0 + 4y_1 + 2y_2]$

(D) None of these

25. If the function $f(x, y)$ is bounded in some region about the point (x_0, y_0) and if $f(x, y)$ satisfies Lipschitz condition $|f(x, y) - f(x, \bar{y})| < k\, |y - \bar{y}|$ (k being a constant), then the sequence $y_1, y_2, \ldots$ the solution of initial value problem $y' = \dfrac{dy}{dx} = f(x, y)$, with $y(x_0) = y_0$—

(A) Converges (B) Diverges
(C) Oscillate (D) None of these

26. In Picard's method, given initial value problem $y' = \dfrac{dy}{dx} = f(x, y)$, with $y(x_0) = y_0$, n-th approximation is—

(A) $y_n = y_0 + \int_{x_0}^{x} f(x, y_n) dx$

(B) $y_n = y_0 + \int_{x_0}^{x} f(x, y) dx$

(C) $y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}) dx$

(D) None of these

27. In Predictor-Corrector method, given initial value problem $y' = \dfrac{dy}{dx} = f(x, y)$, where $y(x_0) = y_0$. Predicated and corrected values are respectively given by—

(A) $y^*_{n+1} = y_n + hf(x_n, y_n)$ and $y_{n+1} = y_n + h\, [f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]$

(B) $y_{n+1} = y_n + h\, [f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]$ and $y^*_{n+1} = y_n + hf(x_n, y_n)$

(C) $y^*_{n+1} = y_n + h$, $y_{n+1} = y_n + h\, [f(x_{n+1}, y^*_{n+1})]$

(D) None of these

28. Uniqueness theorem for initial value problem states—

(A) If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded, $|f| \le k \dfrac{|\partial f|}{\partial y} \le M$ for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(B) If $f(x, y)$ continuous at all points (x, y) is some rectangle R : $|x - x_0| < a$, $|y - y_0| < b$ and bounded in R. If $(x, y) | \le k$, for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(C) Both (A) and (B) above

(D) None of these

29. Newton backward difference formula is—

(A) $f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \nabla f(0)$

(B) $f(x) = \sum_{r=0}^{n} \nabla^r f(0)$

(C) $f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \nabla^r f(0)$

(D) None of these

30. In Euler's method—Given initial value problem $y' = \frac{dy}{dx} = f(x, y)$, with $y(x_0) = y_0$, then approximation is given by—

(A) $y_{n+1} = y_n + hf(x_{n-1}, y_{n-1})$, where $h = \frac{x_n - x_0}{n}$

(B) $y_{n+1} = y_n + hf(x_n, y_n)$, where $h = \frac{x_n - x_0}{n}$

(C) $y_{n+1} = y_n$

(D) None of these

31. Fixed-point theorem states that—

(A) Let g is a continuous function on $[a, b]$ such that $g(x) \in (a, b)$ for all $x \in [a, b]$ suppose in addition that g' exist on (a, b) and positive contant $k < 1$ exist with $|g'(x) \leq k$ for all $x \in (a, b)$, then for any number p in $[a, b]$, the sequence defined by $p_n = g(p_{n-1})$ $n \geq 1$, converges to the unique fixed point p in $[a, b]$

(B) If $f(x, y)$ an $\frac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded $|f| \leq k \left|\frac{\partial f}{\partial y}\right| \leq M$ for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(C) If $f(x, y)$ continuous at all points (x, y) is some rectangle $R : |x - x_0| < a, |y - y_0| < b$ and bounded in R, $/f(x, y) \leq k$, for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(D) None of these

32. Newton-Gregory forward difference interpolation formula is—

(A) $f(x + nh) = \sum_{r=0}^{n} {}^nC_r \Delta^r f(x)$

(B) $f(x + nh) \sum_{r=0}^{n} \Delta^r f(x)$

(C) $f(x + nh) = \sum_{r=0}^{n} {}^nC_r f(x)$

(D) None of these

33. Newton-Gregory advance difference formula is—

(A) $f(x) = \sum_{r=0}^{n} \Delta^r f(0)$

(B) $f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \Delta f(0)$

(C) $f(x) = \sum_{r=0}^{n} \frac{x^{(r)}}{r!} \Delta^r f(0)$

(D) None of these

34. Existence theorem for initial value problem states that—

(A) If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded $|f| \leq k \left|\frac{\partial f}{\partial y}\right| \leq M$ for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(B) If $f(x, y)$ continuous at all points (x, y) is some rectangle $R : |x - x_0| < a, |y - y_0| < b$ and bounded in R, $|f(x, y)| \leq k$, for all $(x, y) \in R$, then the initial value problem has at least one solution $y(x)$

(C) Both (A) and (B) above

(D) None of these

35. If $f(x)$ is three times differentiable and f', f'' are not zero at a solution of $f(x) = 0$, then for x_0 sufficiently close to s—

(A) Newton's method is of first order

(B) Newton's method is of second order*

(C) Newton's method is of third order

(D) None of these

36. Newton-Raphson method states that—

(A) If $f(x) = 0$, where f is assumed to have a continuous derivative

$$f', x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} *$$

(B) If $f(x) = 0$, where f is assumed to have a continuous derivative

$$f', x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$$

(C) If $f(x) = 0$, where f is assumed to have a continuous derivative $f', x_{n+1} = \frac{x(x_n)}{f'(x^n)}$

(D) None of these

37. The divided difference are……in all their arguments—

(A) Asymmetrical (B) Symmetrical
(C) Inverse (D) None of these

38. If g is continuous on interval $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$. If $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exist with $|g'(x)| \leq k$ or $g'(x) \leq k$ for all $x \in (a, b)$, then—

(A) The fixed point in $[a, b]$ is unique
(B) The fixed point in $[a, b]$ is not unique
(C) There is no fixed point in $[a, b]$
(D) None of these

39. If g is continuous on interval $[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then—

(A) g has a fixed point in $[a, b]$
(B) g has not a fixed point in $[a, b]$
(C) g has a fixed point in $[a, b]$
(D) None of these

40. A fixed point for a given function g is a number p for which—

(A) $g(p) = p$ (B) $g(p) = 0$
(C) $g(p) \neq p$ (D) None of these

41. Convergence of fixed point iteration states—

(A) Let $x = s$ be a solution of $x = g(x)$ and suppose that g has a continuous derivative in some interval J containing s. Then if $|g'(x)| \leq k < 1$ in J, the iteration process defined by $x_{n+1} = g(x_n)$ converges for any x_0 in J

(B) If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous for all (x, y) in the rectangle R and bounded $|f| \leq k \left|\frac{\partial f}{\partial y}\right| \leq M$ for all $(x, y) \in$ R, then the initial value problem has at least one solution $y(x)$

(C) If $f(x, y)$ continuous at all points (x, y) is some rectangle. R : $|x - x_0| < a$, $|y - y_0| < b$ and bounded in R, $|f(x, y)| \leq k$, for all $(x, y) \in$ R, then the initial value problem has at least one solution $y(x)$

(D) None of these

42. Fixed point theorem states : Let g is a continuous function on $[a, b]$ such that $g(x) \in [a, b]$ for all $x \in [a, b]$. Suppose, in addition, that g' exist on (a, b) and positive constant $k < 1$ exist with $|g'(x)| \leq k$ for all $x < (a, b)$. Then for any number p in $[a, b]$, the sequence defined by $p_n = g(p_{n-1})$ $n \geq 1$—

(A) Converges to the unique fixed point p in $[a, b]$
(B) Diverges to the unique fixed point p in $[a, b]$
(C) Converges to the different fixed point p in $[a, b]$
(D) Diverges to the different fixed point p in $[a, b]$

43. Everett's formula truncated after second differences is equivalent to……truncated after third differences—

(A) String's formula
(B) Bessel's formula
(C) Everett's formula
(D) None of these

44. $$y_u = v_{y_0} + \frac{v(v^2 - 1^2)}{3\,!}\Delta^2 y_{-1} + \frac{v(v^2 - 1^2)(v^2 - 2^2)}{5\,!}\Delta^4 y_{-2} + \ldots\ldots + uy_1 + \frac{u(u^2 - 1^2)}{3\,!}\Delta^2 y_0 + \frac{u[u^2 - 1^2(u^2 - 2^2)]}{5!}\Delta^4 y_{-1} + \ldots\ldots,$$

where $v = 1 - u$, represents—

(A) String's formula
(B) Bessel's formula
(C) Everett's formula
(D) None of these

45. Shifting the origin in Gauss's backward formula one have—

(A) String formula
(B) Bessel's formula
(C) Everett's formula
(D) None of these

46. $y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots,$

where $u = \frac{x - x_0}{h}$ represents—

(A) String's formula
(B) Bessel's formula
(C) Everett's formula
(D) None of these

47. The.........of Gauss's forward and backward and formula is known as Sterling's formula—

(A) Arithmetic mean
(B) Geometric mean
(C) Harmonic mean
(D) None of these

48. Gauss's backward formula for equal intervals is—

(A) $y_u = y_0 + {}^uC_1\,\Delta y_0 + {}^uC_2\,\Delta^2 y_{-1} + {}^{u+1}C_3\,\Delta^3 y_{-1} + {}^{u+1}c_4\Delta^4 y_{-2} + \ldots\ldots$

(B) $y_u = y_0 + {}^uC_1\,\Delta y_{-1} + {}^{u+1}c_2\Delta^2 y_{-1} + {}^{u+1}C_3\,\Delta^3 y_{-2} + {}^{u+1}C_4\,\Delta^4 y_{-2} + \ldots\ldots$

(C) $y_u = y_0 + \Delta y_{-1} + \Delta^2 y_{-1} + \Delta^3 y_{-2} + \Delta^4 y_{-2} + \ldots\ldots$

(D) None of these

49. Gauss's forward formula for equal intervals—

(A) $y_u = y_0 + {}^uC_1\,\Delta y_0 + {}^uC_2\,\Delta^2 y_{-1} + {}^{u+1}C_3\,\Delta^3 y_{-1} + {}^{u+1}c_4\Delta^4 y_{-2} + \ldots\ldots$

(B) $y_u = y_0 + {}^uC_1\,\Delta y_{-1} + {}^{u+1}c_2\Delta^2 y_{-1} + {}^{u+1}C_3\,\Delta^3 y_{-2} + {}^{u+1}C_4\,\Delta^4 y_{-2} + \ldots\ldots$

(C) $y_u = y_0 + \Delta y_{-1} + \Delta^2 y_{-1} + \Delta^3 y_{-2} + \Delta^4 y_{-2} + \ldots\ldots$

(D) None of these

50. Let $\tilde{a}$ an approximation of exact value a, then error bound β is defined as—

(A) $|a - \tilde{a}| = \beta$

(B) $|a - \bar{a}| \geq \beta$

(C) $|a - \tilde{a}| \leq \beta$

(D) None of these

51. $f(x) = f(0) + x\ \nabla f(0) + \frac{x(x+1)}{2!}\nabla^2 f(0) + \ldots\ldots + \frac{x(x+1)\ldots\ldots(x+n-1)}{n!}\nabla^n f(0)$

represents—

(A) Newton backward difference formula
(B) Newton forward difference formula
(C) Gauss's forward formula
(D) None of these

52. There is at most one polynomial of degree $\leq n$—

(A) Which interpolates, $f(x)$ at $(n+1)$ distinct points $x_0, x_1 \ldots\ldots, x_n$
(B) Which interpolates, $f(x)$ at n distinct points, $x_1 \ldots\ldots x_n$
(C) Which interpolates, $f(x)$ at $(n-1)$ distinct points $x_0,\ x_1, \ldots\ldots, x_{n-2}$
(D) None of these

53. Let h be the finite difference, factorial notation is defined as—

(A) $x^{(n)} = x(x-h), (x-2h)\ldots(x - \overline{n-1})h$

(B) $x^{(n)} = \frac{x!}{(x-n)!}, (n < x)$

(C) Both (A) and (B)
(D) None of these

54. In divided difference, the value of any difference is......... of the order of their arguments.

(A) Independent
(B) Dependent
(C) Inverse
(D) None of these

55. Which of the following is true for backward difference operator ?

(A) $\nabla^n f(x) = \sum_{r=0}^{n} {}^nC_r\, f(x - rh)$

(B) $\nabla^n f(x) = \sum_{r=0}^{n} (-1)^{n-r} f(x - rh)$

(C) $\nabla^n f(x) = \sum_{r=0}^{n} (-1)^{n-r}\,{}^nC_r\, f(x - rh)$

(D) None of these

56. Which of the following is true for backward difference operator ?

(A) $\nabla^2 f(x) = f(x-2h) - 2f(x-h) + f(x)$

(B) $\nabla^2 f(x) = f(x-2h) + 2f(x-h) + f(x)$
(C) $\nabla^2 f(x) = f(x-2h) - 2f(x-h) - f(x)$
(D) None of these

57. In interpolation if $x_0, x_1, \ldots\ldots x_n$ are $(n+1)$ distinct value of real valued function $f(x)$, then—
(A) One has a polynomial $p_n(x_i) \approx f(x)$ of degree n or more
(B) One has a polynomial $p_n(x_i) \approx f(x)$ of degree n exactly
(C) One has a polynomial $p_n(x_i) \approx f(x)$ of degree n or less
(D) None of these

58. Interpolation means—
(A) To find exact value of function $f(x)$ for an x between different x values $x_0, x_1, \ldots\ldots x_n$ at which the value $f(x)$ are given
(B) To find approximate value of function $f(x)$ for an x between different x values $x_0, x_1, \ldots\ldots x_n$ at which the value of $f(x)$ are given
(C) To find approximate value of function $f(x)$ for an x outside different x values $x_0, x_1, \ldots\ldots x_n$ at which the value of $f(x)$ are given
(D) To find exact value of function $f(x)$ for an x outside different x values $x_0, x_1, \ldots\ldots x_n$ at which the value of $f(x)$ are given

59. Which of the following is true ?
(A) $\mu = \frac{1}{2}[E^{1/2} + E^{1/2}]$
(B) $\mu = [E^{1/2} + E^{-1/2}]$
(C) $\mu = [E^{1/2} - E^{-1/2}]$
(D) None of these

60. Which of the following is true ?
(A) $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$
(B) $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$
(C) $\mu = \frac{1}{2}[E^{1/2} E^{-1/2}]$
(D) None of these

61. The central difference operator δ^2 is equal to—
(A) $\Delta \nabla$ (B) $\nabla \Delta$
(C) $\Delta - \nabla$ (D) All of these

62. Which of the following relation is true ?
(A) $\delta^n f(x) = \nabla f\left(x + \frac{1}{2}nh\right)$
(B) $\delta^n f(x) = \nabla^n f\left(x - \frac{1}{2}nh\right)$
(C) $\delta^n f(x) = \nabla^n f\left(x + \frac{1}{2}nh\right)$
(D) None of these

63. Which of the following relation is true ?
(A) $\delta^n f(x) = \Delta^n f\left(x + \frac{1}{2}nh\right)$
(B) $\delta^n f(x) = \Delta f\left(x + \frac{1}{2}nh\right)$
(C) $\delta^n f(x) = \Delta^n f\left(x - \frac{1}{2}nh\right)$
(D) None of these

64. Relation between central forward and shift operator is—
(A) $\delta \equiv \nabla E^{1/2} \equiv E^{1/2}\nabla$
(B) $\delta \equiv \nabla - E^{1/2} \equiv E^{1/2} - \nabla$
(C) $\delta \equiv \nabla + E^{1/2} \equiv E^{1/2} + \nabla$
(D) None of these

65. Relation between central forward and shift operator is—
(A) $\delta \equiv \Delta E^{-1/2} \equiv E^{-1/2}\Delta$
(B) $\delta \equiv \Delta - E^{-1/2} \equiv E^{-1/2} - \Delta$
(C) $\delta \equiv \Delta + E^{-1/2} = \equiv E^{-1/2} + \Delta$
(D) None of these

66. Central difference equivalent to shift operator—
(A) $\delta \equiv E^{1/2} - E^{-1/2}$ (B) $\delta \equiv E^{1/2} + E^{-1/2}$
(C) $\delta \equiv E^{1/2} E^{-1/2}$ (D) None of these

67. Which of the following relation is true ?
(A) $E \equiv \nabla^{-1}$ (B) $E \equiv (1+\nabla)^{-1}$
(C) $E \equiv (1-\nabla)^{-1}$ (D) None of these

68. Weierstrass approximation theorem states : for any continuous function $f(x)$ on an interval J : $a \le x \le b$ and error bound $\beta > 0$, there is a polynomial $p_n(x)$ (of sufficiently high degree) such that—

(A) $|f(x) - p_n(x)| = \beta$ for all $x \in J$
(B) $|f(x) - p_n(x)| > \beta$ for all $x \in J$
(C) $|f(x) - p_n(x)| < \beta$ for all $x \in J$
(D) None of these

69. Which of the following relation is true ?
(A) $E^{-1} \equiv 1 + \nabla$ (B) $E^{-1} \equiv \nabla$
(C) $E^{-1} \equiv 1 - \nabla$ (D) None of these

70. If $f(x)$ is a polynomial of degree n in x, then $(n + 1)$ and higher difference, this polynomial are—
(A) Constant (B) Variable
(C) Zero (D) None of these

71. If $f(x)$ is a polynomial of degree n in x, then nth difference of this polynomial is—
(A) Constant (B) Variable
(C) Zero (D) None of these

72. Which of the following relation is true ?
(A) $E^n f(x) = f(x + nh) \equiv \sum_{r=0}^{n} {}^{n-r}C_r \Delta^r f(x)$
(B) $E^n f(x) = f(x + nh) \equiv \sum_{r=0}^{n} {}^nC_r \Delta f(x)$
(C) $E^n f(x) = f(x + nh) \equiv \sum_{r=0}^{n} {}^nC_r \Delta^r f(x)$
(D) None of these

73. Which of the following relation is true ?
(A) $\Delta \equiv E - 1$
(B) $E^n \equiv (1 + \Delta)^n$
(C) Both (A) and (B)
(D) None of these

74. Relation between ∇, Δ and E is—
(A) $\nabla + E \equiv E + \nabla \equiv \Delta + E$
(B) $\nabla/E \equiv E/\nabla \equiv E/\Delta$
(C) $\nabla E \equiv E\nabla \equiv \Delta$
(D) None of these

75. Which of the following relation is true ?
(A) $E = 1 + \Delta$
(B) $\Delta^n \equiv (E - 1)^n$
(C) Both (A) and (B)
(D) None of these

76. The Averaging operator μ is—
(A) $\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{1}{2}h\right)\right]$
(B) $\mu f(x) = \frac{1}{2}\left[f\left(x - \frac{1}{2}h\right)\right]$
(C) $\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{1}{2}h\right) + f\left(x - \frac{1}{2}h\right)\right]$
(D) None of these

77. The central difference operator's δ is equivalent to—
(A) $\delta^n f(x) = \Delta^n f(xh)$
(B) $\delta^n f(x) = \Delta^n f\left(x + \frac{1}{2}nh\right)$
(C) $\delta^n f(x) = \Delta^n f\left(x - \frac{1}{2}nh\right)$
(D) None of these

78. The central difference operator δ is defined as—
(A) $\delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$
(B) $\delta f(x) = f\left(x + \frac{1}{2}h\right)$
(C) $\delta f(x) = f\left(x - \frac{1}{2}h\right)$
(D) None of these

79. Backward differences is defined as—
(A) $\nabla f(x) = -f(x - h)$
(B) $\nabla f(x) = f(x) + f(x - h)$
(C) $\nabla f(x) = f(x) - f(x - h)$
(D) None of these

80. Let h be the finite difference, then which of the following is true for shift operator ?
(A) $E^n f(x) = f(x)$
(B) $E^n f(x) = f(x + nh)$
(C) $E^n f(x) = f(x - nh)$
(D) None of these

81. Let h be the finite difference, then sift operator is defined as—
(A) $E f(x) = f(x)$
(B) $E f(x) = f(x - h)$
(C) $E f(x) = f(x + h)$
(D) None of these

82. Let h be the finite difference, then which of the following is true for forward difference operator ?
(A) $\Delta^n f(x) = \sum_{r=0}^{n} (-1)^{n-r}\, {}^nC_r f(x + rh)$

(B) $\Delta^n f(x) = \sum_{r=0}^{n} {}^nC_r f(x+rh)$

(C) $\Delta^n f(x) = \sum_{r=0}^{n} (-1)^{n-r} f(x+rh)$

(D) None of these

83. Let h be the finite difference, then which of the following is true for forward difference operator ?

(A) $\Delta^3 f(x) = f(x+3h) + 3f(x+2h) + 2f(x+h) + f(x)$

(B) $\Delta^3 f(x) = f(x+3h) - 3f(x+2h) + 2f(x+h) - f(x)$

(C) $\Delta^3 f(x) = f(x+3h) - 3f(x+2h) - 2f(x+h) - f(x)$

(D) None of these

84. Let h be the finite difference, then which of the following is true for forward difference operator ?

(A) $\Delta^2 f(x) = f(x+2h) + f(x)$

(B) $\Delta^2 f(x) = f(x+2h) + 2f(x+h) + f(x)$

(C) $\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$

(D) None of these

85. Let h be the finite difference, then f forward difference operator is defined as—

(A) $\Delta f(x) = f(x+h)$

(B) $\Delta f(x) = f(x+h) + f(x)$

(C) $\Delta f(x) = f(x+h) - f(x)$

(D) None of these

Answers

1. (B)	2. (B)	3. (C)	4. (B)	5. (B)
6. (D)	7. (A)	8. (B)	9. (D)	10. (C)
11. (B)	12. (A)	13. (A)	14. (B)	15. (C)
16. (B)	17. (A)	18. (A)	19. (A)	20. (A)
21. (A)	22. (A)	23. (C)	24. (A)	25. (A)
26. (C)	27. (A)	28. (A)	29. (C)	30. (B)
31. (B)	32. (A)	33. (C)	34. (A)	35. (B)
36. (A)	37. (B)	38. (A)	39. (A)	40. (A)
41. (A)	42. (A)	43. (B)	44. (C)	45. (B)
46. (A)	47. (A)	48. (B)	49. (A)	50. (C)
51. (A)	52. (A)	53. (C)	54. (A)	55. (C)
56. (A)	57. (C)	58. (B)	59. (A)	60. (A)
61. (D)	62. (C)	63. (C)	64. (A)	65. (A)
66. (A)	67. (C)	68. (C)	69. (C)	70. (C)
71. (A)	72. (C)	73. (C)	74. (C)	75. (C)
76. (C)	77. (C)	78. (A)	79. (C)	80. (C)
81. (C)	82. (A)	83. (B)	84. (C)	85. (C)

Partial Differential Equations

Introductions

A partial differential equation for $u(x, y, \ldots\ldots)$ is a relationship between u and its partial derivatives $u_x, u_y, u_{xy}, u_{yy}, \ldots$

It can be written as

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots\ldots) = 0$$

where F some is function, $x, y, \ldots\ldots$ are independent variables and $u(x, y, \ldots\ldots)$ is called a dependent variable.

Order of partial differential equation—Order of the partial differential equation is the order of the highest derivative appears in it.

Homogeneous and non-homogeneous partial differential equation—If each term of a partial differential equation contains either the dependent variable or its partial derivative, the equation is called homogeneous otherwise non-homogeneous.

First order partial differential equation—

$$F(x, y, u, u_x, u_y) = 0$$

Second order partial differential equation :

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$$

Linear partial differential equation— A partial differential equation is linear, if it is linear in the unknown function and all its derivatives with coefficients depends only on the independent variable.

Quasi-linear partial differential equation— It is quasi-linear, if it is linear in the highest order derivative of the unknown function.

Operator form—The partial differential equation can also be written in the operator form, as $L_x\, u(x) = f(x)$, where L_x is an operator.

(1) The operator L_x is linear operator, for any two function u and v and for any two constant a and b, if

$$L_x(au + bv) = a\, L_x u + b\, L_x v$$

(2) $L_x\, u(x) = f(x)$ is linear if L_x is a linear operator.

(3) $L_x\, u(x) = f(x)$ is in homogeneous (non-homogeneous) equation.

(4) An equation which is not linear is non-linear equation.

Solution of partial differential equation—A classical solution of partial differential equation $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots\ldots) = 0$ is an ordinary function $u \approx u(x, y, \ldots\ldots)$ defined in some domain D, which is continuously differentiable such that its all partial derivatives involved in the equation exist and satisfy $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots\ldots) = 0$ identically.

Weak (or generalised) solution—The solution $u = u(x, y, \ldots\ldots)$ is called a weak (or generalized) solution of $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots\ldots) = 0$ if u or its partial derivatives are discontinued in some or all points in D.

General solution—A general solution of partial differential equation is an expression that involves arbitrary functions.

For two independent variables x, y the solution $u = u(x, y)$ of partial differential equation $F(x, y, u, u_x, u_y) = 0$ is visualized geometrically as a surface and called an integral surface in (x, y, u) space.

Linear superposition principle—The general solution of linear homogeneous ordinary differential equation of order n is a linear combination of n linearly independent solutions with n arbitrary constants.

(1) If $u_1(x), u_2(x), \ldots\ldots u_n(x)$ are n independent (linear) solutions of nth order linear homogeneous ordinary differential equation $L_u(x) = 0$. Then for any arbitrary constants $c_1, c_2, \ldots\ldots, c_n$, $u(x) = \sum_{k=1}^{n} c_k u_k(x)$ represents the most general solution of $L_\mu(x) = 0$. This is called the linear superposition principle for ordinary differential

equation. The general solution of $L_\mu(x) = 0$ depends an exactly n-arbitrary constants.

(2) In linear homogeneous partial differential equation $L_x u(x) = 0$ the general solution depends on arbitrary function rather that arbitrary constants. So there are infinite many solution of $L_x u(x) = 0$. If $u_1(x)$, $u_2(x)$, ... are infinite solutions, then $u(x) = \sum_{k=1}^{\infty} c_k, u_k(x)$ where c_k are arbitrary constants, is a solution if the infinite series is convergent.

Boundary Conditions

(1) **Dirichlet conditions**—Where u is prescribed by each point of a boundary δD of a domain D. The problem of finding the solution of a given equation $L_x u(x) = 0$ inside D with prescribed values of u on δD is called the Dirichlet boundary value problem.

(2) **Neumann conditions**—Where value of normal derivative $\frac{\delta u}{\delta n}$, on the boundary δD are specified. The problem is called Neumann boundary value problem.

(3) **Robin condition**—Where $\left(\frac{\delta u}{\delta n} + au\right)$ is specified on δD. The problem is called Robin boundary value problem.

Well posed problems—A problem described by a partial differential equation in a given domain with a set of initial and/or boundary conditions (or other supplementary conditions) is said to be well posed (or properly posed) provided the following criteria are satisfied—

(1) **Existence**—There exists at least on solution of the problem.

(2) **Uniqueness**—There is at most one solution.

(3) **Stability**—The solution must be stable in the sense that it depends continuously on the data, *i.e.*, small change in the given data must produce a small change in the solution.

Classical linear model equations—

(1) **Wave equation**—$u_{tt} - c^2 \nabla^2 u = 0$,

where $\nabla^2 \equiv \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2}$ and c is a constant.

(2) **Heat (Diffusion equation)**—$u_t - k\nabla^2 u = 0$, k is constant of diffusivity.

(3) **Laplace equation**—$\nabla^2 u = 0$

(4) **Poisson equation**—$\nabla^2 u = f(x, y, x)$ $\nabla^2 u = f(z, y, z)$

(5) **Helmholtz equation**—$\nabla^2 u + \lambda u = 0$, λ is constant.

(6) **Telegraph equation**—$u_{tt} + au_t + b\,u = u_{xx}$ a, b are constant.

(7) **Klien-Gordon equation**—

$$\psi + \left(\frac{mc^2}{h}\right)\psi = 0,$$

where $\equiv \frac{\delta^2}{\delta t^2} - c^2\nabla^2$ is the d' Alembertian.

(8) **Korteweg-derives (or $k\,dv$) equation**—

$$u_t + a\,u_x + b\,u_{xx} = 0,$$

where a, b are constant.

(9) **Linear Boussinesq equation**—

$$u_{tt} - a^2\nabla^2 u - b^2\nabla^2 u_{tt} = 0,$$

where a, b are constant.

(10) **Bi-hormonic wave equation**—

$$u_{tt} + c^2\nabla^4 u = 0,$$

where a is a constant.

Partial differential equation of first order

First order partial differential equation—A first order partial differential equation for $z(x, y)$ is a relationship between z and its partial derivatives $p = \frac{\delta z}{\delta x}$, $q = \frac{\delta z}{\delta y}$. It can be written as, $F(x, y, z, p, q) = 0$, where F some is function x, y are independent variable and $z(x, y)$ is called a dependent variable.

Formulation of partial differential equation—

(1) **Elimination of arbitrary constants**—Given $f(x, y, u, a, b) = 0$, where a, b arbitrary constants are. One can form $F(x, y, z, p, q) = 0$ be eliminating arbitrary constants a, b.

(2) **Elimination of arbitrary functions**—Given $\phi(u, v) = 0$ where $u = u(x, y, z)$ and $v = v(x, y, z)$. One can form a partial differential equation $Pp + Qq = R$, by eliminating arbitrary function $\phi(u, v)$. Here P, Q, R are the functions of x, y, z.

Solution to first order partial differential equation—F $(x, y, z, p, q) = 0$

(1) **Complete integral**—Any relation $f(x, y, u, a, b) = 0$ where a, b arbitrary constants, is called a complete integral.

(2) **Particular integral**—A solution obtained from the complete integral by assigning particular values to arbitrary constants is called a particular integral.

(3) **General integral**—In complete integral $f(x, y, u, a, b) = 0$ where $b = \phi(a)$, one have an envelope of the family of surface $f(x, y, u, a, \phi(a)) = 0$, then the solution containing an arbitrary function $\phi(a)$ is called the general integral.

(4) **Singular integral**—An envelope of the family of surface $f(x, y, u, a, b) = 0$, with parameters a, b, if exist is called a singular integral. Singular integral differs from the particular integral in the sense that they can not be obtained from complete integral by giving particular value to the constants.

Linear first order partial differential equation—If degree of p and q are unity, then first order partial differential equation is linear.

Lagrange's equation—(1) A first order linear partial differential equation of the form $Pp + Qq = R$.

(2) If P, Q, R are the functions of x, y, z, then it is called (quasi linear) Lagrange's equation.

(3) If P, Q, R are the functions of x and y only, then it is called (linear) Lagrange's equation.

(4) Equation $Pp + Qq = R$ gives subsidiary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Find two independent integrals of subsidiary equation (say) $u = a$ and $v = b$, the general integral of the equation is given by $\phi(u, v) = 0$ or $u = \phi(v)$, where ϕ is arbitrary function.

Non-linear first order partial differential equation—If degree of p and q are other than unity, then first order partial differential equation is non-linear.

(1) Given F $(p, q) = 0$, complete integral $z = ax + by + c$, where F $(a, b) = 0$.

(2) Given F $(z, p, q) = 0$, let $u = x + ay$, then $p = \frac{\delta z}{\delta u}$, $q = a\frac{\delta z}{\delta u} \Rightarrow F\left(z, \frac{\delta z}{\delta u}, a\frac{\delta z}{\delta u}\right)$.

This is the ordinary differential equation of first order $\frac{dz}{du} = \phi(z, a)$ integration gives a complete integral $f(z, a) = u + b$.

(3) Given $f(x, p) = F(y, q)$. Let $f(x, p) = F(y, q) = a$.

Solving for p, q gives $p = \phi(x)$, $q = \phi(y)$ and using the relation, $dz = p\,dx + q\,dy$, the complete integral is $z = ax + by + F(a, b)$.

(4) **Charpits method**—Given F $(x, y, z, p, q) = 0$. Subsidiary equations

$$\frac{dx}{-\frac{\delta F}{\delta p}} = \frac{dy}{-\frac{\delta F}{\delta q}} = \frac{dz}{-p\frac{\delta F}{\delta p} - q\frac{\delta F}{\delta q}} = \frac{dp}{\frac{\delta F}{\delta x} + p\frac{\delta F}{\delta z}}$$

$$= \frac{dq}{\frac{\delta F}{\delta y} + q\frac{\delta F}{\delta z}} = \frac{dF}{0}.$$

Solving for p, q and substituting in $dz = p\,dx + q\,dy$, a complete integral is $f(x, y, z, a, b) = 0$.

Partial differential equation with constant coefficients—Homogeneous linear partial differential equation of n order with constant coefficient :

$$\frac{\delta^n z}{\delta x^n} + A_1\frac{\delta^n z}{\delta x^{n-1}\,\delta y} + A_2\frac{\delta^n z}{\delta x^{n-2}\,\delta y^2} + A_3\frac{\delta^n z}{\delta x^{n-3}\,\delta y^3} + \ldots\ldots + A_n\frac{\delta^n z}{\delta y^n} = F(x, y),$$

where F some is function, x, y are independent variables, $z(x, y)$ is a dependent variable and $A_1, A_2, \ldots\ldots A_n$ are arbitrary constants.

(1) The expression is equivalent to

$$(D^n + A_1 D^{n-1} D^1 + A_2 D^{n-2} D^2 + A_3 D^{n-3} D^3 + \ldots\ldots + A^n D^m)\,z$$

$$= F(x, y) \Leftrightarrow f(D, D^1)\,z$$

$$= F(x, y),$$

where $D^r \equiv \frac{\delta}{\delta x^r}, D^{1r} \equiv \frac{\delta}{\delta y^r}$

(2) It is homogeneous because all terms contain derivatives of the same order. Solution is same as for ordinary differential equation with constant coefficients.

(3) Complete integral = complementary function + particular integral = $z_k + z_p$.

(4) **Complementary function**—$(D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + A_3 D^{n-3} D'^3 + \ldots\ldots + A_n D'^n)\,z = 0$.

Gives auxilliary equation—$m^n + A_1 m^{n-1} + A_2 m^{n-2} + A_3 m^{+ \cdots\cdots +} A_n = 0$.

(5) $m_1, m_2, m_3, \ldots\ldots m_n$ (distinct roots) the general solution is $z_k = f_1(y + m_1 x) + f_2(y + m_2 x) + \ldots\ldots + f_n(y + m_n x)$.

(6) $m_1, \ldots\ldots m_1, m_{r+1}, \ldots\ldots m_n$ (r equal roots) then complementary function is

$$z_k = f_1(y + m_1 x) + x f_2(y + m_1 x) + \ldots\ldots + x^{r-1} f_r(y + m_1 x) f_{r+1}(y + m_{r+1} x) + f_n(y + m_n x)$$

Particular integral—

F (x, y)	Z_p	
e^{ax+by}	$\frac{1}{f(D, D')} e^{ax+by}$	Substitute $D = a, D' = b$
$\sin(mx + ny)$	$\frac{1}{f(D^2, DD', D'^2)} \sin(mx + ny)$	Substitute $D^2 = m^2, D'^2 = -n^2, DD' = mn$
$\cos(mx + ny)$	$\frac{1}{f(D^2, DD', D'^2)} \cos(mx + ny)$	Substitute $D^2 = m^2, D'^2 = -n^2, DD' = mn$
$x^m y^n$	$\frac{1}{f(D, D')} x^m y^n$	Expand $\frac{1}{f(D, D')}$ and operate on $x^m y^n$ term by term.

Non-Homogeneous linear partial differential equation of n order with constant coefficient—If in equation $f(D, D') z = F(x, y)$ the polynomial $f(D, D')$ is not homogeneous, then equation is called non-homogeneous equation. Complete integral = complementary function + particular integral = $Z_k + Z_p$.

(2) **Complementary function**—Factorized $f(D, D')$ into factors of the form

$$(D - mD' - c) z = 0$$

$$\Rightarrow \quad (P - mq) = cz$$

$$\Rightarrow \quad \frac{dz}{1} = \frac{dy}{-m} = \frac{dz}{cz}$$

Integration gives $y + mx = a$ and $z = be^{cx}$

Take $b = \phi(a)$ which gives $z = e^{cx} \phi(y + mx)$

Partial Differential Equation of Second Order

Second order partial differential equation—The general second order partial differential equation for $z(x, y)$ is given by

$$Rr + Ss + Tt + Pp + Qq + Zz = F,$$

where R, S, T, P, Q, Z, F are function of x, y or constant and

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial p}{\partial x}, s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}, t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial q}{\partial y}.$$

Monge's Method

(1) Given second order partial differential equation

$$Rr + Ss + Tt = F,$$

where R, S, T, F are function of x, y or constant.

Then Monge's subsidiary equation are

$$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0 \quad \ldots(1)$$

$$R\,dP\,dy + T\,dq\,dx - F\,dx\,dy = 0 \quad \ldots(2)$$

Since (2) is quadratic it gives two roots m_1, m_2 and one have $dy - m_1\,dx = 0$

and $\quad dy - m_2\,dx = 0$

or $\quad m_1 \neq m_2$

(1) and $dy - m_1\,dx = 0$ gives integral $u_1 = f(v_1)$

(2) (2) and $dy - m_2\,dx = 0$ gives integral $u_2 = f(v_2)$ which on solving gives general integral.

or $\quad m_1 = m_2$:

The equation reduces to Lagrange's equation $Pp + Qq = R$.

Classification of second order linear partial differential equations

Method of characteristics—The general second order linear differential equation in two independent variables x, y is given by

$$Rr + Ss + Tt + Pp + Qq + Zz = F,$$

where R, S, T, P, Q, Z F are function of x, y or constant. The given equation is

(1) **Hyperbolic equation**—$S^2 - 4RT > 0$

(2) **Parabolic equation**—$S^2 - 4RT = 0$

(3) **Elliptic equation**—$S^2 - 4RT < 0$

Characteristic equations—The characteristic equation of $Rr + Ss + Tt + Pp + Qq + Zz = F$ are

$$\frac{dy}{dx} = \frac{1}{2R}(S \pm \sqrt{S^2 - 4RT})$$

and their solutions are called the **characteristic curves** or simply the characteristic of equation

$$Rr + Ss + Tt + Pp + Qq + Zz = F$$

Characteristics curves—The solution of two ordinary differential equations

$$\frac{dy}{dx} = \frac{1}{2R}(S \pm \sqrt{S^2 - 4RT})$$

defines two distinct families of characteristic curves $\phi(x, y) = c_1$ and $\phi(x, y) = c_2$. There are three possible case to consider.

Hyperbolic equation : $S^2 - 4RT > 0$

Integrating $\frac{dy}{dx} = \frac{1}{2R}(S \pm \sqrt{S^2 - 4RT})$ gives two real and distinct famelies of characteristics curves $\phi(x, y) = C_1$ and $\phi(x, y) = c_2$ where c_1, c_2 are constants of intergration

Parabolic equation : $S^2 - 4RT = 0$

There is only one family of real characteristic curves whose slope, due to

$$\frac{dy}{dx} = \frac{1}{2R}(S \pm \sqrt{S^2 - 4RT}) \text{ is given by}$$

$$\frac{dy}{dx} = \frac{S}{2R}$$

Intergration of this equation gives $\phi(x, y) = \xi$ = constant or $\phi(x, y) = \eta$ = constant

Elliptic equation : $S^2 - 4RT < 0$

In this case equation $\frac{dy}{dx} = \frac{1}{2R}(S \pm \sqrt{S^2 - 4RT})$ have no real solutions; there are two families of complex characteristics.

Usually boundary value problems are associated with elliptic equations and initial-value problems arises in connection with heperbolic and parabolic equations.

Cononical Forms

Given general second order linear differential equation in two independent variables x, y is given by $Rr + Ss + Tt + Pp + Qq + Zz = F$ ...(1)

where R, S, T, P, Q, Z, F are function of x, y or constant.

Let $\xi = \xi(x, y)$ and $\eta(x, y)$ ...(2)

Then the transformation gives

$$\overline{R}\, z_{\xi\xi} + \overline{S}\, z_{\xi\eta} + \overline{T}\, z_{\eta\eta} + \overline{P}\, z_{\xi} + \overline{Q}\, z_{\eta} + \overline{Z}\, z = \overline{F} \quad ...(3)$$

where

$$\overline{R} = R\xi^2_x + B\xi_x\xi_y + C\xi_y^2$$

$$\overline{S} = 2R\xi x\eta x + S(\xi x\eta y + \xi_y\eta_x) + 2T\xi y\eta y$$

$$\overline{T} = R\eta_x^2 + S\eta x\eta y + T\eta y^2$$

$$\overline{P} = R\xi xx + S\xi xy + T\xi yy + P\xi x + Q\xi y \quad ...(4)$$

$$\overline{Q} = R\eta_{xx} + S\eta_{xy} + T\eta_{yy} + P\eta_x + Q\eta_y$$

$$\overline{Z} = Z$$

$$\overline{F} = F$$

Also since the classification of (1) depends only on R,S,T equation (1) can be reduced to

$Rr + Ss + Tt = H(x,y,z,p,q)$ ...(5)

which under the transformation of (2) gives there canonical forms,

Hyperbolic equation :

$Z_{\xi\xi} - Z_{\eta\eta} = \phi(\xi, \eta, Z, Z_\xi, Z_\eta)$

or $Z_{\xi\eta} = \phi(\xi, \eta, Z, Z_\xi, Z_\eta)$

Elliptic equation :

$Z\xi\xi + Z\eta\eta = \phi(\xi, \eta, Z, Z\xi, Z\eta)$

Parabolic equation :

$Z\xi\xi = \phi(\xi, \eta, Z, Z\xi, Z\eta)$

or $Z\eta\eta = \phi(\xi, \eta, Z, Z\xi, Z\eta)$

Laplace Equations :

$$\nabla^2 = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Separation of Variable :

Given $\nabla^2 = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ ...(1)

Let $Z(x,y) = X(x)\, Y(y)$ be a solution of (1) substituting it in (1), one have

$$\frac{d^2X}{dx^2}Y + X\frac{d^2Y}{dy^2} = 0$$

and separating the variables gives

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{-1}{Y}\frac{d^2y}{dy^2} = k \quad ...(2)$$

Case I. $k = P^2$, P is real

$$\therefore \quad \frac{d^2X}{dx^2} - Px^2X = 0$$

and $$\frac{d^2y}{dy^2} + P^2y = 0$$

which gives $X = c_1e^{Px} + c_2e^{-Px}$

and $y = c_3 \cos Py + c_4 \sin Py$

$\therefore$ The solution is

$$z(x, y) = (c_1e^{Px} + c_2e^{-Px})(c_3 \cos Py + c_4 \sin Py)$$

Case II. $k = 0$

$$\frac{d^2x}{dx^2} = 0 \text{ and } \frac{d^2y}{dy^2} = 0$$

which gives $X = c_5x + c_6$ and $Y = c_7y + c_8$

The solution is

$$Z(x, y) = (c_5x + c_6)(c_7y + c_8)$$

Case III. $k = -P^2$, P is real

$$\therefore \quad \frac{d^2X}{dx^2} + P^2X = 0$$

and $$\frac{d^2y}{dy^2} - P^2Y = 0$$

which gives $X = c_q \cos px + c_{10} \sin px$ and $Y = c_{11}\, e^{Py} + c_{12}\, e^{-Py}$

$\therefore$ The solution is $Z(x, y) = (c_9 \cos Px + c_{10} \sin Px)(c_{11}\, e^{Py} + c_{12}\, e^{-Py})$

Here all c_i (i = 1, 2, ……, 12) can be calculated by using boundary conditions.

Dirichlet Problem for Rectangle

Given partial differential equation :

$$\nabla^2 = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,\ 0 \le x \le a,\quad 0 \le y \le b$$

Boundary conditions : $Z(x, b) = Z(a, y) = 0$,

$Z(0, y) = 0,\ Z(x, 0) = f(x)$

The possible solution is

$$Z(x, y) = (c_9 \cos Px + c_{10} \sin Px)(c_{11}\, e^{Py} + c_{12}\, e^{-Py})$$

and required solution is

$$Z(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \sin\left(n\pi \frac{y-b}{a}\right)$$

where $$A_n = \frac{2}{a \sin h\left(\frac{-n\pi b}{a}\right)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

Neumann Problem for a Rectangle

Given Partial differential equation :

$$\nabla^2 = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,\ 0 \le x \le a,\quad 0 \le y \le b$$

Boundary Condition :

$P(x, b) = q(a, y) = 0,$

$q(x, 0) = 0,\ q(x, b) = f(x)$

The possible solution is :

$$Z(x, y) = (c_1 \cos Px + c_2 \sin Px)(c_3\, e^{Py} + c_4\, e^{-Py})$$

The required solution is :

$$Z(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right) \cosh\left(\frac{n\pi y}{a}\right)$$

where $$An = \frac{2}{a} \frac{1}{\sin h\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx$$

Wave Equation

Wave equation is

$$\frac{\partial^2 z}{\partial y^2} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Separation of varibles :

Given $$\frac{\partial^2 z}{\partial y^2} = c^2 \frac{\partial^2 z}{\partial x^2} \quad \ldots(1)$$

Let $Z(x, y) = X(x)\, Y(y)$ be a solution of (1) substituting it in (1) one have

$$\frac{d^2X}{dx^2} + kX = 0 \text{ and } \frac{d^2Y}{dy^2} - kc^2y = 0$$

Case I. $k = P^2$, P is real

$$X = c_1e^{Px} + c_2e^{-Px}$$

and $$Y = c_3e^{cPy} + c_4e^{-cPy}$$

$\therefore$ The solution is

$$Z(X, Y) = (c_1e^{Px} + c_2e^{-Px})(c_3e^{cPy} + c_4e^{-cPy})$$

Case II. $k = 0$

$$X = c_5x + c_6 \text{ and } Y = c_7y + c_8$$

$\therefore$ The solution is

$$Z(x, y) = (c_5x + c_6)(c_7y + c_8)$$

Case III. $k = -P^2$, P is real

$$X = c_9 \cos Px + c_{10} \sin Px$$

$$Y = c_{11} \cos cPy + c_{12} \sin cPy$$

$\therefore$ The solution is

$$Z(X, Y) = (c_9 \cos Px + c_{10} \sin Px)(c_{11} \cos cPy + c_{12} \sin cPy)$$

Here all c_i (i = 1, 2, …… 12) can be calculated by using boundary condition.

Vibrating String :

Given $\frac{\delta^2 z}{\partial y^2} = c^2 \frac{\partial^2 z}{\partial x^2}; 0 \le x \le a, 0 > y$

Boundary Conditions :

$$Z(0, y) = Z(a, y) = 0$$

Initial condition

$$Z(x, 0) = f(x), q(x, 0) = g(x)$$

The possible solution is

$$Z(x, y) = (c_9 \cos px + c_{10} \sin px)(c_{11} \cos cpy + c_{12} \sin cpy)$$

The required solution is

$$Z(x, 0) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right)$$

where $A_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right)dx$

$$\frac{\partial z(x, 0)}{\partial y} = g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)$$

where $B_n = \frac{2}{n\pi c}\int_0^a g(x) \sin\frac{n\pi x}{a}dx$

Heat Equation

Wave equation is :

$$\frac{\partial z}{\partial y} = c^2 \frac{\partial^2 z}{\partial x^2}$$

Separation of variable :

Given $\frac{\partial z}{\partial y} = c^2 \frac{\partial^2 z}{\partial x^2}$ ...(1)

Let $Z(x, y) = X(x) Y(y)$ be a solution of (1) substituting it in (1) one have

$$\frac{d^2X}{dx^2} + kX = 0$$

and $\frac{dY}{dy} - kc^2 y = 0$ ...(2)

Case I. $k = P^2$, P is real

$$X = c_1 e^{Px} + c_2 e^{-Px} \text{ and } y = c_3 e^{c^2 p^2 y}$$

$\therefore$ The solution is

$$Z(x, y) = (c_1 e^{Px} + c_2 e^{-Px}) c_3 e^{c^2 p^2 y}$$

Case II. $k = 0$

$$X = c_4 x + c_5 \text{ and } Y = c_6$$

$\therefore$ The solution is $Z(x, y) = (c_4 x + c_5)c_6$

Case III. $k = -P^2$, P is real

$$X = c_7 \cos Px + c_8 \sin Px$$

and $Y = c_9 e^{-c^2 p^2 y}$

$\therefore$ The solution is

$$Z(x, y) = (c_9 \cos Px + c_{10} \sin Px)\, c_9 e^{-c^2 P^2 y}$$

Here all $c_i (i = 1, 2, \ldots\ldots, 12)$ can be calculated by using boundary conditions.

(B) Fourier Transform

(I) Fourier Series and Integral

Periodic functions—A function $f(x)$ is called periodic if it is defined for real $x \in R$ and if there is any positive number p, Such that $f(x + p) = f(x)$, the number P is called a period

e.g., $\sin x$, $\cos x$, $f(x)$ = constant

(1) If function $f(x)$ is periodic then $f(x + np) = f(x)$ where P is period and $n \in Z$ an integer.

(2) A function $h(x) = af(x) + bg(x)$ is a periodic function given $f(x)$ and $g(x)$ are periodic function of period of P.

(3) If a function $f(x)$ has a smallest period $P > 0$, then this is called fundamental period of $f(x)$
e.g. $\sin x$, $\cos x$ have fundamental period 2π. $\sin 2x$, $\cos 2x$ have fundamental period is π, $f(x)$ = constant has no fundamental period

(4) If $f(x)$ is a periodic function of x, of period P, $f(ax)$, $a \ne 0$ is a periodic function of x of period $\frac{P}{a}$ and $f\left(\frac{x}{b}\right)$, $b \ne 0$ is a periodic function of x of period bq.

5. Trigonometric Series

The series $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, where $a_0, \ldots\ldots a_n$ and $b_1 \ldots\ldots b_n$ are coefficient of the series

Euler formula and Fourier Series

The trigonometric series $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ with coefficient (called Euler formulas)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

is called Fourier series of $f(x)$

Orthogonality of the Trignometric Series

Real function f_m and f_n are orthogonal nality in the interval (a, b) if $\int_a^b f_m(x)\, f_n(x)\, dx = 0$, $m \ne n$ and $\int_a^b [f_n(x)]^2 = 1$, for all n.

The trigonometric system sin x, cos x, sin $2x$, cos $2x$, …… is orthogonal on interval $-\pi \le x \le \pi$

Convergence and Sum of Fourier Series

If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \le x \le \pi$ and has a left hand derivative and right hand derivative at each point of that intervals, then the Fourier series $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ of $f(x)$ with coefficient (Euler's formula)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

is convergent. Its sum is $f(x)$, expect at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left and righ hand limits of (fx) at x_0.

Functions of any Period P = 2L

If a function $f(x)$ of period P = 2L has fourier series then, the series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with the Fourier coefficients of $f(x)$ is given Euler formulas.

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\frac{\cos n\pi x}{L}dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\frac{\sin n\pi x}{L}dx$$

Even functions : A function $f(x)$ is even functions, if

$$f(-x) = f(x)$$

Odd function : A function $f(x)$ is odd function,

if $\quad f(-x) = -f(x)$

1. If $f(x)$ is even function, then

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

2. If $f(x)$ is odd function, then

$$\int_{-L}^{L} f(x)dx = 0$$

3. The product of even and odd function is odd

4. The graph of even function $y = f(x)$ is symmetric about y-axis

Fourier Sine Series

The fourier series of an even function of period 2L is a fourier cosine series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{L},$$

where $\quad a_0 = \frac{1}{L}\int_0^L f(u)\,du$

and $\quad a_n = \frac{2}{L}\int_0^L f(x)\frac{\cos n\pi x}{L}dx.$

Fourier Cosine Series

The Fourier series of an odd function of period 2L is a Fourier sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where $\quad b_0 = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

1. The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding fourier coefficient of f_1 and f_2
2. The Fourier coefficient of cf are c times the corresponding Fourier coefficients of f.

Complex Fourier Series

1. A function $f(x)$ with period 2π complex Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $\quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}dx,\ n = 1, 2$

2. A function $f(x)$ of period P = 2L has complex Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

where $\quad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{in\pi x}{L}}dx,\ n = 1, 2$

Fourier Integral

If $f(x)$ is piecewise continuous, in every limit interval and has a right hand derivative and a left hand derivative at each point and if the intergal

$\int_{-\infty}^{\infty} |f(x)| dx$ exists, then $f(x)$ can be represented by a fourier integral

$$f(x) = \int_0^{\infty} [A(W) \cos Wx + B(W) \sin (W)x\, d\omega$$

where A(W) and B(W) are Fourier coefficients of Fourier series defined as

$$A(W) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \cos Wv\, dV$$

$$B(W) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \sin Wv\, dV$$

At the point where $f(x)$ is discountinuous the value of fourier integral equal the average of the left and right hand limits of $f(x)$ at that point.

Fourier sine and cosine integrals

1. If $f(x)$ is even function the $B(\omega) = 0$ and $A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v) \cos Wv\, dv$. Fourier cosine intgral is

$$f(x) = \int_0^{\infty} A(W) \cos Wx\, dW.$$

2. If $f(x)$ is odd function then $A(W) = 0$, $B(W) = \frac{2}{\pi}\int_0^{\infty} f(v) \sin (\omega v) dv$ Fourier sine integral is

$$f(x) = \int_0^{\infty} B(W) \sin Wx\, dW$$

(II) Fourier sine and cosine transforms

Fourier cosine transforms

1. Fourier cosine transforms of $f(x)$

$$F_c [f(x)] = \tilde{f}_c (W) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \cos Wx\, dx$$

2. Inverse Fourier cosine transforms of $\tilde{f}_c (W)$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \tilde{f}(W) \cos Wx\, dx$$

Fourier sine Transforms

1. Fourier sine transform of $f(x)$

$$F_s [f(x)] = \tilde{f}_s (W) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \sin Wx\, dx$$

2. Inverse Fourier sine tranform of $\tilde{f}_s (W)$

$$f_s (x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \tilde{f} (W) \sin Wx\, dw$$

3. If $f(x)$ is absolutely integrable on positive x-axis and piecewise continuous on finite interval, then the Fourier cosine and sine transforms of $f(x)$ exist.

Fourier cosine and sine transform of derviatives—

If $f(x)$ is continuous and absolutely integrable on the x-axis, and $f'(x)$ is piecewise continuous on each interval, also $f(x) \to 0$ as $x \to \infty$, then

$$F_C \{f'(x)\} = WF_C \{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$$

$$F_S \{f'(x)\} = -WF_S \{f(x)\}$$

$$F_C \{f''(x)\} = -W^2 F_C \{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$$

$$F_S \{f''(x)\} = -W^2 F_S \{f(x)\} + \sqrt{\frac{2}{\pi}} Wf(0)$$

Fourier sine transform

fx	$f_S \{f(x)\}$
$\begin{cases} 1, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\left[\frac{1 - \cos a\omega}{W}\right]$
$e^{-ax} (a > 0)$	$\frac{W}{a^2 + W^2}$
$xe^{-ax^2} (a > 0)$	$\frac{W}{\sqrt{2a}} e^{-w^2/4a}$
x^{a-1}, $0 < a < 1$	$\sqrt{\frac{2}{\pi}}\left[\Gamma(a) W^{-a} \sin \frac{a\pi}{2}\right]$
$\cos ax^2$	$\sqrt{\frac{\pi}{2a}}\left[\sin \frac{W^2}{4a} \cos\left(\frac{W}{\sqrt{2\pi a}}\right) - \cos \frac{W^2}{4a} \sin\left(\frac{W}{\sqrt{2\pi a}}\right)\right]$
$\sin ax^2$	$\sqrt{\frac{\pi}{2a}}\left[\cos \frac{W^2}{4a} \cos\left(\frac{W}{\sqrt{2\pi a}}\right) + \sin \frac{W^2}{4a} \sin\left(\frac{W}{\sqrt{2\pi a}}\right)\right]$

Fourier casine transform

fx	$fe\ [f(x)]$
$\begin{cases} 1, 0<x<a \\ 0, \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin aW}{W}$
$e^{-ax}\ (a>0)$	$\sqrt{\frac{2}{\pi}}\frac{a}{a^2+w^2}$
$e^{-ax^2}\ (a>0)$	$\frac{1}{\sqrt{2a}}e^{-w^2/4a}$
$\cos ax^2$	$\frac{1}{\sqrt{2\pi}}\cos\left(\frac{W^2}{4a}-\frac{\pi}{4}\right)$
$\sin ax^2$	$\frac{1}{\sqrt{2\pi}}\cos\left(\frac{W^2}{4a}+\frac{\pi}{4}\right)$
$x^{n-1}\ (0<a<1)$	$\Gamma(a)\ W^{-a}\cos\frac{a\pi}{2}$

Fourier Transform

1. Fourier transform of $f(x)$

$$F(w) = \int_{-\infty}^{\infty} f(x)\, e^{iwx}\, dx$$

2. Inverse Fourier transform of F(W)

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(W)\, e^{-iwx}\, dW$$

3. If $f(x)$ is absoluetly integrable on positive x-axis and piecewise continuous on finite interval, then the Fourier transform at $f(x)$ exists.

Fourier trnasform of derivatives

If $f(x)$ is continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$ $f'(x)$ is absolutely integrable on then axis.

$$F\{f'(x)\} = iWF\{f(x)\}$$

$$F\{f''(x)\} = -W^2F\{f(x)\}$$

$$F\left\{\frac{d^nf}{dx^n}\right\} = (-iW)^n F\{f(x)\}$$

Relation between Fourier Transforms

If F(W) is the Fourier transform of $f(x)$, $-\infty < x < \infty$ then

$f(x)$ is even $\Rightarrow F(W) = 2F_C(W)$

$f(x)$ is odd $\Rightarrow F(W) = 2i\, F_S(W)$

Fourier Transforms and Initial Bondary-Value Problem

1. The Fourier transform of $u(x, t)$, $x \in R$ is denoted by $F\{u(x, t)\} = U(w, t)$ and is defined by the integral

$$F\{\mu(x, t)\} = U(w, t)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwx}\, u(x, t)dx$$

where w is real and is called the transform variable.

2. The inverse Fourier transform is

$$f\{U(w,t)\} = \mu(x, t)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iwx}U(w, t)dw$$

3. If $\mu(x, t) \to 0$ as $|x| \to \infty$, then

$$f\left\{\frac{\partial u}{\partial x}\right\} = iw\, U(w, t)$$

4. If $\mu(x, t)$ is continuously n-time differentiable and $\frac{\partial^k \mu}{\partial u^k} \to 0$, as $|x| \to \infty$ for $k = 1, 2, 3, \ldots,$ $(n-1)$, then

$$F\left\{\frac{\partial^n u}{\partial x^n}\right\} = (-iw)^n$$

$$F\{u(x,t)\} = (-iw)^n\, U(w, t)$$

$$F\left\{\frac{\partial u}{\partial t}\right\} = \frac{dU}{dt}$$

$$F\left\{\frac{\partial u}{\partial t} = \frac{\partial^2 U}{\partial t^2}, \ldots F\left\{\frac{\partial^n u}{\partial t^n} = \frac{\delta^n U}{\partial t^n}\right.\right.$$

5. A sufficient condition for $\mu(x, t)$ to have a fourier transform is that $\mu(x, t)$ is absoluetly integrable in $-\infty < x < \infty$

1. Convolution Theorem—If

$$F\{f(x)\} = F(W)$$

and $F\{g(x)\} = G(W)$ then

$$F\{(x)^*\, g(W)\} = \sqrt{2\pi}\, F(W)\, G(W)$$

2. $\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau = \int_{-\infty}^{\infty} e^{iwx}$ $F(W)G(W)dW$

3. $f(x)^* \sqrt{2\pi}\,\delta(x) = f(x) = \sqrt{2\pi}\,\delta(x)^*\, f(x)$ (identity)

4. The fourier transform are useful in solving a wide variety of initial boundary value problem goverened by linear differential equations.

Some important Fourier transforms

$f(t)$	F(W)	
$f(t)$	$\text{F(W)} = \int_{-\infty}^{\infty} f(t) e^{-iWt} dt$	
$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\text{W})e^{i\text{W}t} d\text{W}$	F(W)	
$af(t) + bg(t)$, a, b constant	$a\text{F}(t) + b\text{G}(t)$	Lineartiy Property
$f(at)$	$\frac{1}{\lvert a\rvert}\text{F}\left(\frac{\text{W}}{a}\right)$	
$f(-t)$	F(– W)	
$f(t - \text{T})$, Tisreal	e^{-iWt} F(W)	
$e^{irt} f(t)$ r is real	F(W – r)	
Ft	$2\pi f(-\text{W})$	
$\left(\frac{d}{dt}\right)^n f(t)$	$(i\text{W})^n$ F(W)	
$(-it)^n f(t)$	$\left(\frac{d}{d\text{W}}\right)^n$ F(W)	
$\int_{-\infty}^{\infty} f(\tau)d\tau$	$\frac{\text{F(W)}}{i\text{W}} + \pi\text{F}(0)\,\delta(\text{W})$	
$f(t)^* g(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau)d\tau$	$\sqrt{2\pi}$ F(W)G(W)	Convolution
$f(t)\ g(t)$	$\frac{1}{2\pi}$ F(W)* G(W)	
$\delta(t)$	1	
$\delta^{(n)}(t)$	$(i\text{W})^n$	
$\delta^{(n)}(t - \text{T})$	$(i\text{W})^n e^{-iWt}$ $(n = 0, 1, 2, \ldots\ldots)$	
1	$2\pi\,\delta(\text{W})$	
t^n	$2\pi i^n\,\delta^{(n)}(\text{W})$ $(n = 0, 1, 2, \ldots)$	
t^{-n}	$\frac{\pi(-i)^n}{(n-1)!}\text{W}^{n-1}$ δgnW $(n = 1, 2, 3)$	
$\theta(t) = \text{H}(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$	$\frac{1}{i\text{W}} + \pi\,\delta(\text{W})$	
$t^n\theta(t)$	$\frac{n!}{(i\text{W})^{n+1}} + \pi i^n\,\delta^{(n)}(\text{W})$ $n = 1,$	

OBJECTIVE TYPE QUESTIONS

1. Given wave equation $\frac{\partial z}{\partial y} = c^2\,\frac{\partial^2 z}{\partial x^2}$. By separation of variable let $z(x, y) = X(x)Y(y)$ be a solution. Substituting it in $\frac{\partial z}{\partial y} = c^2\frac{\partial^2 z}{\partial x^2}$ one have $\frac{d^2\text{X}}{dx^2} + k\text{X} = 0$ and $\frac{d\text{Y}}{dy} - kc^2 y = 0$ if $k = -p^2$, p is real then the solution is (c's are constant)
   (A) $z(x, y) = (c_9 \cos px)c_9\ c^{-c^2p^2y}$
   (B) $z(x, y) = (c_9 \cos px + c_{10} \sin px)c_9\ c^{-c^2p^2y}$
   (C) $z(x, y) = e^{-c^2p^2y}$
   (D) $z(x, y) = (c_{10} \sin px)c_9\ c^{-c^2p^2y}$

2. Given wave equation $\frac{\partial z}{\partial y} = c^2\,\frac{\partial^2 z}{\partial x^2}$. By separation of variable, Let $z(x, y) = X(x)\,Y(y)$ be a solution. Subsitituting it in $\frac{\partial z}{\partial y} = c^2\frac{\partial^2 z}{\partial x^2}$ are have $\frac{d^2\text{X}}{dx^2} + k\text{X} = 0$ and $\frac{d\text{Y}}{dy} - kc^2\text{Y} = 0$. If then the solution is (c's are constant)—
   (A) $z(x, y) = c_6$
   (B) $z(x, y) = (c_4x + c_5)$
   (C) $z(x, y) = (c_4x)$
   (D) $z(x, y) = (c_4x + c_5)c_6$

3. Given wave equation : $\frac{\partial z}{\partial y} = c^2\,\frac{\partial^2 z}{\partial x^2}$ By separation of variable let $Z(x, y) = X(x), x\,Y(y)$ be a solution substituting it in $\frac{\partial z}{\partial y} = c^2\frac{\partial^2 z}{\partial x^2}$ are have $\frac{d^2\text{X}}{dx^2} + k\text{X} = 0$ and $\frac{d\text{Y}}{dy}$ –

$kc^2Y = 0$. If $k = P^2$, p is real, then the solution is (c's are constant)—

(A) $Z(x, y) = (c_1 + c_2 e^{-px})e^{c^2p^2y}$

(B) $Z(x, y) = e^{c^2p^2y}$

(C) $Z(x, y) = (c_1 e^{px} + c_2 e^{-px})c_3 e^{c^2p^2y}$

(D) $Z(x, y) = (c_1e^{px} + c_2 e^{-px})$

4. Given $\frac{\partial^2 z}{\partial y^2} = c^2 \frac{\partial^2 z}{\partial x^2}$ $0 \le x \le a$, $0 > y$ with boundary conditions $Z(0, y) = Z = (a, y) = 0$ and initial conditions :
$Z(x, 0) = f(x)$, $q(x, 0) = g(x)$.
If the possible solution is
$Z(x, y) = (c_9 \cos px + c_{10} \sin px)$
$(c_{11} \cos cpy + c_{12} \sin cpy)$
then x required solution is given by.

(A) $Z(x, 0) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{a}\right)$
where $A_n = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{m\pi x}{a}\right) dx$

(B) $\frac{\partial z}{\partial y}(x, 0) = g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)$,
where $B_n = \frac{2}{n\pi c}\int_0^a g(x) \sin\frac{n\pi x}{a} dx$

(C) Both (A) and (B) a bove

(D) None of these

5. Given the wave equation $\frac{\partial^2 z}{\partial y^2} = c^2 \frac{\partial^2 z}{\partial x^2}$. By separation of variable Let $Z(x, y) = X(x) Y(y)$ be a solution of wave equation subsitiuting it in wave equation, one have $\frac{d^2X}{dx^2} + kX = 0$ and $\frac{d^2Y}{dy^2} - Kc^2Y = 0$. If $k = -P^2$, P is real then the solution is (c's are constant)

(A) $Z(x, y) = (c_9 \cos Px + c_{10} \sin Px)$

(B) $Z(x, y) = (c_9 \cos Px + c_{10} \sin Px) + (c_{11} \cos cPy + c_{12} \sin cPy)$

(C) $Z(x, y) = (c_9 \cos Px - c_{10} \cos Px)$
$(c_{11} \cos cPy + c_{12} \cos Py)$

(D) $Z(x, y) = (c_9 \cos Px + c_{10} \sin Px)$
$(c_{11} \cos cPy + c_{12} \sin cPy)^*$

6. Given the wave equation $\frac{\partial^2 z}{\partial y^2} = c^2 \frac{\partial^2 z}{\partial x^2}$. By separation of variable. Let $Z(x, y) = X(x) Y(y)$ be a solution of wave equation. Substituting it in wave equation, one have $\frac{d^2X}{dx^2} + kX = 0$ and $\frac{d^2Y}{dy^2} - kc^2Y = 0$. If $k = 0$ then the solution is (c's are constant)

(A) $Z(x, y) = (c_5x + c_6)$

(B) $Z(x, y) = (c_5 x)$

(C) $Z(x, y) = (c_5x + c_6) + (c_7x + c_8)$

(D) $Z(x, y) = (c_5x + c_6)(c_7y + c_8)$

7. Given the wave equation $\frac{\partial^2 z}{\partial y^2} = c^2 \frac{\partial^2 z}{\partial x^2}$. By separation of variable let $Z(x, y) = X(x) Y(y)$ be a solution of wave equation substituting it in wave equation one have $\frac{d^2X}{dx^2} + kX = 0$ and $\frac{d^2Y}{dy^2} - kc^2Y = 0$. If $k = P^2$, P is real, then the solution is (c's are constant)—

(A) $Z(x, y) = (c_1 + c_2 e^{-Px})$

(B) $Z(x, y) = (c_1 e^{Px})(c_2 e^{cPy})$

(C) $Z(x, y) = (c_1 e^{Px} + c_2 e^{-Px})$

(D) $Z(x, y) = (c_1 e^{Px} + c_2 e^{-Px})$
$(c_3 e^{cPy} + c_4 e^{-cPy})$

8. Given Laplace equation $\nabla^2 = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$
Let $Z(x, y) = X(x) Y(y)$ be a solution of given equation. Substituting it in given equation one have $\frac{d^2X}{dx^2} Y + X\frac{d^2Y}{dy^2} = 0$ and separating the variable gives $\frac{1}{X}\frac{d^2X}{dx^2} = \frac{-1}{Y}\frac{d^2Y}{dx^2} = k$
If $k = -P^2$, P is real the solution is (c's are constant)

(A) $Z(x, y) = (c_9 \cos Px + c_{10} \sin Px)$

(B) $Z(x, y) = (c_9 \cos Px + c_{10} \sin Px)$
$(c_{11} e^{Py} + c_{12} e^{-Py})$

(C) $Z(x, y) = (c_9 \cos Px \sin Px)$

(D) $Z(x, y) = (\cos Px \sin Px)(e^{Py} e^{-Py})$

9. Given Laplace equation $\nabla^2 = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.
Let $Z(x, y) = X(x) Y(y)$ be a solution of given equation. Substituting it in given equation one have $\frac{d^2X}{dx^2} Y + X\frac{d^2Y}{dy^2} = 0$ and separating the

variables given $\frac{1}{X}\frac{d^2X}{dx^2} = \frac{-1}{Y}\frac{d^2Y}{dy^2} = k$. It $k = 0$ the solution is (c's are constant)

(A) $Z(x, y) = (c_5x + c_6)$
(B) $Z(x, y) = (c_5\, xy)$
(C) $Z(x, y) = (c_5x + c_6)(c_7y + c_8)^*$
(D) None of these

10. Given Laplace equation $\nabla^2 = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$. Let $Z(x, y) = X(x)\, Y(y)$ be a solution of given equation. Substituting it in given equation one have $\frac{d^2X}{dx^2}Y + X\frac{d^2Y}{dy^2} = 0$ and separating the variables gives $\frac{1}{X}\frac{d^2X}{dx^2} = \frac{-1}{Y}\frac{d^2Y}{dy^2} = K$. If $K = P^2$, P is real, then solution is (c's are constant)
(A) $Z(x, y) = (c_1 e^{Px} + c_2 e^{-Px}) + (c_3 \cos Py + c_4 \sin Py)$
(B) $Z(x, y) = \frac{(c_1 e^{Px} + c_2 e^{-Px})}{(c_3 \cos Py + c_4 \sin Py)}$
(C) $Z(x, y) = (c_1 e^{Px} + c_2 e^{-Px})$
(D) $Z(x, y) = (c_1 e^{Px} + c_2 e^{-Px})(c_3 \cos Py + c_4 \sin Py)$

11. Given second order partial differential equation $Rr + Ss + Tt = F$. Where R, S, T, F are functions of x, y or constant. Then Monge's subsidiary equations are—
(A) $Rdy + Sdxdy + Tdx = 0$ and $Rdpdy + Tdqdu - Fdxdy = 0$
(B) $Rdy - Sdxdy - Tdx = 0$ and $Rdp + Tdq - Fdx = 0$
(C) $Rdy^2 - Sdxdy + Tdx^2 = 0$ and $Rdpdy + Tdqdx - Fdxdy = 0$
(D) $Rdy\, Sdx + Tdx^2 = 0$ and $Rdy + Tdx - F = 0$

12. A first order linear partial differential equation of the form $F(x, y, z, p, q) = 0$ solved by Charpit's method have subsidiary equation—
(A) $\frac{dx}{\frac{\partial F}{\partial p}} = \frac{dy}{\frac{\partial F}{\partial q}} = \frac{dz}{p\frac{\partial F}{\partial q} - q\frac{\partial F}{\partial q}} = \frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dF}{0}$
(B) $\frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dF}{0}$
(C) $\frac{dx}{\frac{dF}{dp}} = \frac{dy}{\frac{\partial F}{\partial q}} = \frac{dz}{p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial p}} = \frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dF}{0}$
(D) None of these

13. A First order linear partial differential equation of the form $Pp + Qq = R$ called (linear) Lagrange's equation has a subsidiary equation—
(A) $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ (B) $\frac{dx}{P} = \frac{dy}{q} = \frac{dz}{r}$
(C) $\frac{1}{P} = \frac{1}{q} = \frac{1}{R}$ (D) None of these

14. A first order linear partial differential equation of the form $Pp + Qq = R$ is called (linear) Lagrange's equation—
(A) If P, Q, R are the functions of x, y, z
(B) If P, Q, R are the functions of x and y only
(C) If P, Q, R are the function of y and z only
(D) None of these

15. A first order linear partial differential equation of form $Pp + Qq = R$ is called (Quasi-Linear) Lagrange's equation—
(A) If P, Q, R are the function of x, y, z
(B) If P, Q, R are the function of x and y only
(C) If P, Q, R are the function of x and z only
(D) None of these

16. In first order partial differential equation $F(x, y, z, p, q) = 0$ if degree of p and q are …, then first order partial differential equation is linear—
(A) One (B) Two
(C) Three (D) Four

17. In first order partial differential euqation $F(x, y, z, p, q) = 0$. General integral is—
(A) A solution obtained from the complete integral $f(x, y, u, a, b) = 0$ by assigning

particular values to arbitrary constants a, b

(B) In complete integral $f(x, y, u, a, b) = 0$ where $b = \phi(a)$ one have an envelope of the family of surface $f(x, y, u, a, \phi(a)) = 0$

(C) In complete integral $f(x, y, \mu, a, b) = 0$ is only constant

(D) None of these

18. In first order partial differiential equation, $F(x, y, z, p, q) = 0$. Particular integral is—
    (A) A solution obtained from the complete integral $f(x, y, u, a, b) = 0$ by assinging particular ... values to arbitrary constant a, b
    (B) In complete integral $f(x, y, u, a, b) = 0$ where $b = \phi(a)$ one have an envelope of the family of surface $f(x, y, z, p, q) = 0$
    (C) In complete integral $f(x, y, z, P, Q) = 0$ is constant
    (D) None of these

19. For the first order partial differiential equation $F(x, y, z, p, q) = 0$.
    Complete integral is—
    (A) Any relation $f(x, y, u, a, b) = 0$ where a, b arbitary constant
    (B) An envelope of the family surface $f(x, y, u, a, b) = 0$ with parameters a, b if exist
    (C) Any relation $f(x, y, z, p, q) = 0$ is only constant
    (D) None of these

20. Bi-harmonic wave equation is—
    (A) $u_t + au_x + bu_{xx} = 0$, where a, b are constant
    (B) $u_{tt} - a^2\nabla^2 u - b^2\nabla^2 u_{tt} = 0$, where a, b are constant
    (C) $u_{tt} + c^2\nabla^4 u = 0$, where a is constant
    (D) None of these

21. Linear Boussinesq equation is—
    (A $u_t + au_x + bu_{xx} = 0$, where a, b are constant
    (B) $u_{tt} - a^2\nabla^2 u - b^2\nabla^2 u_{tt} = 0$ where a, b are constants
    (C) $u_{tt} + c^2\nabla^4 u = 0$, where a is constant
    (D) None of these

22. Korteweg derives (or kdV) equation is—
    (A) $u_t + au_x + bu_{xx} = 0$ where a, b are constants
    (B) $u_{tt} - a^2\nabla^2 u - b^2\nabla^2 u_{tt} = 0$, where a, b are constants
    (C) $u_{tt} + c^2\nabla^4 u = 0$, where a is constant
    (D) None of these

23. Klien-Gordon equation is represented by—
    (A) $\nabla^2 u + \lambda u = 0$; where λ is constant
    (B) $u_{tt} + au_t + bu = u_{xx}$, where a, b are constants
    (C) $\psi + \left(\frac{mc^2}{h}\right)\psi = 0$, where $\frac{\partial^2}{\partial t^2} - c^2\nabla^2$ is the d' Alembertian
    (D) None of these

24. Telegraph equation is represented by—
    (A) $\nabla^2 u + \lambda u = 0$ where λ is constant
    (B) $u_{tt} + au_t + bu = u_{xx}$ where a, b are constant
    (C) $\psi\left(\frac{mc^2}{h}\right)\psi = 0$, where $\frac{\partial^2}{\partial t^2} - c^2\nabla^2$ is the d' Alembertian
    (D) None of these

25. Helmholtz equation is represented by—
    (A) $\nabla^2 u + \lambda u = 0$. where λ is constant
    (B) $u_{tt} + au_t + bu = u_{xx}$ where a, b are constants
    (C) $\psi\left(\frac{mc^2}{h}\right)\psi = 0$. where $= \frac{\partial^2}{\partial t^2} - c^2\nabla^2$ is the d' Alembertian
    (D) None of these

26. Poisson equation is represented by—
    (A) $u_{tt} - c^2\nabla^2 u = 0$, c is constant
    (B) $u_t - k\nabla^2 u = 0.$,k is constant
    (C) $\nabla^2 u = 0$
    (D) $\nabla^2 u = f(x, y, z)$

27. Laplace equation is represented by—
    (A) $u_{tt} - c^2\nabla^2 u = 0$, c is constant
    (B) $u_t - k\nabla^2 u = 0$, k is constant
    (C) $\nabla^2 u = 0$
    (D) $\nabla^2 u = f(x, y, z)$

28. Heat (Diffusion equation) is represented by—
    (A) $u_{tt} - c^2\nabla^2 u = 0$, where c is a constant
    (B) $u_t - k\nabla^2 u = 0$, where k is constant
    (C) $\nabla^2 u = 0$
    (D) $\nabla^2 u = f(x, y, z)$

29. Wave equation is represented by—
    (A) $u_{tt} - c^2\nabla^2 u = 0$, where c is constant
    (B) $u_t - k\nabla^2 u = 0$, where k is constant
    (C) $\nabla^2 u = 0$
    (D) $\nabla^2 u = f(x, y, z)$

30. A problem described by a partial differential equation in a given domain with a set of intial and/or boundry conditions (or other supplementry conditions is said to be well posed (or properly posed) provided which of the following criteria are satisfied ?
    (A) Existence : There exists at least one solution of the problem
    (B) Uniquences : There is at most one solution.
    (C) Stability : The solution must be stable in the sense that it depends continuously on the data
    (D) All the above

31. In Robin conditions—
    (A) u is prescribed by each point of a boundary ∂D of a domain D
    (B) Where value of normal derivative $\frac{\partial \mu}{\partial n}$ on the boundry ∂D are sepcified
    (C) $\left(\frac{\partial u}{\partial n} + au\right)$ is specified on ∂D
    (D) None of these

32. In Neumann condition—
    (A) μ is prescribed by each point of boundary ∂D of a domain D
    (B) Where value of normal derivative $\frac{\partial u}{\partial n}$ on the boundary ∂D are specified*
    (C) $\left(\frac{\partial u}{\partial n} + au\right)$ is specified ∂D
    (D) None of these

33. In Dirichlet conditions—
    (A) U is prescribed by each point of a boundary ∂D of a domain D
    (B) Where value of normal derivative $\frac{\partial u}{\partial n}$ on the boundary ∂D are specified
    (C) $\left(\frac{\partial u}{\partial n} + au\right)$ is specified on ∂D
    (D) None of these

34. Following expression $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0$ represents [where $u = u(x, y)$]—
    (A) First order partial differential equation
    (B) Second order partial defferential equation*
    (C) Third order partial differential equation
    (D) None of these

35. In linear homogeneous partial differential equation, the general solution depends on—
    (A) arbitrary function
    (B) Arbitrary constants
    (C) Only constant
    (D) None of these

36. Linear superposition principle states that—
    (A) The general solution of a linear homogeneous ordinary differential equation of order n is linear combination of n dependent solutions
    (B) The general solution of linear homogeneous ordinary differential equation of order n is a linear combination of n linearly independent solutions with n arbitrary constants
    (C) The general solution of linear homogeneous ordinary differential equation of order n is a linear combination of n linearly in dependent solutions with no arbitrary constants
    (D) None of these

37. Non-homogeneous partial differential equation is—
    (A) If each term as a partial differential equation contains either the dependent variable or its partial derivative
    (B) If each term of a partial differential equation does not contains either the dependent variable or its partial derivative
    (C) If each term of a partial differential equation contains the dependent variable but not its partial derivative
    (D) None of these

38. For the independent variables x, y, the solution $\mu = \mu(x, y)$ of partial differential equation $F(x, y, u, \mu_x, \mu_y) = 0$ is visualized geometrically as a surface and called—
    (A) An integral surface in (x, y, u) space

(B) Differential surface in (x, y, u) space
(C) An solid surface in (x, y, u) space
(D) None of these

39. A general solution of a partial differential equation is an expression—
(A) That involves arbitary functions*
(B) That does not involves arbitary functions
(C) Both (A) and (B)
(D) None of these

40. The solution $u = u(x, y, \ldots)$ is called a weak (or generalized) solution of $F(x, y, u, u_x, \mu_y, \mu_{yy}, u_{xx}, u_{xy}, u_{yy} \ldots) = 0$. if—
(A) u or its partial derivatives are discontinued in some or all points in D
(B) u or its partial derivative are continued in some or all points in D
(C) u or its partial derivatives do not exist in some or all point in D
(D) None of these

41. A classical solution (solution, of partial defferential equation $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy} \ldots) = 0$ is an ordinary function $u = \mu(x, y, \ldots)$ defined in some domain D, which—
(A) Is countinuously differentiable such that its all partial derivatives involved in the equation exist
(B) Satisfies $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy} \ldots) = 0$ identically
(C) Both (A) and (B) above
(D) None of these

42. Quasi-linear partial differential equation is—
(A) If it is linear in the lowast order derivative of the known function
(B) If it is linear in the any derivative of the unknown function
(C) If it is linear in the hightest order derivative of the unknown function
(D) None of these

43. A partial differential equation is linear—
(A) If it is linear in the known function and all its derivatives with coefficients depends only on the dependent variable
(B) If it is linear in the unknown function and all its derivatives with coefficients depends only on the independent variable
(C) if it is linear in the known function and all its derivatives with coefficients depends only on the dependent variable
(D) None of these

44. Following expression $F(x, y, u, u_x, u_y) = 0$ represents where $u = u(x, y)$—
(A) First order partial differential equation
(B) Second order partial differential equation
(C) Third order parital differential equation
(D) None of these

45. Homogeneous partial differential equation is—
(A) If each term of a partial differential equation contains either the dependent variable or its partial derivatives
(B) If each term of a partial differential equation does not contains either the dependent variable or its partial derivative
(C) If each term of partial differential equation contains the dependent variable but not its partial derivative
(D) None of these

46. The general second order linear differential equation in two independent variable x, y is given by $Au_{xx} + Bu_{xy} + Cu_{yy} + D\mu_x + Eu_y + Fu = G$. where A, B, C, D, E and F are given functions of x and y or constant. The characteristic equation is given by—
(A) $\frac{dy}{dx} = (B \pm \sqrt{B^2 - 4AC})$
(B) $\sqrt{B^2 - 4AC}$
(C) $\frac{dy}{dx} = \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC})$
(D) None of these

47. The general second order linear differential in two independent variable x, y is given by $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, where A, B, C, D, E and F are given functions of x and y or constant. The equation is elliptic equation if —
(A) $B^2 - 4AC > 0$ (B) $4AC = 0$
(C) $B^2 - 4AC < 0$ (D) None of these

48. The general second order linear differential equation in two independent variable x, y is

given by $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, where A, B, C, D, E and F are given functions of x and y or constant. The equation is parabolic equation if—

(A) $B^2 - 4AC > 0$ (B) $4AC = 0$
(C) $B^2 - 4AC < 0$ (D) None of these

49. The general second order linear differential equation in two independent variable x, y is given by $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$. Where A, B, C, D, E and F are given functions of x and y or constant. The equation in Hyperbolic equation if—

(A) $B^2 - 4AC > 0$ (B) $B^2 - 4AC = 0$
(C) $B^2 - 4AC < 0$ (D) None of these

50. Order of the partial differential equation is—

(A) The order of the highest derivative appears in it
(B) The order of the lowest derivative appears in it
(C) The order of any derivative appears in it
(D) None of these

51. A function $f(x)$ is called periodic if it is defined for real $x \in R$ and—

(A) if there is any positive number p, such that $f(x + p) = f(x) + f(p)$
(B) if there is any positive number p, such that $f(x + p) > f(x)$
(C) if there is any positive number p, such that $f(x + p) = f(x)$
(D) None of these

52. Which of the following function is a periodic function ?

(A) $\sin x$ (B) $\cos x$
(C) $f(x) =$ constant (D) All of these

53. If a function $f(x)$ is periodic, then for p period and $n \in z$.

(A) $f(x + np) = f(x + n)$
(B) $f(x + np) = f(x)$
(C) $f(x + np) = f(x + p)$
(D) None of these

54. The fundamantal period of $f(x)$ is—

(A) Smallest period $p > 0$
(B) Largest period $p > 0$
(C) Any period $p > 0$
(D) None of these

55. The trigonometric function $\cos x$ have fundamental period—

(A) π (B) 2π
(C) 3π (D) 4π

56. The trigonometric function $\sin x$ have fundamental period—

(A) π (B) 2π
(C) 3π (D) 4π

57. The trigonometric function $\cos 2x$ have fundametal period—

(A) π (B) 2π
(C) 3π (D) 4π

58. The trigonometric function $\sin 2x$ have fundametal period—

(A) π (B) 2π
(C) 3π (D) 4π

59. The function $f(x) =$ constant—

(A) have no fundamental period
(B) have fundamental period π
(C) have fundamental period 2π
(D) have fundamental period 3π

60. If $f(x)$ is a periodic function of x of period p, then—

(A) $f(ax)$, $a \neq 0$ is a periodic function of x of period p/a
(B) $f(ax)$, is a periodic function of x of period p
(C) $f(ax)$, $a \neq 0$ is a periodic function of x of period $1/a$
(D) None of these

61. If $f(x)$ is a periodic function of x of period p, then—

(A) $f\left(\frac{x}{b}\right)$, $b \neq 0$ is a periodic function of x of period bp
(B) $f\left(\frac{x}{b}\right)$, is a periodic function of x of period p
(C) $f\left(\frac{x}{b}\right)$, $b \neq 0$, is a periodic function of x of period b
(D) None of these

62. In Fourior series $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, Euler formula is—

(A) $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$

(B) $a_0 = \int_{-\pi}^{\pi} f(x)\,dx$

(C) $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx$

(D) None of these

63. In Fourier series $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, Euler formula is—

(A) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

(B) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

(C) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

(D) None of these

64. In Fourier series $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, Euler formula is—

(A) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

(B) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

(C) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

(D) None of these

65. Real function f_m and f_n are orthogonal in the interval (a, b)—

(A) if $\int_a^b f_m(x) f_n(x)\,dx = 0$ and for all m, n

(B) if $\int_a^b f_m(x) f_n(x)\,dx = 0$, $m \neq n$ and for all $n \int_a^b [f_n(x)]^2 = 0$

(C) if $\int_a^b fm(x) f_n(x)\,dx = 1$, $m \neq n$ and for all $n, \int_a^b [f_n(x)]^2 = 0$

(D) None of these

66. The trigonometric system $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$ … is—

(A) Orthogonal an interval $-\pi \leq x \leq \pi$

(B) Not orthogonal on interval $-\pi \leq x \leq \pi$

(C) Both (A) and (B)

(D) None of these

67. If a periodic function $f(x)$ with period 2π is piecewise continuous in the interval $-\pi \leq x \leq \pi$ and has a left hand derivative and right hand derivative at each point of that interval, then the Fourier series is—

(A) Convergent (B) Divergent

(C) Constant (D) None of these

68. If a function $f(x)$ of a period $p = 2L$ has Fourier series, then in series $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ the Fourier coefficients a_0 is given by—

(A) $\frac{1}{2L}\int_{-L}^{L} f(x)\,dx$ (B) $\int_{-L}^{L} f(x)\,dx$

(C) $\frac{1}{2}\int_{-L}^{L} f(x)\,dx$ (D) None of these

69. If a function $f(x)$ of period $P = 2L$ has Fourier series, then in series $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, then Fourier coefficients a_n is given by—

(A) $\int_{-L}^{L} f(x)\frac{\sin n\pi x}{L}\,dx$

(B) $\frac{1}{L}\int_{-L}^{L} f(x)\frac{\cos n\pi x}{L}\,dx$

(C) $\int_{-L}^{L} f(x)\frac{\cos n\pi x}{L}\,dx$

(D) None of these

70. If a function $f(x)$ of period $P = 2L$ has Fourier series, then in series $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, the Fourier coefficients b_n is given by—

(A) $\frac{1}{L}\int_{-L}^{L} f(x)\frac{\sin n\pi x}{L}\,dx$

(B) $\frac{1}{L}\int_{-L}^{L} f(x)\frac{\cos n\pi x}{L}\,dx$

(C) $\int_{-L}^{L} f(x)\frac{\sin n\pi x}{L}\,dx$

(D) $\int_{-L}^{L} f(x)\frac{\cos n\pi x}{L}\,dx$

71. A function $f(x)$ is even function, if—
    (A) $f(-x) = f(x)$
    (B) $f(-x) = -f(x)$
    (C) $f(-x) = \text{constant}$
    (D) None of these

72. A function $f(x)$ is odd function, if—
    (A) $f(-x) = f(x)$
    (B) $f(-x) = -f(x)$
    (C) $f(-x) = \text{constant}$
    (D) None of these

73. If $f(x)$ is even function, then—
    (A) $\int_{-L}^{L} f(x)\, dx = 2\int_{0}^{L} f(x)\, dx$
    (B) $\int_{-L}^{L} f(x)\, dx = 0$
    (C) $\int_{-L}^{L} f(x)\, dx = \int_{0}^{L} f(x) dx$
    (D) None of these

74. If $f(x)$ is odd function, then—
    (A) $\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$
    (B) $\int_{-L}^{L} f(x)dx = 0$
    (C) $\int_{-L}^{L} f(x)dx = \int_{0}^{L} f(x)dx$
    (D) None of these

75. The product of even and odd function is—
    (A) Odd (B) Even
    (C) Constant (D) None of these

76. The graph of even function $y = f(x)$ is symmetric about—
    (A) x-axis
    (B) y-axis
    (C) Both (A) and (B) above
    (D) None of these

77. The Fourier series of an even function of period 2L is a—
    (A) Fourier cosine series
    (B) Fourier sine series
    (C) Fourier complex series
    (D) None of these

78. The Fourier series of an odd function of period 2L is a—
    (A) Fourier cosine series
    (B) Fourier sine series
    (C) Fourier complex series
    (D) None of these

79. The Fourier series of an even function of period 2L is given by—
    (A) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \frac{\cos n \pi x}{L}$, where $a_0 = \frac{1}{L}\int_0^L f(x)dx$, $a_n = \frac{2}{L}\int_0^L f(x) \frac{\cos n \pi x}{L} dx$
    (B) $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{L}$, where $b_0 = \frac{2}{L}\int_0^L f(x) \frac{\sin n \pi x}{L} dx$
    (C) $f(x) = \sum_{n=-\infty}^{\infty} e_n e^{inx}$, where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}dx$, $n = 1, 2, \ldots\ldots$
    (D) None of these

80. The Fourier series of an odd function of period 2L is given by—
    (A) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \frac{\cos n \pi x}{L}$, where $a_0 = \frac{1}{L}\int_0^L f(x)dx$, $a_n = \frac{2}{L}\int_0^L f(x) \frac{\cos n \pi x}{L} dx.$
    (B) $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{L}$, where $b_0 = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx$
    (C) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}dx$, $n = 1, 2, \ldots\ldots$
    (D) None of these

81. The Fourier cosine series of function of period 2L is given by—
    (A) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \frac{\cos n \pi x}{L}$, where $a_0 = \frac{1}{L}\int_0^L f(x)dx$, $a_n = \frac{2}{L}\int_0^L f(x) \frac{\cos n \pi x}{L} dx$
    (B) $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{L}$, where $b_0 = \frac{2}{L}\int_0^L f(x) \sin \frac{n \pi x}{L} dx$

(C) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)$ $e^{-inx}dx$, $n = 1, 2, \ldots\ldots$

(D) None of these

82. The Fourier sine series of a function of period 2L is given by—

(A) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{L}$, where $a_0 = \frac{1}{L}\int_0^L f(x)dx$, $a_n = \frac{2}{L}\int_0^L f(x)\frac{\cos n\pi x}{L}dx$

(B) $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$, where $b_0 = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

(C) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$, $n = 1, 2, \ldots\ldots$

(D) None of these

83. The Fourier complex series of a function of period 2L is given by—

(A) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \frac{\cos n\pi x}{L}$, where $a_0 = \frac{1}{L}\int_0^L f(x)dx$, $a_n = \frac{2}{L}\int_0^L f(x)\frac{\cos n\pi x}{L}dx$

(B) $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$, where $b_0 = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

(C) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)$ $e^{-inx}dx$, $n = 1, 2, \ldots\ldots$

(D) None of these

84. A function $f(x)$ with period 2π have a complex Fourier series—

(A) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$, where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$, $n = 1, 2, \ldots\ldots$

(B) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$, where $c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{in\pi x}{L}}dx$, $n = 1, 2, \ldots\ldots$

(C) $f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$

(D) None of these

85. $f(x)$ can be represented by a Fourier integral if $f(x)$—

(A) is piecewise continuous, in every finite integral

(B) has a right hand derivative and a left hand derivative at each point in a finite integral

(C) in the integral $\int_{-\infty}^{\infty} |f(x)|dx$ exists

(D) All the above

86. If $f(x)$ represented by Fourier integral $f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\, d\omega_y$ then $A(w)$ is defined as—

(A) $\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$

(B) $\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$

(C) $\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$

(D) $\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$

87. If $f(x)$ represented by Fourier integral $f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega$, then $B(\omega)$ is defined as—

(A) $\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$

(B) $\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\, dv$

(C) $\int_{-\infty}^{\infty} f(v)\cos\omega v\, dv$

(D) $\int_{-\infty}^{\infty} f(v)\sin\omega v.dv$

88. At the point where $f(x)$ is discontinuous, the value of the Fourier integral equals—

(A) Average of the left and right hand limits of $f(x)$ at that point

(B) Sum of the left and right hand limits of $f(x)$ at that point

(C) Product of the left and right hand limits of $f(x)$ at that point

(D) Difference of the left and right hand limits of $f(x)$ is even function

89. If $f(x)$ represented by Fourier integral and if $f(x)$ is odd funtion, then $f(x)$ is—
(A) Fourier cosine integral
(B) Fourier sine integral
(C) Both (A) and (B) above
(D) None of these

90. If $f(x)$ represented by Fourier integral and if $f(x)$ is odd function, then $f(x)$ is—
(A) Fourier cosine integral
(B) Fourier sine integral
(C) Fourier tan integral
(D) None of these

91. If $f(x)$ represented by Fourier integral and if $f(x)$ is even function, then $f(x)$ is—
(A) $f(x) = \int_0^{\infty} n(\omega) \cos \omega x \, d\omega$
(B) $f(x) = \int_0^{\infty} B(\omega) \sin \omega x \, d\omega$
(C) Both (A) and (B) above
(D) None of these

92. If $f(x)$ represented by Fourier integral and if $f(x)$ is odd function, then $f(x)$ is—
(A) $f(x) = \int_0^{\infty} A(\omega) \cos \omega x \, d\omega$
(B) $f(x) = \int_0^{\infty} B(\omega) \sin \omega x \, d\omega$
(C) If $f(x)$ is even function, then $B(\omega) = 0$ and
(D) If $f(x)$ is odd function, then and $B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \sin \omega v \, dv$

93. If $f(x)$ represented by Fourier integral $f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega$ and $f(x)$ is even function, then—
(A) $B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \sin \omega v \, dv$
(B) $B(\omega) = 0$
(C) Both (A) and (B)
(D) None of these

94. If $f(x)$ represented by Fourier integral $f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega$ and $f(x)$ is odd function, then—
(A) $B(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \sin \omega v \, dv$
(B) $B(\omega) = 0$
(C) Both (A) and (B)
(D) None of these

95. If $f(x)$ represented by Fourier integral $f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega$ and $f(x)$ is odd function, then—
(A) $A(\omega) = 0$
(B) $A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \cos \omega v \, dv$
(C) Both (A) and (B) above
(D) None of these

96. If $f(x)$ represented by Fourier integral $f(x) = \int_0^{\infty} A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega$ and $f(x)$ is odd function, then—
(A) $A(\omega) = 0$
(B) $A(\omega) = \frac{2}{\pi} \int_0^{\infty} f(v) \cos \omega v \, dv$
(C) Both (A) and (B) above
(D) None of these

97. Fourier cosine transform of $f(x)$ is—
(A) $\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx$
(B) $\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x \, dx$
(C) Both (A) and (B) above
(D) None of these

98. Fourier sine transform of $f(x)$ is—
(A) $\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx$
(B) $\sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x \, dx$
(C) Both (A) and (B) above
(D) None of these

99. The Fourier cosine and sine tranforms of $f(x)$ exist if—
(A) $f(x)$ is absolutely integrable on positive x-axis
(B) $f(x)$ piecewise continuous on finite interval
(C) Both (A) and (B)
(D) None of these

100. If $f(x)$ is continuous and absolutely integrable on the x-axis, and $f'(x)$ is piecewise continuous an each interval, also $f(x) \to 0$ as $x \to \infty$. Then Fourier cosine transform—

(A) $F_C\{f''(x)\} = -\omega^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$

(B) $F_C\{f''(x)\} = f(x) - \sqrt{\frac{2}{\pi}} f'(0)$

(C) $F_C\{f''(x)\} = -\omega^2 f(x) - \sqrt{\frac{2}{\pi}} f(0)$

(D) None of these

101. If $f(x)$ is continuous and absolutely integrable on the x-axis and $f'(x)$ is piecewise continuous on each interval, also $f(x) \to 0$ as $x \to \infty$. Then Fourier cosine transform—

(A) $F_C\{f'(x) = \omega F_C\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$

(B) $F_C\{f'(x) = \omega F_C\{f(x)\}$

(C) $F_C\{f'(x)\} = \omega\{f(x)\} - f(0)$

(D) None of these

102. If $f(x)$ is continuous and absolutely integrable on the x-axis, and $f'(x)$ is piecewise continuous an each interval, also $f(x) \to 0$ as $x \to \infty$. Then Fourier sine transoform—

(A) $F_S\{f'(x)\} = w f(x)$

(B) $F_S\{f'(x)\} = F_S\{f(x)\}$

(C) $F_S\{f'(x)\} = -wF_S\{f(x)\}$

(D) None of these

103. If $f(x)$ is continuous and absolutely integrable on the x-axis, and $f'(x)$ is piecewise continuous on each interval, also $f(x) \to 0$ as $x \to \infty$. Then Fourier sine transform—

(A) $F_S\{f''(x)\} = \sqrt{\frac{2}{\pi}}\, \omega f(0)$

(B) $F_S\{f''(x)\} = -\omega^2 F_S\{f(x)\} + \sqrt{\frac{2}{\pi}}\, \omega f(0)$

(C) $F_S\{f''(x)\} = \omega^2 F_S\{f(x)\} + \sqrt{\frac{2}{\pi}} f(0)$

(D) None of these

104. Fourier sine transform of e^{-ax} $(a > 0)$ is—

(A) $\frac{\omega}{a^2 + \omega^2}$

(B) $\frac{\omega}{\sqrt{2a}} e^{-\omega^2/4a}$

(C) $\sqrt{\frac{2}{\pi}}\left(\Gamma(a)\omega^{-a} \sin\frac{a\pi}{2}\right)$

(D) None of these

105. Fourier sine transform of xe^{-ax^2} $(a > 0)$ is—

(A) $\frac{\omega}{a^2 + \omega^2}$

(B) $\frac{\omega}{\sqrt{2a}} e^{-\omega^2/4a}$

(C) $\sqrt{\frac{2}{\pi}}\left(\Gamma(a)\omega^{-a} \sin\frac{a\pi}{2}\right)$

(D) None of these

106. Fourier sine transform of x^{a-1} $a < 0 < 1$ is—

(A) $\frac{\omega}{a^2 + \omega^2}$

(B) $\frac{\omega}{\sqrt{2a}} e^{-\omega^2/4a}$

(C) $\sqrt{\frac{2}{\pi}}\left(\Gamma(a)\omega^{-a} \sin\frac{a\pi}{2}\right)$

(D) None of these

107. Fourier sine transform of $\cos ax^2$ is—

(A) $\sqrt{\frac{\pi}{2a}}\left[\sin\frac{\omega^2}{4a}\cos\left(\frac{\omega}{\sqrt{2\pi a}}\right) - \cos\frac{\omega^2}{4a}\sin\left(\frac{\omega}{\sqrt{2\pi a}}\right)\right]$

(B) $\sqrt{\frac{\pi}{2a}}\left[\cos\frac{\omega^2}{4a}\cos\left(\frac{\omega}{\sqrt{2\pi a}}\right) + \sin\frac{\omega^2}{4a}\sin\left(\frac{\omega}{\sqrt{2\pi a}}\right)\right]$

(C) $\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}$

(D) None of these

108. Fourier sine transform of $\sin ax^2$ is—

(A) $\sqrt{\frac{\pi}{2a}}\left[\sin\frac{\omega^2}{4a}\cos\left(\frac{\omega}{\sqrt{2\pi a}}\right) - \cos\frac{\omega^2}{4a}\sin\left(\frac{\omega}{\sqrt{2\pi a}}\right)\right]$

(B) $\sqrt{\frac{\pi}{2a}}\left[\cos\frac{\omega^2}{4a}\cos\left(\frac{\omega}{\sqrt{2\pi a}}\right) + \sin\frac{\omega^2}{4a}\sin\left(\frac{\omega}{\sqrt{2\pi a}}\right)\right]$

(C) $\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos a\omega}{\omega}\right]$

(D) $\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}$

109. Fourier sine transform of $\begin{cases} 1, 0 < x < a \\ 0, \text{otherwise} \end{cases}$ is—

(A) $\sqrt{\frac{\pi}{2a}}\left[\sin\frac{\omega^2}{4a}\cos\left(\frac{\omega}{\sqrt{2\pi a}}\right) - \cos\frac{\omega^2}{4a}\sin\left(\frac{\omega}{\sqrt{2\pi a}}\right)\right]$

(B) $\sqrt{\frac{\pi}{2a}}\left[\cos\frac{\omega^2}{4a}\cos\left(\frac{\omega}{\sqrt{2\pi a}}\right) + \sin\frac{\omega^2}{4a}\sin\left(\frac{\omega}{\sqrt{2\pi a}}\right)\right]$

(C) $\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos a\omega}{\omega}\right]$

(D) $\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}$

110. Fourier cosine transform of $\begin{cases} 1, 0 < x < a \\ 0, \text{otherwise} \end{cases}$ is—

(A) $\sqrt{\frac{\pi}{a}}\left[\sin\frac{\omega^2}{4a}\cos\left(\frac{\omega}{\sqrt{2\pi a}}\right) - \cos\frac{\omega^2}{4a}\sin\left(\frac{\omega}{\sqrt{2\pi a}}\right)\right]$

(B) $\sqrt{\frac{\pi}{2a}}\left[\cos\frac{\omega^2}{4a}\cos\left(\frac{\omega}{\sqrt{2\pi a}}\right) + \sin\frac{\omega^2}{4a}\sin\left(\frac{\omega}{\sqrt{2\pi a}}\right)\right]$

(C) $\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos a\omega}{\omega}\right]$

(D) $\sqrt{\frac{2}{\pi}}\frac{\sin a\omega}{\omega}$

111. Fourier Cosine transform of e^{-ax} $(a > 0)$ is—

(A) $\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}$

(B) $\frac{1}{\sqrt{2a}}e^{-\omega^2/4a}$

(C) $\frac{1}{\sqrt{2\pi}}\cos\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)$

(D) $\frac{1}{\sqrt{2\pi}}\cos\left(\frac{\omega^2}{4a}+\frac{\pi}{4}\right)$

112. Fourier cosine transform of e^{-ax^2} $(a > 0)$ is—

(A) $\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}$

(B) $\frac{1}{\sqrt{2a}}e^{-\omega^2/4a}$

(C) $\frac{1}{\sqrt{2\pi}}\cos\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)$

(D) $\frac{1}{\sqrt{2\pi}}\cos\left(\frac{\omega^2}{4a}+\frac{\pi}{4}\right)$

113. Fourier Cosine transform of $\cos ax^2$ is—

(A) $\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}$

(B) $\frac{1}{\sqrt{2a}}e^{-\omega^2/4a}$

(C) $\frac{1}{\sqrt{2\pi}}\cos\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)$

(D) $\frac{1}{\sqrt{2\pi}}\cos\left(\frac{\omega^2}{4a}+\frac{\pi}{4}\right)$

114. Fourier cosine transform of $\sin ax^2$ is—

(A) $\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}$

(B) $\frac{1}{\sqrt{2a}}e^{-\omega^2/4a}$

(C) $\frac{1}{\sqrt{2\pi}}\cos\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)$

(D) $\frac{1}{\sqrt{2\pi}}\cos\left(\frac{\omega^2}{4a}+\frac{\pi}{4}\right)$

115. Fourier cosine transform of x^{n-1} $(0 < a < 1)$ is—

(A) $\Gamma(a)\,\omega^{-a}\cos\frac{a\pi}{2}$

(B) $\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\omega^2}$

(C) $\frac{1}{\sqrt{2a}}e^{-\omega^2/4a}$

(D) $\frac{1}{2\pi}\cos\left(\frac{\omega^2}{4a}-\frac{\pi}{4}\right)$

116. Fourier transform of $f(x)$ is—

(A) $F(\omega)=\int_{-\infty}^{\infty} f(x)\, e^{i\omega x}dx$

(B) $F(\omega)=\int_{-\infty}^{\infty} f(x)\, dx$

(C) $F(\omega)=\int_{-\infty}^{\infty} e^{i\omega x}\, dx$

(D) None of these

117. Inverse Fourier transform of $F(\omega)$—

(A) $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega e^{i\omega x}d\omega$

(B) $f(x)=\int_{-\infty}^{\infty} F(w)\, e^{-iwx}dw$

(C) $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{-iwx}d\omega$

(D) None of these

118. Fourier transforms of $f(x)$ exist if—

(A) If $f(x)$ is absolutels integrable on positive x-axis

(B) If $f(x)$ is piecewise continuous on finite interval

(C) Both A and B

(D) None of these

119. If $f(x)$ is continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. $f'(x)$ is absolutely integrable on the x-axis. Then Fourier transform of derivative—

(A) $F\{f'(x)\} = i\,\omega\, F\{f(x)\}$

(B) $F\{f'(x)\} = F\{f(x)\}$

(C) $F\{f'(x)\} = \omega\, F\{f(x)\}$

(D) None of these

120. If $f(x)$ is continuous on the x-axis, and $f(x) \to 0$ as $|x| \to \infty$. $f'(x)$ is absolutely integrable on the x-axis. Then Fourier transform of derivative—

(A) $F\{f''(x)\} = -w^2\, F\{f(x)\}$

(B) $F\{f''(x)\} = w\, F\{f(x)\}$

(C) $F\{f''(x)\} = F\{f(x)\}$

(D) None of these

121. If $f(x)$ is continuous an the x-axis and $f(x) \to 0$ as $|x| \to \infty$. $f'(x)$ is absolutely integrable on the x-axis. Then Fourier transform of derivative—

(A) $F\left\{\frac{d^n f}{dx^n}\right\} = (-i)^n\, F\{f(x)\}$

(B) $F\left\{\frac{d^n f}{dx^n}\right\} = (\omega)^n\, F\{f(x)\}$

(C) $F\left\{\frac{d^n f}{dx^n}\right\} = (-i\omega)^n\, F(f(x))$

(D) $F\left\{\frac{d^n f}{dx^n}\right\} = F\{f(x)\}$

122. If $F(w)$ is the Fourier transform of $f(x)$, $-\infty < x < \infty$, then—

(A) $f(x)$ is even $\Rightarrow F(\omega) = F_C(\omega)$

(B) $f(x)$ is even $\Rightarrow F(\omega) = 2F_C(\omega)$

(C) $f(x)$ is even $\Rightarrow F(\omega) = 4F_C(\omega)$

(D) None of these

123. If $F(w)$ is the Fourier transform of $f(x)$, $-\infty < x < \infty$, then—

(A) $f(x)$ is odd $\Rightarrow F(\omega) = F_S(\omega)$

(B) $f(x)$ is odd $\Rightarrow F(\omega) = 2F_S(\omega)$

(C) $f(x)$ is odd $\Rightarrow F(\omega) = 2iF_S(\omega)$

(D) None of these

124. The Fourier transform of $\mu(x, t)$, $x \in R$ is defined by the integral—

(A) $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwx}\,\mu(x, t)dx$

(B) $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwx}dx$

(C) $\int_{-\infty}^{\infty} e^{-iwx}\,\mu(x, t)dx$

(D) None of these

125. If $F(w)$ is the Fourier transform of $f(x)$, then Fourier transform of $f(t)$ is—

(A) $\frac{1}{|a|}F\left(\frac{\omega}{a}\right)$ (B) $F(-\omega)$

(C) $e^{-iwt}F(\omega)$ (D) $F(\omega - r)$

126. If $F(w)$ is the Fourier transform of $f(x)$. Then Fourier transform of $f(at)$ is—

(A) $\frac{1}{|a|}F\left(\frac{\omega}{a}\right)$ (B) $F(-\omega)$

(C) $e^{-iwt}F(\omega)$ (D) $F(\omega - r)$

127. If $f(w)$ is the Fourier transform of $f(x)$, then Fourier transform of $f(t - \mathrm{T})$, T is real is—

(A) $\frac{1}{|a|}\mathrm{F}\left(\frac{\omega}{a}\right)$ (B) $\mathrm{F}(-\omega)$

(C) $e^{-iwt}\,\mathrm{F}(\omega)$ (D) $\mathrm{F}(\omega - r)$

128. If $\mathrm{F}(w)$ is the Fourier transform of $f(x)$, then Fourier transform of $e^{irt} f(t)$, r is real is—

(A) $\frac{1}{|a|}\mathrm{F}\left(\frac{\omega}{a}\right)$ (B) $\mathrm{F}(-\omega)$

(C) $e^{-iwt}\,\mathrm{F}(\omega)$ (D) $\mathrm{F}(\omega - r)$

Answers

1. (B)	2. (D)	3. (C)	4. (C)	5. (D)
6. (D)	7. (A,D)	8. (B)	9. (C)	10. (D)
11. (C)	12. (B)	13. (A)	14. (B)	15. (A)
16. (A)	17. (B)	18. (A)	19. (A)	20. (C)
21. (B)	22. (A)	23. (C)	24. (B)	25. (A)
26. (A)	27. (A)	28. (B)	29. (A)	30. (A)
31. (C)	32. (B)	33. (A)	34. (B)	35. (A)
36. (B)	37. (B)	38. (A)	39. (A)	40. (A)
41. (C)	42. (C)	43. (B)	44. (A)	45. (A)
46. (C)	47. (B)	48. (B)	49. (A)	50. (A)
51. (C)	52. (D)	53. (B)	54. (A)	55. (B)
56. (B)	57. (A)	58. (A)	59. (A)	60. (A)
61. (A)	62. (A)	63. (C)	64. (A)	65. (B)
66. (A)	67. (A)	68. (A)	69. (B)	70. (A)
71. (A)	72. (B)	73. (A)	74. (B)	75. (A)
76. (B)	77. (A)	78. (B)	79. (A)	80. (B)
81. (A)	82. (B)	83. (C)	84. (A)	85. (D)
86. (A)	87. (B)	88. (A)	89. (A)	90. (B)
91. (A)	92. (B)	93. (B)	94. (A)	95. (A)
96. (B)	97. (A)	98. (B)	99. (C)	100. (A)
101. (A)	102. (C)	103. (B)	104. (A)	105. (B)
106. (C)	107. (A)	108. (B)	109. (C)	110. (D)
111. (A)	112. (B)	113. (C)	114. (D)	115. (A)
116. (A)	117. (C)	118. (C)	119. (A)	120. (A)
121. (C)	122. (B)	123. (C)	124. (A)	125. (B)
126. (A)	127. (A)	128. (D)		

Mechanics

Mechanics—It is a branch of physics which deals with physical objects in motion and at rest under the influence of external and internal interaction.

Classical Mechanics—The mechanics based on Newton's laws of motion and alternatively developed by Lagrange, Hamilton and others. Semi permeable dipole moment, enthalpy, toluene internal enthalpy of vaporisation, sublimation.

1. **Newtonian (vectorial) Mechanics**—Classical mechanics deals with the Newton's laws and their consequences. Here vector quantities such as force, acceleration, momentum etc. are used.
2. **Analytical Mechancis**—It is an alternative and superior schemes in classical mechanics, developed by D' Alembert, Lagrange, Hamilton and others. Here scalar quantities such as energy rather than vector quantities are used and dynamical relations are obtained by systematic process of differentiation.

Co-ordinate Systems

Two dimensional systems :

1. Cartesian co-ordinate system (x, y)
2. Polar co-ordinate system (r, θ)
3. Relation between cartesian and polar co-ordinate systems.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

and $$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} y/x$$

Three dimensional systems :

1. Cartesian co-ordinate system (x, y, z)
2. Cylindrical co-ordinate system (r, θ, z)
3. Spherical coordinate system (r, θ, ϕ)
4. Relation between cartesian and cylindrical co-ordinate systems

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1} y/x \\ z = z \end{cases}$$

5. Relation between cartesian and spherical co-ordinae systems.

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\sqrt{\dfrac{x^2 + y^2}{z}}\right) \\ \phi = \tan^{-1} y/x \end{cases}$$

Frame of Reference

Point and event—If the physical phenomena occur in a space, its position is known as a point, the time of occurrence and the point taken together are called an event.

Frame of Reference—If a co-ordinate system is attached to a rigid body and one describe the position of any particle relative to it, then such co-ordinate system is called frame of reference.

Space-time reference system—The event which is represented by a position (point) and time is referred as space-time reference system.

Newton's Laws of Motion

1. **Law of inertia (First law)**—A body continues in its state of rest or uniform motion, unless no external force is applied to it.
2. **Law of force (Second law)**—The time rate of change of momentum is proportional to impressed force.

$$F = \frac{dP}{dt} = ma.$$

where P is momentum, m is mass and a an acceleration.

3. **Law of action and reaction (Third law)**—For every action, there is always equal and opposite reaction.

$$F_{ij} = -F_{ji}$$

Inertial frames (Galilean frames of reference)—A frame of reference in which law of inertia holds.

1. All those frames, which are moving with constant velocity relative to an initial frame are also inertial.

2. An inertial frame is non-accelerated.

   Non-inertial frames : an accelerated frame of reference.

Mechanics of a Particle

1. **Conservation of linear momentum**—In absence of external force, linear momentum of a particle is conserved.

   i.e. $$\frac{dP}{dt} = 0$$

   $$\Rightarrow \quad P = mv = \text{constant}$$

   here v is the velocity of a particle.

2. **Conservation of angular momentum**—In absence of external torque, angular momentum of a particle is constant of motion.

   $$\tau = \frac{dJ}{dt} = 0$$

   $$\Rightarrow \quad J = \text{constant}$$

Conservation of Energy

1. **Work energy theorem**—If W_{12} is the work done by an external force F upon a particle in displacing from point 1 to 2 and the change in kinetic energy from point 1 to 2 is T_1 and T_2 respectively, then

   $$W_{12} = \int_1^2 F.dr = T_2 - T_1$$

   where $$T_i = \frac{1}{2} mv_i^2$$

   (i = 1, 2) is kinetic energy

2. **Conservative force**—If the force is conservative, the work done on the particle around a closed path in the force field is zero.

   $$\oint F.dr = 0$$

3. **Work and potential energy**—If W_{12} is the work done by an external force F upon a particle in displacing from point 1 to 2 and the change in potential energy from point 1 to 2 is V_1 and V_2 respectively, then

   $$W_{12} = \int_1^2 F.dr = V_1 - V_2$$

   where $$V(r) = -\int_\infty^r F.dr$$

4. **Energy Conservation law**—The sum of kientic energy and potential energy (*i.e.*, total mechanical energy) of a particle remains constant in a conservative force field.

Mechanics of System of Particles

1. **External and internal forces**—If a mechanical system consists of N (N > 1) particles, then the force on the i-th particle is

   $$F_i = F_i^e + \sum_{j=1}^{N} F_{ij}$$

   where F_i^e is the external force acting on i-th particle due to sources outside the system and F_{ij} is the internal force on i-th particle due to j-th particle.

2. **Centre of mass**—The centre of mass is

   $$R = \frac{\sum_i m_i r_i}{\sum_i m_i} = \frac{\sum_i m_i r_i}{M}$$

   where $\sum_i m_i = M$ (total mass of system)

3. **Conservation of linear momentum**—If the total external force F^e on the system is zero, the total linear momentum is constant of motion.

   $$P = \sum_i m_i v_i = \text{constant}$$

4. **Centre of mass-frame of reference**—In centre of mass-frame of reference total linear momentum is zero.

   $$P = \sum_i m_i v_i = 0$$

5. **Conservation of total angular momentum**—The total angular momentum of a system of particles is conserved.

   $$\Sigma J_i = \text{constant}$$

6. **Relation between angular momentum (J) and angular momentum about centre of mass Jcm**—The total angular momentum of a system of particles about a point is the angular momentum of the system about the centre of mass plus the angular momentum about the reference point of the system mass concentrated at the centre of the mass.

Dynamical System

Dynamical System—It's a system of particles.

Configuration—The set of positions of all the particles.

Degree of Freedom—The minimum number of independent coordinates (or variables) required to specify the system.

Constraints—Limitations imposed on the motion of a system.

1. **Holonomic Constraints**—If the constraints can be expressed as equation form then holonomic constraints otherwise non-holonomic constraints.
2. **Rheonomous and Scleronomous Constraints**—Equation of constraints containing time as explicit variable then rheonomous otherwise scleronomous constraints.
3. **Conservative and Dissipative Constraints**—In conservative constraints total mechanical energy of the system is conserved during constrained motion and constraint forces do not do any work otherwise dissipative constraints.

Forces of Constraints—Constraints are always related to force which restrict the motion of the system. These forces are called forces of constraints.

Generalized Coordinates—A set of independent coordinates sufficient in number to describe completely the state of configuration of a dynamical system. These coordinates are denoted as $q_1, q_2, \ldots\ldots q_n$.

Principle of Virtual Work—The work done is zero in the case of an arbitrary displacement of a system from a position of equilibrium

$$\delta W = \sum_{i=1}^{N} F_i \cdot \delta r_i = 0$$

where F_i is the total of force on i-th particle and δr_i is the virtual displacement.

D' Alembert's Principle

$$\sum_{i=1}^{N} (F_i - P_i) \cdot \delta r_i = 0$$

where $P_i = \frac{\delta F_i}{\delta t}$ is the reversed effective force.

Components of Generalized Force—Let $q_1, q_2, \ldots, q_n$ are generalized coordinates and $r_i = r_i(q_1, q_2, \ldots, q_n)$, then for a system of n-particles, the components of generalized force Q_k associated with qk is—

$$Q_k = \sum_{i=1}^{N} F_i \cdot \frac{\delta r_i}{\delta q_k} = -\frac{\delta V}{\delta q_K}, \quad (K = 1, \ldots, n).$$

Lagrangian—$L = T - V$, where T is kinetic energy and V potential energy.

Lagrange's equation for holonomic system

$$\frac{d}{dt}\left(\frac{\delta T}{\delta q_k}\right) - \frac{\delta T}{\delta q_k} = Q_k \, (k = 1, \ldots, n)$$

Lagrange's equation for a conservative, holonomic dynamical system

$$\frac{d}{dt}\left(\frac{\delta T}{\delta q_k}\right) = \frac{\delta L}{\delta q_k} \qquad (k = 1, \ldots, n)$$

where $L = T - V$, a Lagrangian.

Lagrange's equations for conservative, non-holonomic system—

$$\frac{d}{dt}\left(\frac{\delta T}{\delta q_k}\right) - \frac{\delta L}{\delta q_k} = \sum_{j=1}^{m} \lambda_i a_{ik}, \quad (k = 1, \ldots, n)$$

where a_{ik} are function of q_k's and t and λ_t are constants ($i = 1, \ldots, m$).

Hamilton Principle

Hamilton's Principle—For a conservative holonomic system, the motion of a particle from time t_1 to time t_2 is such that line integral

$$S = \int_{t_1}^{t_2} L \, dt$$

has stationary (external) value for the correct path of motion. Here S is called Hamilton's principle function.

Conjugate momentum (Canonical momentum)—$P_k = \frac{\delta L}{\delta q_k}$ whenever the Lagrangian function does not contain a coordinate q_k explicitly, the generalized momentum P_k is a constant of motion *i.e.*, $P_k = \frac{\delta_L}{\delta q_k}$ = constant. The coordinate q_k are called cyclic or ignorable coordinate.

Hamiltonian Function

$$H = \sum_k P_k q_k - L$$

1. If time t does not appear in Lagrangian L explicity, then Hamiltonian function is constant in time and Hamiltonian function is conserved.

$$\frac{dH}{dt} = 0$$

$$\Rightarrow \quad H = \sum_k P_k q_k - L = \text{const.}$$

2. Hamiltonian H represents the total energy to the system and is conserved, provided the system is conservative.

$$\mathrm{H} = \mathrm{T} + \mathrm{V}$$

Hamilton's Equation

$$q_k = \frac{\delta \mathrm{H}}{\delta \mathrm{P}_k} \text{ and } -\mathrm{P}_k \frac{\delta \mathrm{H}}{\delta q_k}$$

If any coordinate q_k is cyclic, *i.e.*, not contained in H, then

$$\frac{\delta \mathrm{H}}{\delta q_k} = 0$$

or, $$\mathrm{P}_k = 0$$

Euler-Lagrange's Equations

1. Let $f(y_1, y_2, \ldots, y_k, \ldots, y_n, y_1', y_2', \ldots, y'_k, \ldots y'_n, x)$ and $y_i(x) = y_i$,

   then Euler-Lagrange's equations are

$$\frac{\delta f}{\delta \mathrm{Y}_k} - \frac{d}{dx}\left(\frac{\delta f}{\delta \mathrm{Y}'_k}\right) = 0,$$

$$k = 1, 2, \ldots, n$$

2. If $f(\mathrm{Y}, \mathrm{Y}', x)$ defines a cure $y = y(x)$, then Euler-Lagrange's equation is

$$\frac{\delta f}{\delta \mathrm{Y}} - \frac{d}{dx}\left(\frac{\delta f}{\delta \mathrm{Y}'_k}\right) = 0$$

Modified Hamilton Principle

$$\delta \int_{t_1}^{t_2} \left(\sum_{k=1}^{n} \mathrm{P}_k\, q_k - \mathrm{H}\right) dt = 0$$

Hamilton's principle for non-conservative, non-holonomic system.

$$\delta \int_{t_1}^{t_2} (\mathrm{T} + \mathrm{W})\, dt = 0$$

Principle of Least Action

1. For a conservative system

$$\Delta \int_{t_1}^{t_2} \sum_{k=1}^{n} \mathrm{P}_k\, q_k\, dt = 0$$

   where $\int_{t_1}^{t_2} \mathrm{P}_k\, q_k\, dt$ is called action.

2. Jacob's form

$$\Delta \int_{t_1}^{t_2} \sqrt{\mathrm{H} - \mathrm{V}(q)}\, d\mathrm{P} = 0$$

3. In terms of arc length of particle trajectory

$$\Delta \int_{t_1}^{t_2} \sqrt{2m(\mathrm{H} - \mathrm{V})}\, ds = 0$$

Equations of Motion of Rigid Bodies

Rigid Body—A rigid body is defined as a system of points subject to the holonomic constraints where the distances between all pairs of points remain constant throughout the motion, *i.e.*

$r_{ij} = c_{ij}$, where r_{ij} is the distance between ith and jth particles and c_{ij} is the constant.

The number of degrees of freedom for the general motion of a rigid body is six.

Body Co-ordinae System—A coordinate system, fixed in rigid body, is called a body coordinate system and its axes are called body set of axes.

Space Co-ordinate System—In this coordinate system axes are fixed in space, called space set of axes.

Eulerian Angles—They are defined as the three successive angles of rotation of rigid body about a point fixed in the body. We first rotate initial system of axes OXYZ (fixed by space) by an angle of counter clockwise about the Z-axis and the resultant coordinate system will be labeled as $\xi n \xi$. Secondly the intermediate axes $\xi n \xi$ are rotated about the ξ-axis counter clockwise by an angle θ to produce set of axes $\xi' n' \xi'$. The ξ'-axis is at the intersection of XY and $\xi' n$ planes and is known as lines of nodes. At least the axes $\xi n \xi$ are rotated counter-clockwise by an angle ψ about ξ axis to produce X′Y′Z′ system of axis (fixed in the body).

Here the angles (ϕ, θ, ψ) are known as Eularian angles.

Components of Angular Velocity—If ϕ, θ, ψ represents Euler's angle then ϕ, θ, ψ represents the angular velocity about the space Z-axis, line of nodes and body Z′ axis respectively, ω_ϕ, ω_θ, ω_ψ represents ϕ, θ, ψ respectively and called components of angular velocity ω.

Components of angular velocity along body set of axes—

$$\omega_x = \phi_x + \theta_x + \psi_x = \phi \sin\theta \sin\psi + \theta \cos\psi$$

$$\omega_y = \phi_y + \theta_y + \psi_y = \phi \sin\theta \cos\psi - \theta \sin\psi$$

$$\omega_z = \phi_z + \theta_z + \psi_z = \phi \cos\theta + \psi$$

Euler's Theorem—The general displacement of a rigid body with one point fixed is a rotation about some axis.

Chasle's Theorem—The general displacement of a rigid body is a translation with a rotation.

Angular Momentum and Inertia Tensor

Moment of Inertia

(a) Moment of inertia about X-axis.

$$I_{xx} = \sum_{i=1}^{n} m_i (y_i^2 + z_i^2)$$

(b) Moment of inertia about Y-axis

$$I_{yy} = \sum_{i=1}^{n} m_i (z_i^2 + y_i^2)$$

(c) Moment of inertia about Z-axis

$$I_{zz} = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2)$$

Inertia Tensor—The elements I_{xx}, I_{xy}, I_{xz}, I_{yx}, I_{yy}, I_{yz} and I_{zx}, I_{zy}, I_{zz} are called components of I and I is called inertia tensor, *i.e.*

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Principal Axes and Principal Moments of Inertia—If one choose the axes of the coordinate system fixed in the body with respect to which off diagonal elements disappear and only the diagonal elements remains in the inertia tensor, then such axes are called principal axes of the body and the corresponding moments of inertia a principal moment of inertia.

Euler's Equation of Motion for a Rigid Body

The x, y, z components of the torque is

$$T_x = I_{11} + (I_3 - I_2)_{23}$$

$$T_y = I_{22} + (I_1 - I_3)_{31}$$

$$T_z = I_{33} + (I_2 - I_1)_{12}$$

Principal of conservation of total rotational kinetic energy (in absence of torque)—

$$\frac{1}{2}\omega . J. = \text{constant}$$

Canonical Transformation—In canonical equation of motions, the generalized coordiantes $q_1, \ldots, q_n$ and generalized momentum $P_1, \ldots, P_n$ of a holonomic system are independent variables. The transformations of these $2n$ variables into a new set of variables

$$Q_i = Q_i(q, P, t) \text{ and } P_i + P_i(q, P, t)$$

are called canonical transformations if the motion equations of the system in new variables preserve the form of canonical equation of motion

$$Q_i = \frac{\delta H'}{\delta P}$$

and $$P_i = -\frac{\delta H'}{\delta Q_i}$$

where $H^i = H^i(Q, P, t)$ is a new Hamiltonian function.

Generating Function of the Canonical Transformation—The neccessary and sufficient condition for the transformation to be canonical is

$$\left[\sum_{i=1}^{n} P_i q_i - H(q, P, t)\, dt\right] - \left[\sum_{i=1}^{n} P_i Q_i - H^i(Q, P, t)\, dt\right] = dF$$

where F is a generating function of the canonical transformation. This generating function can be written in one of the following forms—

(i) $F_1(q, Q, t)$, provided that only q_i, Q_i are treated independent.

(ii) $F_2(q, P, t)$, provided that only q_i, P_i are treated independent

(iii) $F_3(P, Q, t)$, provided that only P_i, Q_i are treated independent.

(iv) $F_4(P, P, t)$, provided that only P_i, P_i are treated independent.

If $F = F_1$

$$P_i = \frac{\delta F_1}{\delta q_i}, \quad P_i = -\frac{\delta F}{\delta Q_i},$$

and $$H^i = H + \frac{\delta F_1}{\delta t}$$

If $F = F_2$

$$P_i = \frac{\delta F_2}{\delta q_i}, \quad Q_i = \frac{\delta F_2}{\delta P_i}$$

and $$H^i = H + \frac{\delta F_2}{\delta t}$$

If $F = F_3$

$$q_i = \frac{\delta F_3}{\delta P_i}, \quad P_i = -\frac{\delta F_3}{\delta Q_i}$$

and $$H^i = H + \frac{\delta F_3}{\delta t}$$

If $F = F_4$

$$q_i = \frac{\delta F_4}{\delta P_i}, \quad Q_i = -\frac{\delta F_4}{\delta P_i}$$

and $$H^i = H + \frac{\delta F_4}{\delta t}$$

Brackets

Lagrange's Bracket :

$$\{u, V\}_{q, P} = \sum_i \left(\frac{\delta q_i}{\delta u}\frac{\delta P_i}{\delta v} - \frac{\delta q_i}{\delta v}\frac{\delta P_i}{\delta u}\right)$$

$$= \sum_i \frac{\delta (q_i, P_i)}{\delta (u, v)}$$

Poisson's Bracket—

$$\{u, v\}_{q, P} = \sum_i \left(\frac{\delta u}{\delta q_i}\frac{\delta v}{\delta P_i} - \frac{\delta u}{\delta P_i}\frac{\delta v}{\delta q_i}\right)$$

$$= \frac{\delta (u, v)}{\delta (q_i, P_i)}$$

Phase Space—Phase space is a $2n$ dimensional space having coordinates $q_1, \ldots q_n$ and $P_1, \ldots P_n$.

Hamilton's equation of motion in Poisson's bracket

$$[q, H]_{q, P} = q$$

and $$[P, H]_{q, P} = P$$

Infinitesimal Contract Transformations—Those transformation in which the new set of coordinates (Q_k, P_k) differ from the old set (q_k, P_k) by infinitesimals.

Some Important Theorems

1. The transformation from variables $(q_1, q_2, \ldots, q_n, P_1, P_2, \ldots P_n)$ to variables $(Q_1, Q_2, \ldots Q_n, P_1, P_2, \ldots, P_n)$ will be a contract or canonical transformation if,

$$\sum_{i=1}^{n} P_r . d\,(Q_r - P_r\, dq_r) = dT$$

where dT is an exact differential.

2. **Poineare theorem**—Under Canonical transformations, the integral $J_1 = \int\int_S \sum_i dg_i\, dP_i$ remains invariant, where S is a two-dimensional surface in phase space.
3. Lagrange's bracket is canonical invariant.
4. Poisson's bracket is invariant under canonical transformation.
5. Jacobi's Identity for Poisson's brackets—If u, v, w are functions of q and P only, then $\{u, [v, w]\} + \{v, [w, u]\} + \{w\,\{u, v]\} = 0$.
6. If u, v are constants of motion, then $[u, v]$ is also constant of motion.

Small Oscillations and Normal Modes

Potential Energy and Equilibrium

System in Equilibrium—The system is said to be an equilibrium if the generalized forces acting on the system are equal to zero.

Stable Equilibrium—A system is in stable equilibrium, if a small displacement of the system from the rest position (after giving small energy) results in a small bounded motion about the equilibrium position.

Unstable Equilibrium—A system is in unstable equilibrium, if a small displacement of the system from the rest position (after giving small energy) results in an unbounded motion about the equilibrium position.

Neutral Equilibrium—A system is in neutral equilibrium, if the system on displacement has no tendency to move about or above the equilibrium position.

Two Coupled Oscillators—

$$m\ddot{x}_1 + Kx_1 + K'(x_1 + x_2) = 0$$

The equation of motion are

$$m\ddot{x}_2 + Kx_2 + K'(x_1 + x_2) = 0.$$

OBJECTIVE TYPE QUESTIONS

1. For a conservative system, generalized force—
   (A) Has necessarily the dimensions of force
   (B) Is a dimensionless quantity
   (C) Can not have dimensions of force
   (D) May have the dimensions of torque

2. Generalized coordinates—
   (A) Depend on each other
   (B) Are independent of each other
   (C) Are spherical coordinates
   (D) None of these

3. A rigid body moving freely in space has degrees of freedom—
   (A) Three (B) Six
   (C) Nine (D) Twelve

4. If the Lagrangian does not depend on time explicity—
   (A) The Hamiltonian is constant
   (B) Hamiltonian not constant
   (C) The kinetic energy is constant
   (D) The potential energy is constant
5. Choose the correct statements—
   (A) The angular momentum is not conserved for systems possessing rotational symmetry
   (B) If the Lagrangian of system is not invariant under translation along a direction, the corresponding linear momentum is conserved
   (C) If the Lagrangian of system is not invariant under translation along a direction, we can not say anything about the corresponding linear momentum
   (D) For a conservative system, the Hamiltonian is equal to the sum of kinetic and potential energies
6. A particle is constrained to move along the inner surface of a fixed hemispherical bowl. The number of degrees of freedom of the particle is—
   (A) One (B) Two
   (C) Three (D) Four
7. Choose the correct statements—
   (A) The angular momentum is not conserved for systems possessing rotational sysmmetry.
   (B) If the Lagrangian of a system is invariant under translation along a direction, the corresponding linear momentum is not conserved.
   (C) If the Lagrangian of a system is invariant under translation along a diection, we can not say anything about the corresponding linear momentum.
   (D) For a non conservative system, the Hamiltonian is equal to the sum of kinetic and potential energies.
8. Choose the correct statements—
   (A) The angular momentum is not conserved for systems possessing rotational symmetry.
   (B) If the Lagrangian of a system is invariant under translation along a direction, the corresponding linear momentum is conserved
   (C) If the Lagrangian system is not invariant under translation along a direction, we can not say anything about the corresponding linear momentum.
   (D) For conservative system, the Hamiltonian is not equal to the sum of kinetic and potential energies.
9. Choose the correct statements—
   (A) The angular momentum is conserved for systems possessing rotational symmetry.
   (B) If the Lagrangian of a system is invariant under translation along a direction, the corresponding linear momentum is not conserved.
   (C) If the Lagrangian of a system is not invariant under translation along a direction, one can not say anything about the corresponding linear momentum.
   (D) For a conservative system, the Hamiltonian is not equal to the sum of kinetic and potential energies.
10. The dimensions of generalized momentum—
    (A) Are always those of linear momentum
    (B) Are always those of angular momentum
    (C) May be those of angular momentum
    (D) None of these
11. The dimensions of generalized momentum—
    (A) Are always those of linear momentum
    (B) Are always those of angular momentum
    (C) May be those of linear momentum
    (D) None of these
12. Whenever the Lagrangian for a system does not contain a coordinate explicity—
    (A) P_k is the cyclic coordinate
    (B) P_k the generalized momentum, is a constant of motion
    (C) q_k is always zero
    (D) None of these
13. Whenever the Lagrangian for a system does not contain a coordinate explicity—
    (A) q_K is cyclic coordinate
    (B) P_K is cyclic coordinate
    (C) P_K is a constant of motion
    (D) None of these

14. 100 Rutherford differential scattering cross-section—
   (A) Has the dimensions of area
   (B) Has the dimension of solid angle
   (C) Is proportional to the square of the kinetic energy of the incident particle
   (D) None of these

15. A particle is moving under central force about a fixed centre of force. Choose the correct statements—
   (A) The motion of the particle is always an a circular path.
   (B) Its angular momentum is conserved
   (C) Its kinetic energy remains constant
   (D) None of these

16. A particle is moving under central force about a fixed centre of force. Choose the correct statements—
   (A) The motion of the particle is always on a circular path
   (B) Its kinetic energy remains constant
   (C) Motion of the particle takes place in a plane
   (D) None of these

17. If P_k and q_k (K = 1, 2, 3) represent the momentum and position coordinates respectively for a particle—
   (A) The phase space is six dimensional
   (B) The configuration space is six dimensional
   (C) The configuration space is three dimensional
   (D) None of these

18. If P_k and q_k (K = 1, 2, 3) represent the momentum and position coordinates respectively for a particle—
   (A) The phase space is six dimensional
   (B) The configuration space is six dimensional
   (C) The phase space is three dimensional
   (D) None of these

19. If the Poisson bracket of a function with the Hamiltonian vanishes—
   (A) The function depends upon time
   (B) The function does not depend on time explicity
   (C) The function is not the constant of motion
   (D) None of these

20. If the Poisson bracket of a function with the Hamiltonian vanishes—
   (A) The function depends upon time
   (B) The function is a constant of motion
   (C) The function is not the constant of motion
   (D) None of these

21. For a one-dimensional harmonic oscillator, the representative point in two dimensional phase space traces—
   (A) An ellipse
   (B) A parabola
   (C) A hyperbola
   (D) Always a straight line

22. Hamilton principal function δ and Hamilton characteristic function W for conservative system are related as, where E is the total energy and t is the time—
   (A) $\delta = W$
   (B) $\delta = W - Et$
   (C) $\delta = W + Et$
   (D) None of these

23. If we make a canonical transformation from the set of variables (P_k, q_k) to new set of variables (P_k, Q_k) and the transformed Hamiltonina is identically zero, then—
   (A) The new variables are cyclic
   (B) The new variables are not constant in time
   (C) The new variables are not cyclic
   (D) None of these

24. If we make a canonical transformation from the set of variables (P_k, q_k) to new set of variables (P_k, Q_k) and the transformed Hamiltonian is identically zero, then—
   (A) The new variables are constnat in time
   (B) The new variables are not constant in time
   (C) The new variables are not cyclic
   (D) None of these

25. In case of two coupled identical pendulums, in general—
   (A) The potential energy is a homogeneous quadratic functions, when expressed in terms of actual displacements.

(B) The potential energy is not a homogeneous quadratic functions, when expressed in terms of normal coordinates.

(C) The potential energy is a homogeneous quadratic functions, when expressed in terms of normal coordinates.

(D) None of these

26. In case of two coupled identical pendulums, in general—

(A) The potential energy is a homogoneous quadratic function, when expressed in terms of actual displacements.

(B) The potential energy is not a homogeneous guadratic functions, when expressed in terms of actual displacements.

(C) The potential energy is not a homogeneous quadratic functions when expressed in terms of normal coordinates

(D) None of these

27. In the case of symmetric mode for two coupled identical spring mass system—

(A) Both masses have equal and opposite displacements in phase

(B) Both masses vibrate with the frequency of single spring mass-system

(C) Both masses vibrate with a frequency different to the frequency of single spring-mass system

(D) None of these

28. In the case of symmetric mode for two coupled identical spring-mass system—

(A) Both masses have equal displacement in phase

(B) Both masses have equal and opposite displacement in phase

(C) Both masses vibrate with a frequency different to the frequency of single-spring-mass system

(D) None of these

29. In case of two coupled identical pendulums, oscillating in a plane—

(A) Each pendulum always execute simple harmonic motion

(B) The general motion can be expressed as a superposition of two simple harmonic motions of the same frequency.

(C) The general motion can be expressed as a superposition of two simple harmonic motions of the different frequency

(D) None of these

30. In case of two coupled identical pendulums, oscillating in a plane.

(A) Each pendulum always execute simple harmonic motion.

(B) Two pendulum always execute simple harmonic motion.

(C) The general motion can be expressed as a superposition of two simple harmonic motions of the same frequency.

(D) None of these

31. An example of stable equilibrium is—

(A) A pendulum in the rest position

(B) An egg standing on one end

(C) Book placed flat anywhere on table

(D) None of these

32. For two coupled oscillators, the equation of motion are—

(A) $m\ddot{x}_1 + Kx_1 + K'(x_1 + x_2) = 0$

$m\ddot{x}_2 + Kx_2 + K'(x_1 + x_2) = 0$

(B) $m\ddot{x}_2 + Kx_2 + K'(x_1 + x_2) = 0$ only

(C) $m\ddot{x}_1 + Kx_2 + K'(x_1 + x_2) = 0$ only

(D) None of these

33. An example of stable equilibrium is—

(A) A hanging spring-mass system in the stationary position

(B) An egg standing an one end

(C) A book placed flat anywhere an a table

(D) None of these

34. If a small displacement of the system from the rest position (after giving small energy) results in a small bounded motion about the equilibrium position. Then system is in—

(A) Stable equilibrium

(B) Unstable equilibrium

(C) Neutral equilibrium

(D) None of these

35. The system is said to be in equilibrium if the generalized forces acting on the system.

(A) Are equal to zero

(B) Are non zero
(C) Are infinite
(D) None of these

36. Poisson's bracket is—
(A) Invariant under canonical transformation
(B) Variant under canonical transformation
(C) Both (A) and (B)
(D) None of these

37. Unstable equilibrium—
(A) If a small displacement of the system from the rest position (after giving small energy) results in a small bounded motion about the equilibrium position
(B) If a small displacement of the system from the rest position (after giving small energy) results in an unbounded motion about the equilibrium position
(C) If the system on displacement has no tendency to move about or above the equilibrium position
(D) None of these

38. Lagrange's bracket is—
(A) Canonical invariant
(B) Cannonical variant
(C) Non-invariant
(D) None of these

39. Neutral equilibrium—
(A) If a small displacement of the system from the rest position (after giving small energy) results in a small bounded motion about the equilibrium position
(B) If a small displacement of the system from the rest position (after giving small energy) results in an unbounded motion about the equilibrium position
(C) If the system on displacement has no tendency to move about or above the equilibrium position
(D) None of these

40. Moment of inertia about z-axis is—
(A) $I_{xx} = \sum_{i=1}^{n} m_1 (y_i^2 + z_i^2)$
(B) $I_{yy} = \sum_{i=1}^{n} m_i (z_i^2 + y_i^2)$
(C) $I_{zz} = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2)$
(D) None of these

41. The following $I_{zz} = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2)$ represents—
(A) Moment of intertia about X-axis
(B) Moment of inertia about Y-axis
(C) Moment of inertia about Z-axis
(D) None of these

42. Stable equilibrium—
(A) If a small displacement of the system from the rest position (after giving small energy) results in a small bounded motion about the equilibrium position
(B) If a small displacement of the system from the rest position (after giving small energy) results in an unbounded motion about the equilibrium position
(C) If the system an displacement has no tendency to move about or above the equilibrium position
(D) None of these

43. Charle's theorem states—
(A) The general displacement of a rigid body with one point fixed is a rotation about some axis
(B) The general displacement of a rigid body is a translation with a rotation
(C) Lagrange's bracket is canonical invariant
(D) Poisson's bracket is invariant under canonical transformation

44. Moment of inertia about Y-axis is—
(A) $I_{xx} = \sum_{i=1}^{n} m_i (y_i^2 + z_i^2)$
(B) $I_{yy} = \sum_{i=1}^{n} m_i (z_i^2 + y_i^2)$
(C) $I_{zz} = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2)$
(D) None of these

45. The following $I_{yy} = \sum_{i=1}^{n} m_i (z_i^2 + y_i^2)$ represent—
(A) Moment of inertia about X-axis
(B) Moment of inertia about Y-axis
(C) Moment of inertia about Z-axis
(D) None of these

46. The general displacement of a rigid body with one point fixed is a rotation about some axis—
    (A) Euler's theorem
    (B) Charle's theorem
    (C) Law of inertia
    (D) Law of force

47. If $f(y, y', x)$ defines a curve $y = y(x)$, then Euler-Lagrange's equation is—
    (A) $\frac{\delta f}{\delta y} \frac{d}{dx}\left(\frac{\delta f}{\delta y'_K}\right) = 0$
    (B) $\frac{\delta f}{\delta y} + \frac{d}{dx}\left(\frac{\delta f}{\delta y'_K}\right) = 0$
    (C) $\frac{\delta f}{\delta y} - \frac{d}{dx}\left(\frac{\delta f}{\delta y'_K}\right) = 0$
    (D) None of these

48. The general displacement of a rigid body is a translation with a rotation—
    (A) Euler's theorem
    (B) Charle's theorem
    (C) Law of inertia
    (D) Law of force

49. Moment of inertia about x-axis is—
    (A) $I_{xx} = \sum_{i=1}^{n} m_i (y_i^2 + z_i^2)$
    (B) $I_{yy} = \sum_{i=1}^{n} m_i (z_i^2 + y_i^2)$
    (C) $I_{zz} = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2)$
    (D) None of these

50. A system is in ………, if the system on displacement has no tendency to move about or above the equilibrium position—
    (A) Stable equilibrium
    (B) Unstable equilibrium
    (C) Neutral equilibrium
    (D) None of these

51. A system is in ………, if a small displacement of the system from the rest position (after giving small energy) results in an unbounded motion about the equilibrium position—
    (A) Stable equilibrium
    (B) Unstable equilibrium
    (C) Neutral equilibrium
    (D) None of these

52. Euler's theorem states—
    (A) The general displacement of a rigid body with one point fixed is a rotation about some axis
    (B) The general displacement of a rigid body is a translation with a rotation
    (C) Lagrange's bracket is canonical invariant
    (D) Poisson's bracket is invriant under canonical transformation

53. Newton's First low of motion states—
    (A) A body continues in its state of rest or uniform motion, unless no external force is applied to it.
    (B) The time-rate of change of momentum is proportional to impressed force.
    (C) For every action there is always equal and opposite reaction
    (D) None of these

54. For every action there is always equal and opposite reaction—
    (A) Law of inertia
    (B) Law of force
    (C) Law of action and reaction
    (D) None of these

55. Lagrange's equations for conservative, non-holonomic system are—
    (A) $\left(\frac{\delta T}{\delta q_K}\right) - \frac{\delta L}{\delta q_K} = \sum_{i=1}^{m} \lambda_i, a_{iK}$, (K = 1, …, n)
    where a_{iK} are function of q_K·s and t and λ_i are constants (i = 1, …, m)
    (B) $\frac{d}{dt}\left(\frac{\delta T}{\delta q_K}\right) = \sum_{i=1}^{m} \lambda_i, a_{iK}$, (K = 1, …, n)
    where a_{iK} are function of q_K·s and t and λ_i are constants (i = 1, …, m)
    (C) $\frac{d}{dt}\left(\frac{\delta T}{\delta q_K}\right) - \frac{\delta L}{\delta q_k} = \sum_{i=1}^{m} \lambda_i, a_{iK}$, (K = 1, …, n) where a_{iK} are function of q_K·s and t and λ_t are constants (i = 1, …, m)
    (D) None of these

56. Let $f(y_1, y_2, \ldots, y_K, \ldots, y_n, y_1', y_2', \ldots, y_K', \ldots, y_x', x)$ and $y_i(x) = y_i$. Then Euler Lagrange's equations are—
    (A) $\frac{\delta f}{\delta y_k} + \frac{d}{dx}\left(\frac{\delta f}{\delta y'_k}\right) = 0$, K = 1, 2, …, n

(B) $\frac{\delta f}{\delta y_k} \frac{d}{dx}\left(\frac{\delta f}{\delta y'_k}\right) = 0$, K = 1, 2, …, n

(C) $\frac{\delta f}{\delta y_k} - \frac{d}{dx}\left(\frac{\delta f}{\delta y'_k}\right) = 0$, K = 1, 2, …, n

(D) None of these

57. If any coordinate q_k is cyclic, *i.e.*, not contained in H, then—
    (A) $\frac{\delta H}{\delta q_K} = 0$
    (B) $P_K = 0$
    (C) (A) and (B) both true
    (D) (A) or (B) are true

58. Hamilton's equation is—
    (A) $q_k = \frac{\delta H}{\delta P_K}$
    (B) $-P_K = \frac{\delta H}{\delta q_K}$
    (C) (A) and (B) both
    (D) None of these

59. Scleronomous constraints are—
    (A) The constraints that can be expressed as equation form
    (B) The constraints that can not be expressed as equation form
    (C) Equation of constraints that contain time as explicit variable
    (D) Equation of constraints that does not contains time as explicit variable

60. The constraints that can not be expressed as equation form—
    (A) Holonomic constraints
    (B) Non-holonomic constraints
    (C) Rheonomous constraints
    (D) Scleronomous constraints

61. Hamiltonion H is defined as—
    (A) The total energy of the system
    (B) The difference in energy of the system
    (C) The product of energy of the system
    (D) None of these

62. Hamiltonian function is—
    (A) $H = L \sum_K P_K q_K$ (B) $H = \sum_K P_K q_K - L$
    (C) $H = \sum_K P_K q_K + L$ (D) None of these

63. The Hamilton's principle states for a conservative holonomic system, the motion of a particle from time t_1 to time t_2 is such that line integral $S = \int_{T_1}^{T_2} L\, dt$ has …… for the correct path of motion. Here S is called Hamilton's principle function—
    (A) Stationary (external) value
    (B) Mean value
    (C) Non-stationary value
    (D) None of these

64. Lagrange's equation for holonomic system are—
    (A) $\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_K}\right)\frac{\delta T}{\delta q_K} = Q_K$ (K = 1, …, n)
    (B) $\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_K}\right) / \frac{\delta T}{\delta q_K} = Q_K$ (K = 1, …, n)
    (C) $\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_K}\right) - \frac{\delta T}{\delta q_K} = Q_K$ (K = 1, …, n)
    (D) None of these

65. Rheonomous constraints are—
    (A) The constraints that can be expressed as equation form
    (B) The constraints that can not be expressed as equation form
    (C) Equation of constraints that contain time as explicit variable
    (D) Equation of constraints that does ot contains time as explicit variable

66. Non-holonomic constraints are—
    (A) The constaints that can be expressed as equation form
    (B) The constraints that can not be expressed as equation form
    (C) Equation of constraints that contain time as explicit variable
    (D) Equation of constraints that does not contain time as explicit variable

67. Lagrange's equations for a conservative, holonomic dynamical system are—
    (A) $\left(\frac{\delta T}{\delta \dot{q}_K}\right) = \frac{\delta L}{\delta q_K}$ (K = 1, …, n)
    (B) $\left(\frac{\delta T}{\delta \dot{q}_K}\right)^2 = \frac{\delta L}{\delta q_K}$ (K = 1, …, n)
    (C) $\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_K}\right) = \frac{\delta L}{\delta q_K}$ (K = 1, …, n)
    (D) None of these

68. Lagrangian is defined as—
    (A) L = T – V, where T is kinetic energy and V potential energy
    (B) L = TV, where T is kinetic energy and V potential energy
    (C) L = T ± V, where T is kinetic energy and V potential energy
    (D) L = T/V, where T is kinetic energy and V potential energy

69. In centre of mass-frame of refrence—
    (A) Total linear momentum is zero
    (B) Total linear momentum is less than zero
    (C) Total linear momentum is greater than zero
    (D) None of these

70. D′ Alembert's principle is—
    (A) $\sum_{i=1}^{N} (F_i - P_i) \cdot \delta r_i \neq 0$, where $P_i = \frac{\delta F_i}{\delta t}$ is the reversed effective force
    (B) $\sum_{i=1}^{N} (F_i - P_i) \cdot \delta r_i = 0$, where $P_i = \frac{\delta F_i}{\delta t}$ is the reversed effective force
    (C) $\sum_{i=1}^{N} (F_i - P_i) \cdot \delta r_i \geq 0$, where $P_i = \frac{\delta F_i}{\delta t}$ is the reversed effective force
    (D) $\sum_{i=1}^{N} (F_i - P_i) \cdot \delta r_i \leq 0$, where $P_i = \frac{\delta F_i}{\delta t}$ is the reversed effective force

71. Principle of virtual work states—
    (A) The work done is zero in the case of an arbitrary displacement of a system from a position of equilibrium
    (B) The work done is non-zero in the case of an arbitrary displacement of a system from a position of equilibrium
    (C) The work done is infinite in the case of an arbitrary displacement of a system from a position of equilibrium
    (D) None of these

72. Generalized coordinates are—
    (A) A set of dependent coordinates sufficient in number to describe completely the state of configuration of a dynamical system
    (B) A set of independent coordinates sufficient in number to describe completely the state of configuration of dynamical system
    (C) A set of dependent coordinates excesses in number to describe completely the state of configuration of a dynamical system
    (D) None of these

73. Centre of mass-frame of reference—
    (A) If the total external force on the system is zero, the total linear momentum is constant of motion
    (B) Total linear momentum is zero
    (C) The total angular momentum of a system of particles is conserved
    (D) None of these

74. Equation of constraints that contain time as explicit variable are referred as—
    (A) Holonomic constraints
    (B) Non-holonomic constraints
    (C) Rheonomous constraints
    (D) Scleronomous constraints

75. The constraints that can be expressed as equation form—
    (A) Holonomic constraints
    (B) Non-holonomic constraints
    (C) Rheonomous constraints
    (D) Scleronomous constraints

76. Limitations imposed on the motion of a system are referame of reference
    (A) Constraints
    (B) Frame of reference
    (C) Degree of freedom
    (D) None of these

77. Holonomic constraints are—
    (A) The constraint that can be expressed as equation form
    (B) The constraint that can not be expressed as equaton form
    (C) Equation of constraints that contain time as explicit variable
    (D) Equation of constraints that does not contain time as explicit variable

78. Equation of constraints that does not contain time as explicit variable are referred as—
(A) Holonomic constraints
(B) Non-holonomic constraints
(C) Rheonomous constraints
(D) Scleronomous constraints

79. Degree of freedom is—
(A) The minimum number of independent coordinates (or variables) required to specify the system
(B) The maximum number of independent coordinates (or variables) required to specify the system
(C) The minimum number of dependent coordinates (or variables) required to specify the system
(D) The maximum number of dependent coordinates (or variables) required to specify the system

80. The total angular momentum of a system of particles about a point—
(A) Is the sum of angular momentum of the system about the centre of mass and the angular momentum about the reference point of the system mass, concentrated at the centre of the mass
(B) Is the difference of a angular momentum of the system about the centre of mass and the angular momentum about the reference point of the system mass, concentrated at the centre of the mass
(C) Is the product of angular momentum of the system about the centre of mass and the angular moment about the reference point of the system mass concentrated at the centre of the mass
(D) None of these

81. Conservation of linear momentum—
(A) If the total external force on the system is zero, the total linear momentum is constant of motion
(B) Total linear momentum is zero
(C) The total angular momentum of a system of particles is conserved
(D) None of these

82. The centre of mass is defined as—
(A) $R = \dfrac{\sum_i m_i r_i}{\sum_i m_i} = \dfrac{\sum_i m_i r_i}{M}$, where $\sum_i m_i = M$ (total mass of system)
(B) $R = \dfrac{\sum_i m_i r_i}{\left(\sum_i m_i\right)^2} = \dfrac{\sum_i m_i r_i}{M^2}$ where $\sum_i m_i = M$ (total mass of system)
(C) $R = \sum_i m_i r_i \sum_i m_i = M \sum_i m_i r_i$, where $\sum_i m_i =$ M (total mass of system)
(D) None of these

83. Conservation of total angular momentum—
(A) If the total external force on the system is zero, the total linear momentum is constant of motion
(B) In centre of mass-frame of reference, total linear momentum is zero
(C) The total angular momentum of a system of particles is conserved
(D) None of these

84. If the work done by forces on the particle around a closed path in the force field is zero—
(A) The force are conservative
(B) The forces are non conservative
(C) Both (A) and (B) are true
(D) None of these

85. Relation between cartesian and spherical coordinate system—
(A) $\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r\cos\theta \end{cases}$
(B) $\begin{cases} x = r \sin\theta \sin\phi \\ y = r \sin\theta \sin\phi \\ z = r\sin\theta \end{cases}$
(C) $\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \end{cases}$
(D) None of these

86. Relation between cartesian and cylindrical coordinate system—
(A) $\begin{cases} r = \sqrt{x^2 y^2 z^2} \\ \theta = \tan^{-1} y/x \\ z = z \end{cases}$
(B) $\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1} y/x \\ z = z \end{cases}$

(C) $\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan y/x \\ z = z \end{cases}$

(D) None of these

87. Energy conservation law states—
    (A) The sum of kinetic energy and potential energy (*i.e.* total mechanical energy) of a particle remains constant in a conservative force field
    (B) The sum of kinetic energy and potential energy (*i.e.* total mechanical energy) of a particle remains not constant in conservative force field.
    (C) The sum of kinetic energy and potential energy *i.e.* (total mechanical energy)of a particle remains constant in non-conservative force field
    (D) None of these

88. If W_{12} is the work done by an external force F upon a particle in displacing from point 1 to 2 and the change in potential energy from point 1 to 2 is V_1 and V_2 respectively. Then—

    (A) $W_{12} = \int_1^2 F.dr = V_1 + V_2$

    (B) $W_{12} = \int_1^2 F.dr = V_1 - V_2$

    (C) $W_{12} = \int_1^2 F.dr = V_1 V_2$

    (D) None of these

89. If the force is conservative, the work done on the particle—
    (A) Around a closed path in the force field is zero
    (B) Around a open path in the force field is zero
    (C) Around a closed path in the force field is non-zero
    (D) None of these

90. If W_{12} is the work done by an external force F upon a particle in displacing from point 1 to 2 and the change in kinetic energy from point 1 to 2 is T_1 and T_2 respectively. Then—

    (A) $W_{12} = \int_1^2 F.dr = T_1 T_2$

    (B) $W_{12} = \int_1^2 F.dr = T_1 + T_2$

    (C) $W_{12} = \int_1^2 F.dr = T_2 - T_1$

    (D) None of these

91. Newton's second law of motion states—
    (A) A body continues in its state of rest or uniform motion, unless no external force is applied to it
    (B) The time rate of change of momentum is proportional to impressed force
    (C) For every action there is always equal and opposite reaction
    (D) None of these

92. The time-rate of change of momentum is proportional to impressed force—
    (A) Law of inertia
    (B) Law of force
    (C) Law of action and reaction
    (D) None of these

93. The following represents the spherical coordinates—
    (A) (x, y, z) (B) (r, θ, ϕ)
    (C) (r, θ, z) (D) None of these

94. The following represents the cylindrical coordinate—
    (A) (x, y, z) (B) (r, θ, ϕ)
    (C) (r, θ, z) (D) None of these

95. The following represents the cartesian coordinate—
    (A) (x, y, z) (B) (r, θ, ϕ)
    (C) (r, θ, z) (D) None of these

96. An inertial frame is—
    (A) Non-accelerated frame of reference
    (B) Accelerated frame of reference
    (C) Both (A) and (B)
    (D) None of these

97. A frame of reference in which law of inertia holds—
    (A) Galilean frames of reference
    (B) Non Newtonian frame of reference
    (C) Accelerated frames
    (D) None of these

98. Newton's third law of motion states—
    (A) A body continues in its state of rest or uniform motion, unless no external force is applied to it

(B) The time-rate of change of momentum is proportional to impressed at force

(C) For every action there is always equal and opposite reaction

(D) None of these

99. A body continues in its state of rest or uniform motion, unless no external force is applied to it—

(A) Law of inertia

(B) Law of force

(C) Law of action and reaction

(D) None of these

100. Non inertial frame is—

(A) Non-accelerated frame of reference

(B) Accelerated frame of reference

(C) Both (A) and (B)

(D) None of these

101. If a coordinate system is attached to a rigid body and one deseribe the position of any particle relative to it, then such coordinate system is called ....

(A) Frame of reference

(B) Space-time reference system

(C) Coordinate system

(D) None of these

102. If the physical phenomena occur in a space, its position is known as .... the time of occurrence and the point taken together are called ....

(A) Point, Event (B) Event, Point

(C) Space, Event (D) None of these

103. Relation between cartesian and spherical coordinate systems—

(A) $\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\sqrt{\dfrac{x^2 + y^2}{z}} \\ \phi = \tan^{-1} y/x \end{cases}$

(B) $\begin{cases} r = \sqrt{x^2y^2z^2} \\ \theta = \tan^{-1}\sqrt{\dfrac{x^2y^2}{z}} \\ \phi = \tan y/x \end{cases}$

(C) $\begin{cases} r = \sqrt{x^2 + y^2/z^2} \\ \theta = \tan yz/x \\ \phi = \tan^{-1} y/x \end{cases}$

(D) None of these

104. Relation between cartesian and cylindrical coordinate system—

(A) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$ (B) $\begin{cases} x = r\sin\theta \\ y = r\cos\theta \\ z = z \end{cases}$

(C) $\begin{cases} x = r\cos\theta \\ y = z \\ z = r\sin\theta z \end{cases}$ (D) None of these

105. The following (r, θ, z) are the coordinate of—

(A) Cartesian coordinate system

(B) Cylindrical coordinate system

(C) Spherical coordinate system

(D) None of these

106. The following (r, θ, ϕ) are the coordinate of—

(A) Cartesian coordinate system

(B) Cylindrical coordinate system

(C) Spherical coordinate system

(D) None of these

107. The following (x, y, z) are the coordiante of—

(A) Cartesian coordinate system

(B) Cylindrical coordinate system

(C) Spherical coordinate system

(D) None of these

108. Relation between cartesian and polar coordinate system.

(A) $r = \sqrt{x^2y^2}$ and $\theta = \tan y/x$

(B) $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} y/x$

(C) $r = \sqrt{x^2/y^2}$ and $\theta = \tan^{-1} y/x$

(D) None of these

109. Relation between cartesian and polar coordinate system—

(A) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

(B) $\begin{cases} x = r\sin\theta \\ y = r\cos\theta \end{cases}$

(C) $\begin{cases} x = r\tan\theta \\ y = r\sin\theta \end{cases}$

(D) None of these

110. An alternative and superior schemes in classical mechanics, developed by D′ Alembert, Langrange, Hamilton and others. Here scalar quantities such as energy rather than vector quantities are used and dynamical

relations are obtianed by systematic process of differentiation—
(A) Newtonian (vectorial) mechanics
(B) Analytical mechanics
(C) Quantum mechanics
(D) None of these

111. Classical mechanics that deals with the Newton's laws and their consequences. Here vector quantities such as force, acceleration, momentum etc. are used in referred as—
(A) Newtonian (vectorial) mechanics
(B) Quantum mechanics
(C) Analytical mechanics
(D) None of these

112. The mechanics based on Newton's law of motion and alternatively developed by Lagrange, Hamilton and others is—
(A) Classical mechanics
(B) Relative mechanics
(C) Quantum mechanics
(D) None of these

113. Following is true for mechanics—
(A) It deals with physical objects in motion under the influence of external and internal interaction
(B) It deals with physical objects at rest under the influence of external and internal interaction
(C) Both (A) and (B)
(D) None of these

Answers

1. (D)	2. (B)	3. (B)	4. (A)	5. (D)
6. (B)	7. (C)	8. (B)	9. (A)	10. (C)
11. (C)	12. (B)	13. (A)	14. (A)	15. (B)
16. (C)	17. (C)	18. (A)	19. (B)	20. (B)
21. (A)	22. (B)	23. (A)	24. (A)	25. (C)
26. (B)	27. (B)	28. (A)	29. (C)	30. (B)
31. (A)	32. (A)	33. (A)	34. (A)	35. (A)
36. (A)	37. (A)	38. (A)	39. (C)	40. (A)
41. (B)	42. (A)	43. (B)	44. (B)	45. (B)
46. (A)	47. (C)	48. (B)	49. (A)	50. (C)
51. (B)	52. (A)	53. (A)	54. (C)	55. (C)
56. (C)	57. (D)	58. (D)	59. (D)	60. (B)
61. (A)	62. (B)	63. (A)	64. (C)	65. (C)
66. (B)	67. (B)	68. (A)	69. (A)	70. (B)
71. (A)	72. (B)	73. (B)	74. (C)	75. (A)
76. (A)	77. (A)	78. (D)	79. (A)	80. (A)
81. (A)	82. (A)	83. (C)	84. (A)	85. (A)
86. (B)	87. (A)	88. (B)	89. (A)	90. (C)
91. (B)	92. (B)	93. (B)	94. (C)	95. (A)
96. (A)	97. (A)	98. (C)	99. (A)	100. (B)
101. (A)	102. (A)	103. (A)	104. (A)	105. (C)
106. (D)	107. (A)	108. (B)	109. (A)	110. (B)
111. (A)	112. (C)	113. (C)		

●●

Topology

Elements of Topological Space

Topology—A topology on a set X is a collection T of subsets of X having following properties—

(a) ϕ and X are in T

(b) The union of the elements of any subcollection of T is in T

(c) The intersection of the elements of any finite subcollection of T is in T

Topology Space—(X, T), A set X for which a topology T is defined.

Open set—(X, T) is a Topological space, U $\subset$ X is open set if U $\subset$ T.

Finer and Strictly Finer Topology—If T and T are two topologies on a set X, T is finer topology (then T) if T $\supset$ T. T is strictly finer if T properly contains T.

Basis—If X is a set, a basis for topology on X is a collection B of subsets of X called basis elements such that

(a) For each $x \in$ X, there is at least one basis element B containing x.

(b) If x belongs to the intersection of two basis elements B_1 and B_2 then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

Topology Generated by B—If B is a basis for a topology on X, the topology T generated by B is defined as : A subset U $\subset$ X is open in (X, T) if for each $x \in$ U, there is a basis element B $\in$ B such that $x \in$ B and B $\subset$ U.

Standard Topology—If the collection of all intervals in the real line $(a, b) = \{x | a < x < b|\}$, the topology generated by B is called the standard topology on the real line.

Subbasis—A subbasis ξ for a topology on X is a collection of subsets of X whose union equals X.

Topology Generated by the Subbasis—Topology generated by the subbasis ξ is defined as the collection T of all unions of finite intersections of elements of ξ.

Order Topology—Let X be a set with order relation. If B basis for X is a collection of all sets of the form—

(a) All open intervals $(a, b) \in$ X.

(b) All intervals of the form $[a_0, b]$, a_0 is the smallest element (if any) of X.

(c) All intervals of the form $[a_0, b_0]$, b_0 is the largest element (if any) of X.

The collection B is a basis for topology on X and is called order topology.

Product Topology : X × Y—If X and Y are topological spaces, the product topology on X × Y is the topology having as basis the collection B of all sets the form U × V, where U is an open subset of X and V is an open subset of Y.

Subspace topology—If X is a topological space with topology T and Y $\subset$ X, the collection $T_y = \{Y \cap U \mid U \in T\}$ is a topology on Y, called subspace topology. (Y, T_Y) is called subspace topology of (X, T).

Closed sets—A subset A of topological space X is closed if (X – A) is open.

Interior set—(X, T) is given a topological space, A $\subset$ X, then interior of A, A° is the union of all open sets contained in A.

Closure set—(X, T) is a given topological space A $\subset$ X, then closure of A, $\bar{A}$ is the intersection of all closed sets containing A.

Neighbourhood of *x*—An open set U containing x.

Limit Point (cluster point)—Given (X, T) a topological space and A $\subset$ X, a point $x \in$ X is a limit point of A if every neighborhood of x intersects A in some point other than x.

Hausdorff Space—A topological space X is Hausdorff space if for each pair $x_1 + x_2 \in X$, there exist neighbourhoods U_1 and U_2 of x_1 and x_2 respectively such that U_1 and U_2 are disjoint,

Discrete Topology—If X is any set, T is a collection of all subsets of X, then (X, T) is a discrete topology.

Indiscrete Topology (trivial topology)—If X is any set, T = {ϕ, X}, then (X, T) is indiscrete topology.

Theorems

1. Let X be a set; let B be a basis for a topology T on X, then T is equal to the collection of all unions of elements of B.
2. Let B and B′ be a basis for the topologies T and T′ respectively on X then following are equivalent.
   (a) T′ is finer than T.
   (b) For each $x \in X$ and each basis element $B \in B$ containing x, there is a basis element $B' \in B'$ such that $x \in B' \subset B$.
3. Let X be a topological space. Suppose that collection C is a collection of open sets of X such that each $x \in X$ and each open set U of X, there is an element $C \in C$ such that $x \in C \subset U$. Then C is a basis for the topology of X.
4. The lower limit topology T′ an real line R is strictly finer than the standard topology T.
5. If B is a basis for the topology of X and C is a basis for the topology of Y, then the collection $D = \{B \times C \mid B \in B \text{ and } C \in C\}$ is a basis for the topology of $X \times Y$.
6. If B is a basis for the topology of X, then the collection $B_Y = \{B \cap Y \mid B \in B\}$ is a basis for the subspace topology on X.
7. Let Y be a subspace of X if U is open in Y and Y is open in X, then U is open in X.
8. If X is an ordered set in the order topology and if Y is an interval or a ray in X, then the subspace topology and order topology on Y are same.
9. If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.
10. If (X, T) is a topological space. Then following condition holds—
    (a) ϕ and X are closed.
    (b) Arbitrary intersections of closed sets are closed.
    (c) Finite unions of closed sets are closed.
11. If Y is a subspace of X then set A is closed in Y iff it is equal to the intersection of a closed set of X with Y.
12. If Y is a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.
13. If Y is a subspace of X, $A \subset Y$ and $\bar{A}$ is a closure of A in X, then closure of A in Y is equal to $A \cap Y$.
14. If A is a subset of topological space X.
15. $x \in \bar{A}$, closure of A in X, iff every open set U containing x, intersects A.
16. Suppose B is a basis for X, then $x \in \bar{A}$ iff every basis element $B \in B$, containing x intersects A.
17. Let A be a subset of topological space X and A′ be the set of all limit points of A. Then $\bar{A} = A \cup A'$.
18. A subset of a topological space is closed iff it contains all its limit points.
19. Any finite point set in a Hausdorff space X is closed.
20. Let X be a Hausdorff space, $A \subset X$. Then a point x is a limit point of A iff every neighbourhood of x contains infinitely many points of A.
21. Every simple order set is a Hausdorff space in the order topology.
22. The product of two Hausdorff spaces is a Hausdorff space.
23. A subspace of a Hausdorff space is a Hausdorff space.

Continuity

Continuous Function—Let X and Y are topological spaces. A function $f : X \to Y$ is a continuous function if for each open subset V of Y, the set f (V) is an open subset of X.

Homeomorphism—Let X and Y are topological spaces. Let $f : X \to Y$ be a one-to-one function. If both the function f and the inverse function $f' : Y \to X$ are continuous, then f is homeomorphism.

Continuity at a point—Let X and Y are topological spaces. A function $f : X \to Y$ is continuous at $x \in X$, if for every neighbourhood N of $f(x)$, there is a neighbourhood M of x such that f (N) $\subset$ M.

Theorems

1. Let X and Y be topological spaces and $f : X \to Y$. Then the following are equivalent :
   (a) f is continuous
   (b) For every $A \subset X, f(\bar{A}) \subset \overline{f(A)}$
   (c) For every closed set $B \subset Y$, set f^{-1} (B) is closed in X
2. Let X, Y and Z be topological spaces :
   (a) The constant function $f : X \to Y$, *i.e.* $f(x) = Y_0$ for all $x \in X$ and single point $Y_0 \in Y$ is continuous function.
   (b) If A is a subspace of X, the inclusion function $f : A \to X$ is continuous.
   (c) If $f : X \to Y$ and $g : Y \to Z$ are continuous then composites $(g o f) : X \to Z$ is continuous.

The Product Topology

Boxtopology—Let $\{X\}_{\alpha \in J}$ be an indexed family of topological spaces let us take a basis for a topology on the product space. $\pi_{\alpha \in J} X_\alpha$, the collection of all sets of the form, $\pi_{\alpha \in J} U_\alpha$, where U_α is open in X_α for each $\alpha \in T$. The topology generated by this basis is called boxtopology.

Projection Mapping—Let $\pi_\beta : \pi_{\alpha \in J} X_\alpha \to X_\beta$ be the function assigning to each element of the product space its βth coordinate, $\pi\beta\ [(x_\alpha)_{\alpha \in J}] = x_\beta$, it is called the projection mapping associated with the index β.

Product Topology and Product Space—Let $\delta_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta$ open in $X_\beta]$ and $\delta = U_{\beta \in J}\ \delta_\beta$. The topology generated by the subbasis δ is called the product topology and $\pi_{\alpha \in J} X_\alpha$ is called product space.

Theorems

1. **Comparison o f the box and product topologies**—The box topology $n\pi X_\alpha$ has a basis all sets of the form πU_α, where U_α is open in X_α for each θ. The product topology on πU_α has a basis all sets of the form πU_α where U_α is open in X_α for each α and U_α equals X_α except for finetely many values of α.
2. Suppose the topology on each space X_α is given by a basis B_α. The collection of all sets of the form $\pi_{\alpha \in J} B_\alpha$, where $B_\alpha \in \beta_\alpha$ for each α will serve as a basis for the box topology on $\pi_{\alpha \in J} X_\alpha$.
3. The collection of all sets of the same form, where $B_\alpha \in B_\alpha$ for finitely many indices α and $B_\alpha = X_\alpha$ for all the remaining indices, will serve as a basis for the product topology on $\pi_{\alpha \in J} X_\alpha$.
4. Let A_α be a subspace of topological space X_α for each $\alpha \in T$. Then πA_α is a subspace of πX_α is both products are given the topology, or if both products are given the product topology.
5. If each space X_α is Hausdorff space, then πX_α is a Hausdorff space in both box and product topologies.
6. If $f : A \to \pi_{\alpha \in J} X_\alpha$ be given by the $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to X_\alpha$ for each α. Let πX_α have the product topology, then the function f is continuous iff each function f_α is continuous.

Metric Topology

Metric—A metric an a set X is a function $d : X \times X \to R$ having the properties :

(a) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ iff $x =$
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$
(c) $d(x, y) + d(y, z) = d(x, z)$, for all $x, y, z, \in X$ (triangular inequality) $\in$-bull centred at x : $Bd(x, \in) = \{Y \mid d(x, y) < \in\}$.

Metric Topology—If d is a metric on the set X, then the collection off all $\in$-balls $B_d(x, \in)$ for $x \in X$ and $\in > 0$, is a basis for a topology on X, called metric topology induced by d.

Or A set U is open in the metric topology induced by d iff for each $Y \in U$, there is $\delta > 0$: $t\, B_d(Y, \delta) \subset U$.

Metrizable Space—If X is a topological space, then X is metrizable iff there exist a metric d on the set X that induces the topology on X.

A metric space is a metrizable space X together with a specific metric d that gives the topology of X.

Bounded Set—Let X be a metric space with metric d. A subset A of X is said to be bounded if there is some number M such that $d(a_1, a_2) \leq M$ for each pair a_1, a_2 of points of A. If A is bounded, the diameter of A is diam A = lub $\{d(a_1, a_2) : a_1, a_2 \in A\}$.

Euclidean Metric—Given $\bar{x} \in R^n$, the norm of $\bar{x}$ is $|\bar{x}| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$

The Euclidean metric d R^n is

$$\begin{aligned} d(\bar{x}, \bar{y}) &= \|\bar{x} - \bar{y}\| \\ &= [(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots \\ &\quad + (x_n - y_n)^2]^{1/2} \end{aligned}$$

Square Metric—Let $\bar{x} \in R^n$, the square metric P is

$$P(\bar{x}, \bar{y}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$$

Standard Bounded Metric (corresponding to *d*)—Let X be a metric space with metric d and $\bar{d} : X \times X \to R$ such that $\bar{d}(x, y) = \min\{d(x, y), 1\}$ is called standard bounded metric corresponding to d.

Uniform Metric—Given an index set J and $\bar{x} = (x_2)_{\alpha \in J}$ and $\bar{y} = (y_2)_{\alpha \in J}$ of R^J. The uniform metric on R^J is

$$\bar{P}(\bar{x}, \bar{y}) = \text{Qub}\{\bar{d}(x_2, y_2)\ \alpha \in J\}$$

where $\bar{d}(x_\alpha, y_\alpha) = \min\{d(x_\alpha, y_\alpha), 1,\}$ the standard bounded metric on R. The topology induced by uniform metric is called uniform topology.

Convergent Sequence—A sequence $\{x_m\}$ of points of X is convergences to a point $x \in X$ if for every neighbourhood U of x there exists a positive integer N such that $x_n \in U$ for all $n \geq N$.

Countable Basis at a Point—A space X have a countable basis at the point x if there is a countable collection $\{U_n\}_{n \in Z}$ of neighbourhood of x such that any neighbourhood U of x contains atleast one of the sets U_n.

First Countable Axiom—A space X that has a countable basis at each of its points is said to satisfy the first countability axiom.

Some Important Theorems

1. Let d and d' be two metrics on the set X and let T and T′ be the topologies they induce respectively. Then T′ is finer than T iff for each $x \in X$ and each $\in > 0$, there exist $\delta \to 0$, such that $B_d(x, \delta) \subset B_d(x, \in)$.
2. There topologies on R^n induced by the euclidean metric d and the square metric P are the same as the product topology on R^n.
3. The uniform topology on R^J is finer than the product topology. They are different if J is infinite.
4. Let $\bar{d}(a, b) = \min\{|a - b|\}$ be the standard bounded metric on R. If x and y are two points of R^w, define
$$D(\bar{x}, \bar{y}) = \text{lub}\left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$$
Then D is a metric that induces the product topology on R^w.
5. Let $f : X \to Y$ also X and Y be metrizable with metrics d_x and d_y respectively. Then continuity of f is equivalent, to given $x \in X$ and $\in > 0$, there exist $\delta > 0$ such that $d_x(x, y) < \delta \Rightarrow d_y(f(x) f(y)) < \in$.
6. **Sequence Lemma**—Let X be a topological space and $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is metrizable.
7. Let $f : X \to Y$ and X be a metrizable. The function f is continuous iff for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$.
8. The addition, subtraction and multiplication operations are continuous function from R × R into R and quatient operation is continuous function from R × (R–∈0) into R.
9. If X is a topological space and if $f, g : X \to R$ are continuous functions, then $f + g$, $f - g$ and $f \cdot g$ are continuous functions. If $g(x) \neq 0$ for all x, then f/g is continuous.
10. **Uniform Limit Theorem**—Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X into the metric space Y. If $\{f_n\}$ converges uniformely to f, then f is continuous.
11. **Weierstrass M-test**—Let $f_n : X \to R$ be a sequence of functions from topological space X into R and $\delta_n(x) = \sum_{i=1}^{n} f_i(x)$. If $|f_i(x)| \leq b_i$ for all $x \in X$ and all $i = 1, \ldots, n$ and if the

series Σb_i is convergent, the sequence $\{\delta_n\}$ converges uniformly to a functions.

The Quotient Topology

Saturated Set—A subset C of a topological space X is saturated (with respect to the surjective map $P : X \rightarrow Y$) if C contains every set $P^{-1}(\{Y\})$ that it intersects.

Quotient Map—Let X and Y be topological spaces, $P : X \rightarrow Y$ be a surjective map the map P is said to be a quotient map. Provided a subset U $\subset$ Y is open in Y iff $P^{-1}(U)$ is open in X.

Equivalently P is quotient map, if P is continuous and P maps saturated open sets of X to open sets of Y (or saturated closed sets of X to closed sets of Y).

Open Map—A map f : X is open map if for every open set U of X the set $f(U)$ is open in Y.

Closed Map—A map $f : X \rightarrow Y$ is closed if for every closed set of X, the set $f(A)$ is closed in Y.

Quotient Topology—If X is a topological space and A is a set if $P : X \rightarrow A$ is surjective map, then there exist exactly one topology T on A relative to which P is a quotient map and this topology is called quotient topology induced by P.

Quotient Space (decomposition space)—Let X be a topological space and X* be a partition of X into disjoint subsets whose union is X. Let $P : X \rightarrow X^*$ be the surjective map the carries each point of X to the element of X* containing it. In the quotient topology induced by P, the space X* is quotient space of X.

Some Important Theorems

1. Let $P : X \rightarrow Y$ be a quotient map. Let Z be a space and let $g : X \rightarrow Z$ be a continuous map that is constant an each set $P^{-1}(\{Y\})$, for $Y \in y$. Then g induces is continuous map $f : Y \rightarrow Z$ such that $f o P = g$.

$$\begin{array}{c} X \\ P \downarrow \\ Y \end{array}$$

2. Let $g : X \rightarrow Z$ be a surjective continuous map. Let X* be the following collection of subsets of X : $X^* = \{g^{-1}(\{Z\}) \mid Z \in Z\}$ given the quotient topology.
   (a) If z is Hausdorff so is X*
   (b) The map g induces a bijective continuous map $f : X^* \rightarrow z$, which is a homeomorphism iff g is a guotient map.
3. The product of two quotient map need not be a quotient map.

Connectendness

(1) Connected Topological Spaces

Separation of Topological Space—Let X be a topological space. A separation of X is a pair U, V of disjoint nonempty open subsets of X whose union is X.

Connected Topological Space—Let X be a topological space.

(i) The space X is called connected if there does not exist a separation of X.

(ii) A space X is connected iff the only subset of X that are both open and closed in X are empty set and X itself.

Topology Disconnected Topological Space—A topological space X is totally disconnected if its only connected subsets are one-point set.

Some Important Theorems

1. If y is a subspsace of topological space X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, and neither of which contains a limit point of the other.
2. If Y is a subspace of topological space X and Y is connected if there exist no separation of Y.
3. If sets C and D from a separation of topological space X and if Y is a connected subset of X, then Y lies entirely with in either C or D.
4. The union of a collection of connected sets that have a point in common is connected.
5. Let A be a connected subset of topological space X, if $A \subset B \subset \bar{A}$, then B is also connected.
6. The cartensian product of connected topological space is connected.

Connected Sets in the Real Line

Linear Continuum—A simple ordered set L having more than one element is called linear continuum if.

(1) L has the least upper bounded property.

(2) If $x < y$, there exist z such that $x < z < y$.

Path—Let X be a topological space and $x, y \in$ X. The path in X from x to y is a continuous map $f : [a, b] \to$ X of some closed interval $[a, b]$ in the real line into X, such that $f(a) = x$ and $f(b) = y$.

Path Connected (topological) Space— A topological space X is called path connected if each pair of points of X can be joined by a path in X.

Comb (topological) Space—Let

$$K = \left\{ \frac{1}{n} \;\middle|\; n \in Z_+ \right\}$$

and $C = ([0, 1] \times 0) \cup (K \times [0, 1]) \cup (0 \times [0, 1])$

Then C is comb space.

Some Important Theorems

1. Let L is a linear continuum in theorder topology, then L is connected and so is very interval and ray in L.
2. The real line R is connected and so is every interval and ray in R.
3. **Intermediate value theorem**—Let $f : X \to Y$ be a continuous map of the connected topological space X into the ordered set Y, in the order topology.

   If $a, b \in x$ and if $r \in y$ such that $f(a)cr < f(b)$, then exist a point $c \in x$ such that $f(c) = r$.
4. Path connected topological space is connected but converse is not true.
5. The space I × I in the dictionary order topology is connected but not path connected.
6. Comb topological space is connected topological space but not path connected.

Components and Path Components

Components of Topological Space—Let X be a topological space. Given an equivalence relation on X such that $x \sim y$ if there is a connected subset of X containing both x and y. The equivalence classes are called the components (connected components) of X.

Path Components of Topological Space—Let X be a topological space. Given an equivalence relation on X such that $x \sim y$ if there is a path in X from x to y. Then equivalence classes are called the path components of X.

Some Important Theorems

1. The components of topological space X are connected disjoint subsets of X whose union is X such that each connected subsets of X intersects only one of them.
2. The path components of topological space X are path connected disjoint subsets of X whose union is X, such that each path connected subset of X intersects only one of them.

Local Connectedness

Local Connected Space—A topological space X is locally connected at x if for every neighbourhood U of x, there is a connected neighbourhood V of x, $V \subset U$. If X is locally connected at each $x \in$ X, then X is locally path connected space.

Locally Path Connected Space—A topological space X is locally path connected at x iff for every neighbourhood U of x, there is a path connected neighbourhood V of x, U, $V \subset U$.

If X is locally path connected at each $x \in$ X, then X is locally path connected space.

Connected in Kleinen at x—A topological space X is connected in Kleinen at x, if for every neighbourhood U of x, there is a connected subset -Y of U that contains a neighbourhood of x.

Some Important Theorems

1. A topological space X is locally connected iff for every open set U of X, each component of U is open in X.
2. A topological space X is locally path connected iff for every open set U of X each path component of U is open in X.
3. If X is a topological space then each path component of X lies in a component of X.
4. If X is a topological space and locally path connected then the components and the path components of X are the same.

Compactness

Compact Spaces

Cover—Let X be a topological space. A collection of subsets of X is said to cover X (covering of X), if the union of elements of a is equal to X.

Open Covering—A covering of X is called open covering if its elements are open subsets of X.

Compact Space—A topological space X is compact if every open covering a of X contains a finite sub collection that coveres X.

Finite Intersection Condition—A collection G of subsets of topological space satisfies Finite Intersection Condition if for every finite subcollection $\{C_1, C_2, \ldots, C_n\}$ of G, the intersection $\bigcap_{i=1}^{n} G \neq \phi$.

Some Important Theorems

1. Let y be a subspace of topological space X then Y is compact iff every covering of Y be sets open in X contains a finite subcollection covering Y.
2. Every closed subset of a compact space is compact.
3. Every compact set of a Hausdorff space is closed.
4. If Y is a compact subset of the Hausdorff space X and $x_0 \notin Y$, then there exist disjoint open sets U and V of X, such that $x_0 \in U$ and $y \subset V$.
5. The image of a compact topological space under continuous map is continuous.
6. Let $f : X \rightarrow Y$ be bijective continuous function. If X is compact and Y is Hausdorff, then f is homeomorphism.
7. The product of finitely many compact spaces is compact.
8. **Tube Lemma**—Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the 1 slice $x_0 \times Y$, of $X \times Y$, then N contains some tube $\omega \times Y$ about $x_0 \times Y$, where ω is a neighbourhood of $x_0 \in X$.
9. Let X be a topological space. Then X is compact iff for every collection G of closed sets in X satisfying finite intersection condition the intersection $\cup_{C \in G} C \neq \phi$.
10. The topological space X is compact iff for every collect a of subsets of X, satisfying the finite intersection condition, the intersect ion.

    $\cap_{A \in a} \overline{A} \neq \phi$, where $\overline{A}$ is the closure of $A \in a$.

Compact Sets in the Real Line

1. Let X be a simply ordered set having the least upper bound property. In the order topology, each closed interval in X is compact.
2. Every closed interval in R is compact.
3. A subset $A \subset R^n$ is compact iff it is closed and bounded in the euclidean metric d or square metric P.
4. **Maximum and Minimum Value Theorem**—Let $f : X \rightarrow Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there C, $d \in X$ such that $f(C) \leq f(x) \leq f(d)$ for every $x \in X$.
5. Let X be a (non empty) compact Hausdorff space. If every point of X is a limit point of X, then X is uncountable.
6. Every closed interval R is uncountable.

Limit Point Compactness

Limit Point Compact Space—A topological space X is limit point compact if every infinte subset of X has a limit point.

Sequentially Compact—If every sequence in a topological space has a convergent subsequence, then Y is sequentially compact.

Countably Compact Space—A topological space X is countably compact if every countable open covering of X contains a finite subcollection covering X.

Some Important Theorems

1. Compactness implies limit point compactness (converge is not true).
2. **Lebesgue number lemma**—Let a be an open covering of metric space (X, d). If X is compact, there is $\delta > 0$ such that each subset $(x - \delta, x + \delta) \subset X$, there exist $o \in a$ sch that $(x - \delta, x + \delta) \subset 0$.
3. **Uniform Continuity Theorem**—If $f : X \rightarrow Y$ is a continous map of the compact metric space (X, d_x) to the metric space (Y, d_Y). Then f is uniformly continuous *i.e.* given $\in > 0$, there exist $\delta > 0$ such that x, $Y \in X$, $d_x(x, y) < \delta \Rightarrow d_y(f(x), f(y)) \subset \in$.
4. Let X be a metrizable space. Then following are equivalent—
   (i) X is compact
   (ii) X is limit point compact
   (iii) X is sequentially compact

5. If X is countably compact space then it is limit point compact space

Local Compactness

Locally Compact Space—A topological space X is locally compact at X there is some compact subspace C of X that contains a neighbourhood of x.

If topological space X is locally compact at each $x \in$ X, then X is locally compact.

One Point Compactification—Let X be a locally compact Hausdorff space. Let Y = X $\cup$ $\{\infty\}$. The topology Y defines are collection of open sets in Y to be all sets such as—

(1) U–U where U is open subset of X.

(2) Y–C, where C is a compact subset of X.

Then topological space Y is one-point compactification of X.

Some Important Theorems

1. Let X be a locally compact Hausdorff space which is not compact. Let Y be the one point compactification of X. Then (a) Y is compact Hausdorff space (b) X is subspace of Y (c) the set Y – X consists of a single point (d) $\overline{X}$ = Y.
2. Let X be a Hausdorff space. Then X is locally compact at x iff for every neighbourhood U of x, there is neighbourhood V of x, such that $\overline{V}$ is compact and $\overline{V} \subset \overline{U}$.
3. Let X be a locally compact Hausdorff space and Y be a subspace of X. If Y is closed (or open) in X. Then Y is locally compact.
4. A space X is homeomorphic to an open subset of a compact Hausdorff space iff X is locally compact Hausdorff.
5. If P : X $\to$ Y is a quotient map and if Z is a locally compact Hausdorff space, then the map $\pi = P \times i_2 : X \times Z \to Y \times Z$ is a quotient map.
6. Let P : A $\to$ B and g : C $\to$ D be quotient maps. If B and C are locally compact Hausdorff spaces, then $P \times q : A \times C \to B \times D$ is a quotient map.

The Countability Axioms

Countable Basis at x—A topological space X have a countable basis of x if there is a countable collection B of neighbourhood δ of x, such that each neighbourhood of x contains atleast one of the elements of B.

First Countability Axiom—A topological space has a countable basis at each of its points is called first coutability axiom.

Second Countability Axiom—A topological space X satisfies second countability axiom if X has a countable basis for its topology.

Dense Set—A subset Y of topological space X is dense in X if $\overline{Y}$ = X.

Lindel of Space—A topological space for which every open covering contains a countable subcovering.

Separable Space—A topological space having a countable dense subset.

Some Important Theorems

1. Let X be a topological space satisfying first countability axiom.
   (a) The point $x \in \overline{A}$, closure of A $\subset$ X iff there is a sequence of points of A converging to x.
   (b) The function f : X $\to$ Y is continuous iff for every convergent sequence $\{x_n\}$ in X, converging to x the sequence $\{f(x_n)\}$ converges to $f(x)$.
2. A subspace of a first countable space is first countable and a countable product of first countable space is first countable.
3. A subspace of a second countable space is second countable and a product of second countable spaces is second countable.
4. If topological space has a countable basis then
   (a) Every open covering of X contains a countable subcollection covering X.
   (b) There exist a countable subset of X which is dense in X.
5. The product of two Lindel of spaces need not be lindel of.
6. A subspace of a space having a countable dense subset need not have a countable dense subset.

The Separation Axioms

Regular Space—If one point sets are closed in X. Then x is regular if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B respectively.

Normal Space—A topological space X is normal if for each pair A, B, A ∩ B ≠ ϕ of closed sets of X, there exist disjoint open sets θ_1 and θ_2, $\theta_1 \cap \theta_2 \neq \phi$, such that $A \subset \theta_1$ and $B \subset \theta_2$.

The Separation Axioms—Given a topological space, the separation axioms states,

T_1 : Given two distinct points x and y, there is an open set that contains y but not x.

T_2 : Given two distinct points x and y, there are disjoint open sets θ_1 and θ_2 such that $x \in \theta_1$ and $y_1 \in \theta_2$.

T_3 : In addition to T_1, given a closed set F and a point x not in F, there are disjoint open sets θ_1 and θ_2 such that $x \in \theta_1$ and $F \subset \theta_2$.

T_4 : In addition to T_1, given two disjoint closed sets F_1 and F_2 there are disjoint open sets θ_1 and θ_2 such that $F_1 \subset \theta_1$ and $F_2 \subset \theta_2$.

A topological space satisfying T_1 is called a Tychonoff space.
A topological space which satisfies T_2 is called a Hausdorff space.
A topological space which satisfies T_3 is called a regular space.
A topological space which satisfies T_4 is called a normal space.

Here

(i) The condition T_1 is equivalent to the statement that each set consisting of single point is closed.

(ii) $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$, *i.e.* every normal space is regular space, every regular space is Hausdorff space and every Hausdorff space is Tychonoff space.

Some Important Theorems

1. Let X be a topological space. Let one point sets in X be closed.
   (a) X is regular iff given a point $x \in X$ and neighbourhood U of x, there is a neighbourhood V of x such that $\overline{V} \subset U$.
   (b) X is normal iff given a closed set A and open set U, $A \subset V$, there is an open set V, $A \subset U$ such that $\overline{V} \subset U$.
2. (a) A subspace of Hausdorff space is Hausdorff and product of Hausdorff space is Housdorff.
   (b) A subspace of regular space is regular and product of regular spaces is regular.
   (c) A subspace of normal space need not be normal and product of normal spaces need not be normal.
3. Every metrizable space is normal.
4. Every compact Hausdorff space is normal.
5. Every regular space with countable basis is normal.
6. Every well ordered set X is normal in the order topology.

The Urysohn Lemma

Sets Separated by Continuous Function—Let X be a topological space A, $B \subset X$, $f : X \to [0, 1]$ a continuous function such that $f(A) = \{0\}$ and $f(B) = \{1\}$, then sets A and B are separated by continuous function.

Completely regular space—A topological space X is completely regular if one point sets are closed in X and if every $a \in X$, every closed set B, $a \in B$, there exists a continuous function $f : X \to [0, 1]$ such that $f(a) = 0$ and $f(B) = \{1\}$.

Some Important Theorems

1. **Urysohn Lemma**—Let X be a normal space; A and B are disjoint closed subsets of X and $[a, b]$ closed interval of R. Then there exist a continuous map $f : X \to [a, b]$ such that for every $x \in A$, $f(x) = a$ and for every $x \in B$, $f(x) = b$.
2. **Tietze Extension Theorem**—Let X be a normal space. Let A be closed subset of X.
   (a) Any continuous map of A into the closed interval $[a, b]$ of R may be extended to a continuous map of all of X into $[a, b]$.
   (b) Any continuous map of A into R may be extended to a continuous map of X into R.
3. **Strong form of Urysohn Lemma**—Let A and B closed disjoint subsets of the normal space X. There exist a continuous function $f : X \to [0, 1\}$ such that $f^{-1}\{0\} = A$ and $f(B) = \{1\}$ iff A is a G_δ set in X. A is a G_δ set in X if A is the intersection of a countable collection of open sets of X.
   (1) **Urysohn Metrization Theorem**—Every regular space X with countable basis is metrizable.

(2) **Imbedding Theorem**—Let X be Hausdorff space $\{f\}_{\alpha\in J}$ is a collection of continuous function $f_\alpha : X \to R$ satisfying for each $x_0 \in X$ and each neighbourhood U of x_0, there is an index α such that f_α is positive at x_0 and vanishes outside U. Then function $F : X \to R^J$ defined by $F(x) = [f_\alpha(x)]_{\alpha\in J}$ is an imbedding of X in R^J.

Partitions of Unity

Support of ϕ—If $\phi : X \to R$, the support of ϕ is the closure of set ϕ (R–{0}) *i.e.*, If x lies outside the support of ϕ, there is some neighbourhood of x on which ϕ vanishes.

Partition of Unity—Let $\{U_1, U_2, \ldots U_n\}$ be finite indexed open covering of the space X. An indexed family of countable functions.

$\phi_i : X \to [0, 1], i = 1, 2, \ldots, n$ is called partition of unity dominated by $\{U_i\}$ if :

(a) (support) $\subset U_i$ for each i.

(b) $\sum_{i=1}^{n} \phi_i(x) = 1$, for each x.

***m*-manifold**—An m-manifold is a Hausdorff space with a countable basis that each point $x \in X$, has a neighbourhood that is homeomorphic with an open subset of R^m.

Point-Finite Indexed—An indexed collection $\{A_\alpha\}$ of subsets of X is point-finite indexed family if each $x \in X \Rightarrow x \in A_\alpha$ for only finitely many values of α.

Locally Finite Indexed—Let X be a topological space. An indexed family $\{A_\alpha\}_{\alpha\in j}$ of subsets of X is locally finite indexed family if each point of X has a neighbourhood that intersects A_α for only finitely many values of α.

Some Important Theorems

1. **Existence of Finite Partitions of Unity**—Let $\{U_1, \ldots\ldots, U_n\}$ be a finite open covering of the normal space X. Then there exists a partition of unity dominated by $\{U_i\}$.
2. If X is compact m-manifold, then X can be imbedded in R^N for some positive integer N.

Tychonoff Theorem

(1) **Tychonoff Theorem**—Any arbitrary product of compact spaces is compact in the product topology.

(2) **Completely Regular Spaces**—A space X is completely regular if one-point sets are closed in X and for each $x_0 \in X$ and each closed set A, $x_0 \notin A$, there is a continuous function $f : X \to [0, 1]$ such that $f(x_0) = 1$ and $f(A) = \{0\}$.

Some Important Theorems

1. A subspace of a completely regular space is completely regular and the product of completely regular spaces is completely regular.
2. If X is completely regular, then X can be imbedded in $[0, 1]^J$ for some J.
3. Let X be a topological space. Then following are equivalent :
   (a) X is completely regular.
   (b) X is homeomorphic to a subspace of compact Hausdorff space.
   (c) X is homeomorphic to a subspace of a normal space.
4. Every Hausdorff topological group is completely regular.

The Stone-Cech Compactification

Compactification of Space—A compactification of a space X is compact Hausdorff space Y containing X such that X is dense in Y. (*i.e.*, $\overline{X} = Y$).

Equivalent Compactification—Compactification y_1 and y_2 of X are equivalent if there is a homeomorphism $h : Y_1 \to Y_2$ such that $h(x) = x$ for every $x \in X$.

Compactification Y of X, induced by h : If $h : x \to z$ is an imbedding of X in the compact Housdorff space z, then h inducess a compactification y of x.

Stone-Cech Compactification—The compactification of X induced by h is called the stone-cech compactification of X, β (X).

Some Important Theorems

1. Let X be completely regular $\beta(X)$ be its stone-Cech compactification. Then every bounded continuous real valued function of X can be uniquely extended to continuous real valued function of $\beta(X)$.
2. Let $A \subset X$; $f : A \to Z$ be continuous map of A into Hausdorff space Z. There is at most one extension of f to a continuous function.

$$g : \overline{A} \to Z$$

Metrization Theorems

Local Finiteness

Locally Finite Collection—Let X be a topological space. A collection a of subsets of X are locally finite if every point of X has a neighbourhood that intersect only finitely many elements of a.

Countably Locally Finite—A collection B of subsets of X is countably locally finite if B can be written as countable union of collections B_n, each of which is locally finite.

Some Important Theorems

1. Let a be a locally finite collection of subsets of X. Then
   (a) Any subcollection of a is locally finite.
   (b) The collection $B = \{\overrightarrow{A}\}_{A\in a}$ of closures of the elements of a is locally finite.
   (c) $\overline{\cup_{A\in a} A} = \cup_{A\in a} \overline{A}$

The Nagata Smirnov Metrization Theorem

G_δ set—A subset A of topological space X is G_δ set in X if it is equals to the intersection of a countable collection of open subsets of X.

Some Important Theorems

1. Let X be a regular space with a basis B that is countably locally finite. Then X is normal and every closed set of X is G_δ set in X.
2. Let X be a regular space with basis B that is countably locally finite. Then X is metrizable.

The Nagata–Smirnov Theorem

Refinement—Let a be a collection of subsets of the space X. A collection B of subset of X is called refinement of a (refine a) if for each $B \in B$, there is $A \in a$ such that $B \subset A$.

If $B \in B$ are open sets, then B is an open refinement of a.

If $B \in B$ are closed sets, then B is a closed refinement of a.

Locally Discrete Collection—A collection a of subsets of X is locally discrete if each point of X has neighbourhood that intersects at most one element of a.

Countably locally discrete collection— A collection B is countably locally discrete (a-locally discrete) if it is equal to a countable union of locally discrete collections.

Some Important Theorems

1. Let X be a metrizable space. If a is an open covering of X, then there is a collection D of subsets of X such that,
   (a) D is an open covering of X.
   (b) D is a refinement of a.
   (c) D is countably locally finite.
2. Let X be metrizable space. Then X has a basis that is countable locally finite.
3. **Bing metrization theorem**—A space X is metrizable iff it is regular and has a basis that is countably locally discrete.

Para Compactness

Para Compact Space—A space X is para compact if it is Housdorff and every open covering a of X has a locally finite open refinement B that covers X.

Some Important Theorems

1. Every para compact space X is normal.
2. Every closed subspace of a para compact space is para compact.
3. An arbitrary subspace of a para compact space and product of para compact spaces need not be para compact.
4. **Stone's theorem**—Every metrizable space is para compact.
5. Let x be a regular space. Then following are equivalent. Every open covering of x has a refinement. That is—
   (a) An open covering of x and countably locality finite.
   (b) A covering of x and locally finite.
   (c) A closed covering of x and locally finite.
   (d) An open covering of x and locally finite.

Smirnov Metrization Theorems

Locally Metriable Theorem—A space X is locally metrizable if every $x \in X$ has a nighbourhood U that is metrizable in the subspace topology.

Smirnov Metrization Theorem—A space X is metrizable iff it is para compact and locally metrizable.

Homotopy of Path

Homotopy—If f and f' and continuous map of the space X into space Y, then f is homotopic to f' ($f = f'$) if there is one continuous map $F : X \times [0,$

$1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for each $x \in X$.

The map F is called a homotopy between f and f'.

Path Homotopic—Two paths f and f', mapping the interval [0, 1] into X are said to be path homotopic $f =_p f'$ if they have the same intial x_0 and same final path x_1 and if there is continuous map. $F[0, 1] \times [0, 1] \to X$ such that

$$\begin{aligned} F(s, 0) &= f(s) \\ \text{and} \quad F(s, 1) &= f'(s) \\ F(0, t) &= x_0 \\ \text{and} \quad F(1, t) &= x_1, \end{aligned}$$

for each $s \in I$ and each $t \in [0, 1]$. The map F is called path homotopy between f and f'.

Straight Line Homotopy—If $f, g : X \to R^2$ such that the may $F(x, t) = (1 - t) f(x) + tg(x)$, is a homotopy between f and g. Then F is called straight line homotopy.

Composition—If f is a path in X from x_0 to x_1 and g is a path in X from x_1 to x_2 then composition $f^* g$ of f and g is a path h given by.

$$h(s) = \begin{cases} f(2s) & \text{if } S\ \Sigma \left[0, \frac{1}{2}\right] \\ g(2s-1) & \text{if } S\ \Sigma \left[\frac{1}{2}, 1\right] \end{cases}$$

Contractiable—A space is contractible if the identity map $ix : X \to X$ is homotopic to a constant map.

Fundamental Group

Fundamental group (First homotopy group) : Let X be a topological space; let $x_0 \in X$. The path in X, that beings and ends at x_0 is called a loop based at x_0, with the operation * is called fundamental group $\Pi_1(X, x_0)$ of X relative to the base point x_0.

Simply Connected Space—In simply connected space X and two paths having the same initial and final points are path homotopic.

Homeomorphism induced by h—

Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map and $h^* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is such that $h^*([f]) = [h, \text{of}]$.

The map h^* is called homeomorphism induced by h, relative to the base point x_0 zero homeomorphism : Let $A \subset R^n$, $h : (A, a_0) \to (Y, y_0)$. If h is extandable to a continuous map of R^n into y. Then h^* is zero homeomorphism.

Some Important Theorems

1. If X is path connected and $x_0, x_1 \in X$ then π, (X, x_0) is isomorphic to π, (X, x).
2. In a simply connected space X, any two paths having the same initial and final points are path homotopic.
3. If $h : (X, x_0) \to (Y, y_0)$, $K : (Y, y_0) \to (Z, z_0)$, then$(Koh)^* = K^* \; o \; h^*$.
4. If $i : (X, x_0) \to (X, x_0)$ is identity map, then i^* is the identity homeomorphism.
5. If $h : (X, x_0) \to (Y, y_0)$ is homeomorphism of X with Y, then h^* is anisomorphism of π, (X, x_0) with π, (Y, y_0).

Covering Spaces

Evenly Covered Set and Slice—Let $P : E \to B$ be continuous surjective map. The open set $U \subset B$ is said to be evenly convered by p if the inverse p^{-1} (U) can be written as union of disjoint sets $V_2 \subset E$ such that for each 2, the restriction of P to V_2 is a homeomorphism of V_2 onto U. Each of the sets V_2 is called slice.

Covering map and Convering Space—Let $p : E \to B$ be continuous and surjective. If every point $b \in B$ has neighbourhood U that is evenly covered by p, then p is called a covering map and E a covering sapce of B.

Some Important Theorems

1. If $p : E \to B$ is a covering map, then for each $b \in B$, the subject $p^{-1}(b) \subset E$ has the discrete topology.
2. The map $P : R \to S^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is a covering map, where S^1 is the unit sphere in R^2.
3. If $p : E \to B$ is a covering map then P is an open map.

Fundamental Group of the Circle

Lifting of f : Let $p : E \to B$ be a map. If f is continuous mapping of some spaces X into B, a lifting of f is a map $f : X \to E$ such that $p \; o \; f = f$.

Universal covering space of B : If E is a simple connected space and if $p : E \to B$ is a covering map, then E is a universal covering space of B.

Some Important Theorems

1. If $p : E \rightarrow B$ be covering map, let $p(e_0) = b_0$. Any path $f : [0, 1] \rightarrow B$ beginning at b_0 has a unique lifting to a path. f in E beginning at e_0.
2. Let $p : E \rightarrow B$ be a covering map, let $p(e_0) = b_0$ and let $F : [0, 1] \rightarrow B$ be continuous map, with $F(0, 0) = b_0$. There is a lifting of F to continuous map $F : [0, 1] \rightarrow E$ such that $F(0, 0) = e_0$. If F is a path homotopy, then F is a path homotopy.
3. Let $p : E \rightarrow B$ be a covering map; let $p(e_0) = b_0$ let f and g be two paths in B from b_0 to b'_1, let f and g be their respective lifting to paths in E beginning at e_0. If f and g are path homotopic, then f and g end at the same point of E and are path homotopic.
4. The fundamental group of the cricle is finite cyclic.
5. Let $p : (E, e_0) \rightarrow (B, b_0)$be a covering map. If E is path connected, then there is surjection $\phi : \pi, (B, b_0) \rightarrow p^{-1}(b_0)$. If E is simply connected, ϕ is bijection.

The Fundamental Group of Punctured Plane

Strong Deformation Retration—Let A be a subspace of X. Then A is said to be a strong deformation retract of X if three is continuous map $H : X \times [0, 1] \rightarrow X$ such that—

$$H(x, 0) = x \text{ for } x \in X$$
$$H(x, 1) \in A \text{ for } x \in X.$$
$$H(a, 1) = a$$

for $a \in A$ and $t \in [0, 1]$

The map H is called strong deformation retraction.

Some Important Theorems

1. Let $x_0 \in S^1$; the unit sphere in R^2 the inclusion mapping $J : (S^1, x_0) \rightarrow (R_2 - 0, x_0)$ induces an isomorphism of fundamental groups.
2. Let $x_0 \in S^{n-1}$, the unit sphere in R^n the inclusions : $(A, a_0) \rightarrow (X, a_0)$ induces an isomorphism of fundamental groups.
3. Let A be a strong deformation retract of X. Let $a_0 \in A$. Then the inclusion maps : $(A, a_0) \rightarrow (X, a_0)$ induces an isomorphism of fundamental groups.

Fundamental Group of S^n

1. **Special Van Kampen Theorem**—Let $X = U \cup V$, U and V are open in X and $U \cap V$ is path connected. Let $x_0 \in U \cap V$. If both inclusions $i : (U, x_0) \rightarrow (X, x_0)$ and $J : (V, x_0) \rightarrow (X, x_0)$, induces zero homeomorphism of fundamental groups, then $\pi, (X, x_0) = 0$.
2. For $n \geq 2$, the n–sphere S^n is simply connected.
3. $R^n - 0$ is simply connected if $n > 2$.
4. R^n and R^2 are not homeomorphism for $n > 2$.

Fundamental Group of Surfaces :

Projective Plane—The projective plane p^2 is the space obtained from S^2 by identifying each $x \in S^2$ with its antipodal point $-x$.

Some Important Theorems

1. $\pi, (x \times y, x_0 \times y_0)$ is isomorphism with $\pi, (X, x_0) \times \pi, (y, y_0)$.
2. The fundamental group of the torus $T = S' \times S'$ is isomorphic to the group $Z \times Z$.
3. The projective plane P^2 is a surface and the map $P : S^2 \rightarrow P^2$ is a covering map.
4. $\pi, (p^2, y)$ is a group of order 2.
5. The fundamental group of the double torus T_2 is not Abelian.

Essential and Inessential Maps :

Essential and Inessential map—A map $h x \rightarrow y$ is inessential map if h is homotopic to a constant map, otherwise essential.

Some Important Theorems

1. Let $h : S' \rightarrow y$. Then following are equivalent :
   (i) h is inessential
   (ii) h can be extended to a continuous map $g : B^2 \rightarrow y$.
2. Let $h : X \rightarrow y$, If h in essential, then h^* is the zero homomorphism.
3. Let T be a closed triangular region R^2. Let BdT de note the union of the edges of T. There is no continuous map $f : T \rightarrow BdT$ that maps each edge of T into itself.

Homotopy Type

Homotopy Equivalence and Inverse—A continuous map $f : X \rightarrow Y$ is homotopy equivalence, if there is a continuous map $g : y \rightarrow$

X such that gof is homotopic to the identity map i_x of X and fog is homotopic to the identity map i_y of y. The map g is homotopy inverse for the map f.

Theorems

1. Let h, K : X $\to$ y, $h(x_0) = y_0$ and $K(x_0) = y_1$. If h and K are homotopic then there is a path $d \in y$ from y_0 to y_1 such that $K = \overrightarrow{d}\ oh_*$. If $y_0 = y_1$ and if the base point to remains fixed during the homotopy, then $h_* = k_*$.
2. Let h, k : X $\to$ y and $h(x_0) = y_0$, $k(x_0) = y_1$. Suppose that h and Kare homotopic. If K_* is injective (or surjective or zero homomorphism), then so is h_*. If h is homotopic to a constant map, then h_* is the zero homeomorphism.
3. Let f : X $\to$ y be continuous, $f(x_0) = y_0$. If f is homotopy equivalence, then $f_* : \pi_1 (X, x_0) \to \pi_i (Y, y_0)$ is an isomorphism.

   Jordan separation and curvetheorem :

   Arc : An arc is a space homeomorphic to the unit interval [0, 1]. Simple closed curve : A simple closed curve is a space homeomorphic to the circle S′.

Some Important Theorems

1. Let a, b Σ S^2 and A be a compact space. Let $f : A \to S^2 - a - b$ be continuous map. If a and b lie in same component of $S^2 - f(A)$, then f is inessential.
2. **Jordan Separation Theorem**—Let C be a simple closed curve in S^2. Then $S^2 - C$ is not connected.
3. Let A and B be closed connected subsets of S^2 whose intersection consists of precisely two points then A ∪ B separates S^2.
4. **Homotopy existence lemma**—Let X and X_* [0, 1] be normal, Let A be a closed subset of X. If $f : A \to R^2 - 0$ is continuous map and f is inessential, then f can be extended to a continuous may $g : X \to R^2 - 0$.
5. **Borsuk Theorem**—Let x be a compact subset of R^2. If 0 lies in a bounded component of $R^2 - X$, then the inclusion map $j : X \to R^2 - 0$ is essential (and conversely).
6. **Non-Separable Theorem**—No arc separates R^2, no space homeomorphic to ball B^2 separates R^2.
7. **Brouwer theorem on invarience of domain for R^2**—If U is an open subset of R^2 and f : U $\to R^2$ is contiuous and injective, then f(U) is open in R^2 and f is an imbedding.
8. Let X be the union of two open sets U and V suppoose that U ∩ V can be written as the union of two disjoint open sets A and B. Let a ∈ A and b ∈ B and a and b are joined by paths in U and in V. then $\pi_1 (X, a) \neq 0$.
9. Let X = U ∪ V, where U and V are open sets and U ∩ V = A ∪ A′ ∪ B, where A, B, A′ are disjoint open sets. Let $a \in A$, $a' \in A'$ and $b \in B$. Suppose that a, a' and b are joined by paths in U and V-then π; (X, a) is not infinite cyclic.
10. **Non-Separation Theorem**—Let A be an arc in S^2. Then $S^2 - A$ is connected.
11. **Jordan Curve Theorem**—Let C be a simple closed curve in S^2. Then $S^2 - C$ has precisely two components W_1 and W_2 of which C is the common boundary.

OBJECTIVE TYPE QUESTIONS

1. Let X be a set for which a topology T is defined, then—
   (A) Only X is in T (B) Only X is not in T
   (C) ϕ and X are in T (D) None of these
2. Let X be a set for which a topology T is defined, then—
   (A) The union of the elements of finite subcollection of T is in T
   (B) The union of the elements of any subcollection of T is in T
   (C) The union of the elements of any subcollection of T is not in T
3. Let X be a set for which a topology T is defined, then—
   (A) The intersection of the elements of any finite subcollection of T is in T
   (B) The intersection of the elements of any subcollection of T is in T
   (C) The intersection of the elements of any finite subcollection of T is not in T
   (D) None of these

4. Let X be a set for which a topology T is defined then following is true—
   (A) ϕ and X are in T
   (B) The union of the elements of any subcollection of T is in T
   (C) The intersection of the elements of any finite subcollection of T is in T
   (D) None of these
5. Let (X, T) is a topological space, $U \subset X$ is open set—
   (A) U is a subset of T
   (B) U is an element of T
   (C) U is a superset of T
   (D) None of these
6. If B is a basis for a topology on X, the topology T generated by B is defined as—
   (A) Subset $U \subset X$ is closed in (X, T) if for each $x \in U$, there is a basis element $B \in B$ such that $x \in B$ and $B \subset U$
   (B) Subset $U \subset X$ is open in (X, T) if for each $x \in U$, there is a basis element $B \in B$ such that $x \in B$ and $B \subset U$
   (C) Subset $U \subset X$ is open in (X, T) if for some $x \in U$, there is a basis element $B \in B$ such that $x \in B$ and $B = U$
   (D) None of these
7. If B is the collection of all intervals in the real line $(a, b) = \{x \mid a < x < b|\}$, the topology generated—
   (A) Standard topology on the real line
   (B) Discrete topology on real line
   (C) Hausdorff topology on real line
   (D) None of these
8. A subbasis ξ for a topology on X is a collection of subsets of X—
   (A) Whose union equals X
   (B) Where union is subset of X
   (C) Whose union superset of X
   (D) None of these
9. Topology generated by the subbasis ξ is defined as—
   (A) The collection T of all unions of any intersections of elements of ξ
   (B) The collection T of all unions of finite intersections of elements of ξ
   (C) The collection T of all intersection of any intersections of elements of ξ
   (D) None of these
10. Let X be a set with order relation. The collection B is a basis for topology on X and is called order topology. If B basis for X is a collection of all sets of the form—
    (A) All open intervals $(a, b) \in X$
    (B) All intervals of the form $[a_{01}, b]$, a_0 is the smallest element (if any) of X
    (C) All intervals of the form $[a, b_0]$, b_0 is the largest element [if any] of X
    (D) All of these
11. If X and Y are topological spaces. The product topology on $X \times Y$ is—
    (A) Topology having as basis the collection B of all sets of the form $U \times V$, where U is a closed subset of X and V is a closed subset of Y.
    (B) Topology having as basis the collection B of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.
    (C) Topology having as basis the collection B of all sets of the form $U \times V$, where U and V are nulsets.
    (D) None of these
12. If X is a topological space with topology T and $Y \subset X$, the collection $T_Y = \{Y \cap U \mid U \in T]$ is a topology on Y. Then (Y, T_Y) is called .... of (X, T).
    (A) Subspace topology
    (B) Super spacetopology
    (C) Discreate topology
    (D) None of these
13. A subset A of topological space X is closed set if—
    (A) (X–A) is open (B) A is open
    (C) A is closed (D) None of these
14. Let (X, T) is given topological space, $A \subset X$, then interior of A—
    (A) Is the union of all open sets contained in A
    (B) Is the intersection of all open sets contained in A

(C) Is the union of all closed sets contained in A

(D) None of these

15. Let (X, T) is given topological space, A ⊂ X, then closure of A.

(A) Is the intersection of all closed sets containing A

(B) Is the union of all closed sets containing A

(C) Is the intersection of all open sets contianing A

(D) None of these

16. Neighbourhood of x is—

(A) An open set U containing x

(B) An closed set U contianing x

(C) A null set

(D) None of these

17. Given (X, T) a topological space and A ⊂ X, a point $x \in$ X is a limit point of A—

(A) If every neighbourhood of x intersects A in some point other than x

(B) If every neighbourhood of x intersects A in all point other than x

(C) If every neighbourhood of x intersects A in some point including x

(D) None of these

18. A topological space X is Hausdorff space if for each pair $x_1 \neq x_2 \in$ X.

(A) There exist neighbourhood U_1 and U_2 of x_1 and x_2 respectively such that U_1 and U_2 are disjoint

(B) There exist neighbourhood U_1 and U_2 of x_1 and x_2 respectively such that U_1 and U_2 are not disjoint

(C) There exist no neighbourhood U_1 and U_2 of x_1 and x_2 respectively such that U_1 and U_2 are not disjoint

(D) None of these

19. If X is any set, T is a collection of all subsets of X then topology (X, T) is—

(A) A discrete topology

(B) Indiscrete topology

(C) Trivial topology

(D) None of these

20. If X is any set, T = {ϕ, X}, then topology (X, T).

(A) A discrete topology

(B) Indiscrete topology

(C) Trivial topology

(D) Both (A) and (B)

21. Let X be a set; let B be a basis for a topology T on X. Then T is equal to—

(A) The collection of all intersection of elements of B

(B) The collection of all unions of elements of B

(C) The collection of all sum of elements of B

(D) None of these

22. Let B and B′ be basis for the topologies T and T′ respectively on X. Then T′ is finer than T is equivalent to—

(A) For each $x \in$ X and each basis element B ∈ B containing x, there is a basis element B′ ∈ B′ such that $x \in$ B′ ⊂ B.

(B) For some $x \in$ X and some basis element B ∈ B contianing x, there is a basis element B′ ∈ B′ such that $x \in$ B′ ⊂ B.

(C) For $x \in$ X and basis element B ∈ B containing x, there is a basis element B′ ∈ B′ such that $x \in$ B′ ⊂ B.

(D) None of these

23. Let X be a topological space suppose that collection C is a collection of open sets of X such that each $x \in$ X and each open set U of X, there is an element C∈ C such that $x \in$ C ⊂ U.

(A) Then C is not a basis for the topology of X

(B) Then C is a basis for the topology of X

(C) Then C is sub basis for the topology of X

(D) None of these

24. The lower limit topology T′ an real line R—

(A) Is strictly finer than the standard topology T

(B) Is inferior than the standard topology T

(C) Is finer than the standard topology T

(D) None of these

25. If B is a basis for the topology of X and C is a bais for the topology of Y, then—
    (A) The collection $D = \{B \times C \mid B \in B$ and $C \in C\}$ is a basis for the topology of $X \times Y$
    (B) The collection $D = \{B \times C \mid B \in B$ and $C \in C\}$ is not a basis for the topology of $X \times Y$
    (C) The collection $D = \{B \times C \mid B \in B$ and $C \in C\}$ is not a basis for the topology of $X \times X$
    (D) None of these

26. If B is a basis for the topology of X, then—
    (A) The collection $B_Y = \{B \cap Y \mid B \in B\}$ is not a basis for the subspace topology on X
    (B) The collection $B_Y = \{B \cap Y \mid B \in B|$ is a basis for the subspace topology on X
    (C) The collection $B_Y = \{B \cap Y \mid B \in B|$ is a basis for the topology on X
    (D) None of these

27. Let Y be a subspace of X. If U is open in Y and Y is open in X—
    (A) Then U is open in X
    (B) Then U is closed in X
    (C) Then U is null set in X
    (D) None of these

28. If X is an ordered set in the order topology and if Y is an interval or a ray in X—
    (A) Then the subspace topology and order topology on Y are same
    (B) Then the subspace topology and order topology on Y are different
    (C) Then the subspace topology and order topology on Y are open
    (D) None of these

29. If A is a subspace of X and B is a subspace of Y, then—
    (A) Product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$
    (B) Product topology on $A \times B$ is different as the topology $A \times B$ inherits as a subspace of $X \times Y$
    (C) Product topology on $A \times B$ is the same as the topology $A \times B$
    (D) None of these

30. If (X, T) is a topological space. Then—
    (A) ϕ is open (B) ϕ is closed
    (C) ϕ is discrete (D) None of these

31. If (X, T) is a topological space. Then—
    (A) X is open (B) X is closed
    (C) X is discrete (D) None of these

32. If (X, T) is a topological space. Then—
    (A) Arbitrary intersection of closed sets are open
    (B) Arbitrary intersection of closed sets are closed
    (C) Arbitrary intersection of open sets are closed
    (D) None of these

33. If (X, T) is a topological space. Then—
    (A) Finite unions of closed sets are closed
    (B) Finite unions of closed sets are open
    (C) Finite unions of open sets are closed
    (D) None of these

34. If Y is a subspace of X. Then set A is closed in Y iff—
    (A) It is equal to the intersection of a open set of X with Y
    (B) It is equal to the unions of a closed set of X with Y
    (C) It is equal to the intersection of a closed set of X with Y
    (D) None of these

35. If Y is a subspace of X. If A is closed in Y and Y is closed in X.
    (A) Then a is semi-closed in X
    (B) Then A is open in X
    (C) Then A is closed in X
    (D) None of these

36. If Y is a subspace of X. $A \subset Y$ and $\bar{A}$ is a closure of A in X. Then closure of A in Y—
    (A) Is equal to $A \cap Y$
    (B) Is equal to $A \cup Y$
    (C) Is equal to Y
    (D) None of these

37. If A is a subset of topological space, X, Let $x \in A$, closure of A in X iff—
    (A) Every open set U containing x, intersects A

(B) Every open set U that does not contain x, intersects A

(C) Every closed set U containing x, intersects A

(D) None of these

38. If A is a subset of topological space X. Suppose B is a basis for X, then $x \in \bar{A}$, iff—

(A) Every basis element $B \in$ B, containing x does not intersects A.

(B) Every basis element $B \in$ B, containing x intersects A.

(C) Every basis element $B \in$ B, containing x is disjoint to A

(D) None of these

39. Let A be a subset of topological space X and A′ be the set of all limit points of A. Then closure of A—

(A) $\bar{A} = A \cup A'$ (B) $\bar{A} = A - A'$

(C) $\bar{A} = A \cap A'$ (D) None of these

40. A subset of a topological space is closed iff—

(A) It contains none of its limit points

(B) It contains all its limit points

(C) It contains some of its limit points

(D) None of these

41. Let X be a Hausdorff space, $A \subset X$. Then a point x is a limit point of A, iff—

(A) Every neighbourhood of x contains infinitely many points of A

(B) Every neighbourhood of x contains finitely many points of A

(C) Every neighbourhood of x contains no points of A

(D) None of these

42. Every simple ordered set is a Hausdorff space in the—

(A) Order topology

(B) Non-order topology

(C) Discrete topology

(D) None of these

43. The product of two Hausdorff space is a—

(A) Hausdorff space (B) Discrete space

(C) Closed set (D) None of these

44. A subspace of a Hausdorff space is a—

(A) Hausdorff space (B) Discrete space

(C) Closed set (D) None of these

45. Let X and Y are topological spaces. A function $f : X \rightarrow Y$ is a continuous function—

(A) If for each open set V of Y, the set f^{-1} (V) is an closed subset of X

(B) If for each closed subset V of Y, the set f^{-1} (V) is an open subset of X

(C) If for each open subset V of Y, the set f^{-1} (V) is an open subset of X

(D) None of these

46. Let X and Y are topological spaces. Function f is homeomorphism if—

(A) Function $f : X \rightarrow Y$ be a one-to-one function

(B) Function f is continuous

(C) Inverse function $f^{-1} : Y \rightarrow X$ is continuous

(D) All the above

47. Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is open map if—

(A) For every open set U of X, the set f(U) is open in Y

(B) For every closed set U of X, the set f(U) is open in Y

(C) For every open set U of X, the set f(U) is closed in Y

(D) None of these

48. Let X and Y be topological spaces. A map $f : X \rightarrow Y$ is closed map if—

(A) It for every closed set A of X, the set f(A) is closed in Y

(B) For every closed set U of X, the set f(U) is open in Y

(C) For every open set U of X, the set f(U) is closed in Y

(D) None of these

49. If X is a topological space and A is x set if $P : X \rightarrow A$ is surjective map then there exists exactly one topology T on A relative to which P is a quotient map and this topology is called—

(A) Quotient topology induced by P

(B) Hausdorff topology

(C) Discrete topology

(D) None of these

50. Let X and Y are topological spaces. A function $f : X \to Y$ is continuous at $x \in X$—
   (A) If for every neighbourhood N of $f(x)$, there is a neighbourhood M of x such that $f(N) = M$
   (B) If for every neighbourhood N of $f(x)$, there is a neighbourhood M of x such that $f(N) \subset M$
   (C) If for some neighborhood N of $f(x)$, there is a neighborhood M of x such that $f(N) = M$
   (D) None of these

51. Let X and Y be topological spaces and $f : X \to Y$. If f is continuous, then—
   (A) For every closed set $B \subset Y$, set $f^{-1}(B)$ is closed in X
   (B) For every open set $B \subset Y$, set $f^{-1}(B)$ is closed in X
   (C) For every closed set $B \subset Y$, set $f^{-1}(B)$ is open in X
   (D) None of these

52. Let X, Y and Z be topological spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then—
   (A) Composite $(gof) : x \to Z$ is continuous
   (B) Composite $(gof) : x \to Z$ is discontinuous
   (C) Composite $(fog) : x \to Z$ is continuous
   (D) None of these

53. Let X and Y be topological spaces the constant function $f : X \to Y$ is—
   (A) Continuous function
   (B) Discontinuous function
   (C) Inverse function
   (D) None of these

54. If A is a subspace of topological space X, the inclusion function $J : A \to X$ is—
   (A) Continuous function
   (B) Discontinuous function
   (C) Inverse function
   (D) None of these

55. If Y is a subspace of topological space X, a separation of Y is a pair of disjoint non-empty set A and B whose union is Y, and—
   (A) Neither of which contains a limit point of the other
   (B) Each of which contains a limit point of the other
   (C) One of which contains a limit point of the other
   (D) None of these

56. If Y is a subspace of topological space X—
   (A) Y is connected if there exist no separation of Y
   (B) Y is connected if there exist at least one separation of Y
   (C) Y is connected if there exist separation of Y
   (D) None of these

57. If sets C and D from a separation of topological space X and if Y is a connected subset of X, then—
   (A) Y lies entirely with in C
   (B) Y lies entirely with in D
   (C) Y lies entirely with in either C or D
   (D) Y lies entirely with in C and D

58. The union of a collection of connected sets that have a point in common is—
   (A) Connected (B) Separable
   (C) Disconnected (D) None of these

59. Let A be a connected subset of topological space X. If $A \subset B \subset \overline{A}$, then B is—
   (A) Connected (B) Separable
   (C) Disconnected (D) None of these

60. If set B is formed by adjoining to the connected set A of topological space X some or all its limit points, then B is—
   (A) Connected (B) Separable
   (C) Disconnected (D) None of these

61. Let Y be a subspace of topological space X. Then Y is compact, iff—
   (A) Every covering of Y by sets open in X contains a finite subcollection covering Y.
   (B) Every covering of Y by sets closed in X contains a finite subcollection covering Y
   (C) Every covering of Y by sets open in X contains a infinite subcollection covering Y
   (D) None of these

62. Every closed subset of a compact space is—
    (A) Compact space (B) Open set
    (C) Null set (D) None of these

63. Every compact subset of a Hausdorff space is—
    (A) Closed set (B) Open set
    (C) Null set (D) None of these

64. If Y is a compact subset of the Hausdorff space X and $x_0 \notin$ Y, then—
    (A) There exist disjoint open sets U and V of X, such that $x_0 \in$ U and Y$\subset$ V
    (B) There exist overlapping open sets U and V of X, such that $x_0 \in$ U and Y $\subset$ V.
    (C) There exist disjoint closed sets U and V of X, such that $x_0 \in$ U and Y $\subset$ V
    (D) None of these

65. A topological space X is paracompact space if it is Hausdorff space and—
    (A) Every open covering a of X has a locally in finite open refinement B that covers X
    (B) Every open covering a of X has a locally finite open refinement B that covers X
    (C) Every closed covering a of X has a locally infinite open refinement B that covers X
    (D) None of these

66. The image of a compact topological space under continuous map is—
    (A) Continuous (B) Discontinuous
    (C) Constant (D) None of these

67. Let f : X $\to$ Y be bijective continuous function. If X is compact and Y is Hausdorff, then—
    (A) f is automorphism
    (B) f is isomorphism
    (C) f is homeomorphism
    (D) None of these

68. The product of finitely many compact spaces is—
    (A) Compact space (B) Open set
    (C) Null set (D) None of these

69. Let X be a topological space. If every collection G of closed sets in X, satisfy finite intersection condition the intersection $\cap_{C \in G} \subset \neq \phi$. Then X is—
    (A) Compact space (B) Hausdorff space
    (C) Null set (D) None of these

70. Let X be a simply ordered set having the least upper bound property. In the order topology each closed interval in X is—
    (A) Compact space (B) Hausdorff space
    (C) Null set (D) None of these

71. Every closed interval in real line P is—
    (A) Compact space (B) Hausdorff space
    (C) Null set (D) None of these

72. A subset A$\subset$ R^n is compact iff—
    (A) It is closed in the Euclidean metric d or square metric P
    (B) In bounded in the Euclidean metric d or square metric P
    (C) Both (A) and (B)
    (D) None of these

73. Maximum and minimum value theorem states—
    (A) Let f : X $\to$ Y be continuous, where y is an ordered set in the order topology. If X is compact. Then there C, $d \in$ X such that $f(c) \leq f(x) \leq f(\text{D})$ for every $x \in$ X
    (B) Let f : X $\to$ Y be continuous, where Y is an ordered set in the order topology. If X is compact then there C, $d \in$ X such that $f(c) = f(x) = f(\text{D})$ for every $x \in$ X.
    (C) Let f : X $\to$ Y be discontinuous, where Y is unordered set in the order topology. If X is compact, then there C, $d \in$ X such that $f(c) \leq f(x) \leq f(\text{D})$ for every $x \in$ X.
    (D) None of these

74. Let X be a (Non empty) compact Hausdorff space. If every point of X is a limit point of X, then—
    (A) X is uncountable
    (B) X is countable
    (C) X is disjoint
    (D) None of these

75. Let X be a topological space. Let one point sets in X be closed X is ……… if given a point $x \in$ X and neighbourhood U of x, there is a neighbourhood V of x such that $\overline{\text{V}} \subset$ U—

(A) Regular (B) Normal
(C) Disjoint (D0 None of these

76. Let X be a topological space. Let one point sets in X be closed. X is .... iff given a closed set A and open set U, A⊂ U there is an open set V, A⊂ V such that $\overline{V}$⊂ U—
(A) Regular (B) Normal
(C) Disjoint (D) None of these

77. A subspace of Hausdorff space is—
(A) Hausdorff (B) Normal
(C) Regular (D) None of these

78. A topological space X is completely regular—
(A) If one point sets are open in X and if every $a \in$ X, every open set B, $a \in$ B there exists a continuous function f : X → [0, 1] such that $f(a) = f(B) = \{1\}$
(B) If one point sets are closed in X and if every $a \in$ X, every closed set B, $a \in$ B, there exists a continuous function f : X → [0, 1] such that $f(a) = 0$ and $f(B) = \{1\}$
(C) If one point sets are closed in X and if every $a \in$ X, every closed set B, $a \in$ B there exists a discontinuous function f : X → [0, 1] usch that $f(a) = f(B)$
(D) None of these

79. Product of Hausdorff spaces is—
(A) Hausdorff (B) Normal
(C) Disjoint (D) None of these

80. A subspace of regular space is—
(A) Hausdorff (B) Regular
(C) Disjoint (D) None of these

81. A product of regular spaces is—
(A) Hausdorff (B) Regular
(C) Disjoint (D) None of these

82. A subspace of normal space is—
(A) Need not normal
(B) Normal
(C) Hausdorff
(D) Need not Hausdorff

83. Product of normal spaces is—
(A) Need not normal
(B) Normal
(C) Hausdorff
(D) Need not Hausdorff

84. Every metrizable space is—
(A) Hausdorff (B) Normal
(C) Disjoint (D) None of these

85. Every compact Hausdorff space is—
(A) Hausdorff (B) Normal
(C) Disjoint (D) None of these

86. Every regular space with countable basis is—
(A) Hausdorff (B) Normal
(C) Disjoint (D) None of these

87. Every well ordered set X is normal in the—
(A) Order topology
(B) Non order topology
(C) Discrete topology
(D) None of these

88. Let X be a topological space A, B ⊂ X, f : X → [0, 1] a continuous function such that $f(A) = \{0\}$ and $f(B) = \{1\}$ then—
(A) Sets A and B are separated by continuous function
(B) Sets A and B are separated by discontinuous function
(C) Set A and B are separted by step function
(D) None of these

89. A topological space X is completely regular—
(A) If one point sets are open in X and if every $a \in$ X, every open set B, $a \in$ B, there exists a continuous function f : X → [0, 1] such that $f(a) = 0$ and $f(B) = \{1\}$
(B) If one point sets are closed in X and if every $a \in$ X, every closed set B, $a \in$ B, there exists a continuous function f : X → [0, 1] such that $f(a) = 0$ and $f(B) = \{1\}$
(C) If one point sets are closed in X and if every $a \in$ X, every set B $a \in$ B, there exists a discontinous function f : X → [0, 1] such that $f(a) = 0$ and $f(B) = \{1\}$
(D) None of these

90. Urysohn lemma states—
(A) Let X be a normal space. A and B are disjoint closed subsets of X and $[a, b]$ closed interval of R. Then there exists a continuous map f : X → $[a, b]$ such that for every $x \in$ A, $f(x) = a$ and for every $x \in$ B, $f(x) = b$

(B) Let X be a normal space; A and B are open subsets of X and $[a, b]$ closed interval of R. Then there exist a discontinuous map $f : X \to [a, b]$ such that for every $x \in A$, $f(x) = a$ and for every $x \in B$, $f(x) = b$

(C) Let X be a normal space; A and B are disjoint open subsets of X and $[a, b]$ closed interval of R, then there exist a discontinuous map $f : X \to [a, b]$ such that for every $x \in A$, $f(x) = a$ and for every $x \in B$, $f(x) = b$

(D) None of these

91. Tietze extension theorem states : Let X be a normal space. Let A be closed subset of X—

(A) Any continuous map of A into the closed interval $[a, b]$ of R may be extended to a continuous map of all of X into $[a, b]$

(B) Any continuous map of A into R may be extended to a continuous map of X into R

(C) Both (A) and (B)

(D) None of these

92. Set A is a G_δ set in X if—

(A) Set A is the intersection of a countable collection of open sets of X

(B) Set A is the union of a countable collection of open sets of X

(C) Set A is the intersection of a uncountable collection of closed sets of X

(D) Set A is the union of a countable collection of closed sets of X

93. Strong form of Urysohn lemma states—

(A) Let A and B closed disjoint subsets of the normal space X. There exists a continuous function $f : X \to [0, 1]$ such that $f^{-1}\{0\} = A$ and $f(B) = \{1\}$ iff A is a G_δ set in X

(B) Any continuous map of A into the closed interval $[a, b]$ of R may be extended to a continuous map of all of X into $[a, b]$

(C) Every regular space X with countable basis is metrizable

(D) None of these

94. Urysohn metrization theorem states—

(A) Let A and B closed disjoint subsets of the normal space X. There exists a continuous function $f : X \to [0, 1]$ such that $f^{-1}\{0\} = A$ and $f(B) = \{1\}$ iff A is a G_δ set in X

(B) Any continuous map of A into the closed interval $[a, b]$ of R may be extended to a continuous map of all of X into $[a, \text{b}]$

(C) Every regular space X with countable basis is metrizable

(D) None of these

95. Every closed interval of real line R is—

(A) Uncountable (B) Countable

(C) Disjoint (D) None of these

96. The cartesian product of connected topological space is—

(A) Connected (B) Separable

(C) Disconnected (D) None of these

97. Let X be a topological space. A covering of X is called open covering if—

(A) Its elements are open subsets of X

(B) Its elements are open subsets of X

(C) Its elements are null sets

(D) None of these

98. A topological space X is compact if every open covering a of X contains—

(A) A finite subcollection that covers X

(B) A infinite subcollection that covers X

(C) A finite subcollection that does not covers X

(D) None of these

99. A collection G of subjects of topological space satisfies finite intersection condition if for every finite subcollection $\{C_1, C_2 \ldots C_n\}$ of G, the intersection—

(A) Is a null set (B) Is not a null set

(C) Is a set G (D) None of these

100. Let d and d' be two metrics on the set X and let T and T′ be the topologies they induce respectively. Then T′ is … than T iff for each $x \in X$ and $\varepsilon > 0$, there exist $\delta > 0$, such that $B_d(x\ \delta) \subset B_d(x, \varepsilon)$—

(A) Finer (B) Strictly finer

(C) Not finer (D) None of these

101. The topologies on R^n induced by the Euclidean metric d and the square metric pare …… as the product topology on R^n—

(A) Same (B) Dissimilar
(C) Different (D) None of these

102. The uniform topology on R^3 is …… than the product topology. They are different is S is infinite.
(A) Finer (B) Strictly finer
(C) Not finer (D) None of these

103. An arc is—
(A) Space homeomorphic to the unit interval [0, 1]
(B) Not space homeomorphic to the unit interval [0, 1]
(C) Both (A) and (B)
(D) None of these

104. A simple closed curve is—
(A) A space homeomorphic to the circle δ'
(B) Not space homeomorphic to the circle δ'
(C) Both (A) and (B)
(D) None of these

105. Let $a, b \in S^2$ and A be a compact space. Let f^1. $A \to S^2 - a - b$ be continuous map. Then f is in essential—
(A) If a and b lie in same component of $S^2 - f(A)$
(B) If a and b lie in different component of $S^2 - f(A)$
(C) If a and b does not lie in same component of $S^2 - f(A)$
(D) None of these

106. Brouwer theorem on invariance of domain for R^2 states—
(A) If U is an closed subset of R^2 and $f : U \to R^2$ is continuous and injective, then $f(U)$ is open in R^2 and f is an imbedding
(B) If U is an open subset of R^2 and $f : U \to R^2$ is discontinuous and injective, then f (U) is open in R^2 and f is an imbedding
(C) If U is an open subset of R^2 and $f : U \to R^2$ is continuous and injective, then f (U) is open in R^2 and f is an imbedding
(D) None of these

107. Jordan separation theorem states—
(A) Let C be a simple closed curve in S^2. Then $S^2 - C$ is not connected
(B) Let C be a simple closed curve in S^2. Then $S^2 - C$ is connected
(C) Let C be a simple open curve in S^2. Then $S^2 - C$ is not connected
(D) None of these

108. Let A and B be closed connected subset of S^2 whose intersection consists of precisely two points—
(A) Then $A \cup B$ separates S^2
(B) Then $A \cap B$ separates S^2
(C) Then $A - B$ separates S^2
(D) None of these

109. Homotopy existence lemma states—
(A) Let X and X + [0, 1] be normal, let A be a closed subset of X. If $f : A \to R^2 - 0$ is continuous map and f is in essential, then f can be extended to a continuous map $g : X \to R^2 - 0$
(B) Let X and X + [0, 1] be normal, let A be a open subset of X. If $f : A \to R^2 - 0$ is discontinuous map and f is in essential. Then f can be extended to a continuous map $g : X \to R^2 - 0$
(C) Let X and X + [0, 1] be normal, let A be a closed subset of X. If $f : A \to R^2 - 0$ is continuous map and f is in essential. Then f cannot be extended to a continuous map $g : X \to R^2 - 0$
(D) None of these

110. Borsuk theorem states—
(A) Let X be a compact subset of R^2. If 0 does not lies in a bounded component C of $R^2 - X$, then the inclusion map $J : X \to R^2 - 0$ is essential (and conversely)
(B) Let X be a compact subject of R^2. If 0 lies in a bounded component C of $R^2 - X$, then the inclusion map $J : X \to R^2 - 0$ is essential (and conversely)
(C) Let X be a compact subset of R^2. If 0 lies in a bounded component C of $R^2 - X$, then the inclusion map $J : X \to R^2 - 0$ is not essential (and conversely)
(D) None of these

111. Non-separable theorem states—
(A) No arc separates R^2
(B) No space homeomorphism to ball B^2 separates R^2
(C) Both (A) and (B)
(D) None of these

112. Brouwer theorem on invariance of domain for R^2 states—

(A) If U is an closed subset of R^2 and $f: U \to R^2$ is continuous and injective, then f (U) is open in R^2

(B) If U is an open subset of R^2 and $f: U \to R^2$ is discontinuous and injective, then f (U) is open in R^2

(C) If U is open subset of R^2 and $f: U \to R^2$ is continuous and injective then f (U) is open in R^2

(D) None of these

113. Let X be the union of two open sets U and V. Suppose that $U \cap V$ can be written as the union of two disjoint open sets A and B. Let $a \in$ A and $b \in$ B and a and b are joined by paths in U and in V, then—

(A) π; (X, a) is not zero

(B) π; (X, a) is zero

(C) π; (X, a) is positive

(D) None of these

114. Let X = U $\cup$ V, where U and V are open sets and $U \cap V = A \cup A$ 'U where A, B, A' are disjoint open sets. Let $a \in$ A, $a' \in$ A' and $b \in$ B, suppose that a, a' and b are joined by paths is U and in V. Then—

(A) π; (X, a) is not cyclic

(B) π; (X, a) is not infinite cyclic

(C) π; (X, a) is infinite cyclic

(D) None of these

115. Non-separation theorem states—

(A) Let A be an arc in S^2. Then $S^2 - A$ is disconnected

(B) Let A be an arc in S^2. Then $S^2 - A$ is connected

(C) Let A be an arc in S^2. Then $S^2 - A$ is separable

(D) None of these

116. Jordan curve theorem states : Let c be a simple closed curve in S^2—

(A) Then $S^2 - c$ has precisely two components w_1 and w_2 of which c is not the common boundary

(B) Then $S^2 - c$ has precisely two components w_1 and w_2 of which c is the common boundary

(C) Then $S^2 - c$ has no components w_1 and w_2 of which c is the common boundary

(D) None of these

117. Brouwer theorem on in variance of domain for R^2 states—

(A) If U is an closed subset of R^2 and $f: U \to R^2$ is continuous and injective, then f is an imbedding

(B) If U is an open subset of R^2 and $f: U \to R^2$ is discontinuous and injective, then f is an imbedding

(C) If U is an open subset of R^2 and $f: U \to R^2$ is continuous and injective, then f is an imbedding

(D) None of these

118. A continuous map $f: X \to Y$ is homotopy equivalence—

(A) If there is a continuous map $g: y \to X$ such that gof is homotopic to the identity map i_x of X and fog is homotopic to the identity map i_y of Y

(B) If there is a discontinuous map $g: Y \to X$ such that gof is homotopic to the identity map i_x of X and fog is homotopic to the identity map i_y of Y

(C) If there is a constant map $g: Y \to X$ such that gof is homotopic to the identity map i_x of X and fog is homotopic to the identity map i_y of Y

(D) None of these

119. Let continuous map $f: X \to Y$ is homotopy equivalence and if there is a continuous map $g: Y \to X$ such that gof is homotopic to the identity map i_x of X and fog is homotopic to the identity map i_y of y—

(A) The map g is homotopy inverse for the map f

(B) The map g is not homotopy inverse for the map f

(C) The map g is equal to map f

(D) None of these

120. Let h, K : X $\to$ Y, $h(x_0) = y_0$ and $K(x_0) = y_1$. If h and K are homotopic then there is a path $\alpha \in y$ from y_0 to y_1 such that $K = \hat{\alpha}oh$. If $y_0 = y_1$ and if the base point x_0 remains fixed during the homotopy. Then—

(A) h is equal to K
(B) h is not equal to K
(C) Both (A) and (B)
(D) None of these

121. h, K : X $\to$ Y and $h(x_0) = y_0$, $k(x_0) = y_1$. Suppose that h and k are homotopic. If k is …… then so is h—
(A) Injective
(B) Surjective
(C) Zero homomorphism
(D) All the above

122. Let $h : X \to Y$ and $h(x_0) = y_0$. If h is homotopic to a constant map, then—
(A) h is the zero homeomorphism
(B) h is the isomorphism
(C) h is the homeomorphism
(D) None of these

123. Let $f : X \to Y$ be continuous, $f(x_0) = y_0$. If f is Homotopy equivalence, then $f : \pi : (X, x_0) \to \pi : (Y, y_0)$ is—
(A) Zero homeomorphism
(B) Isomorphism
(C) Homeomorphism
(D) None of these

124. Let X be a topological space. A separation of X—
(A) Is a pair U, V of disjoint non empty open subsets of X whose union is X
(B) Is a pair U, V of non disjoint empty open subsets of X whose union is X
(C) Is a pair U, V of disjoint empty closed subsets of X whose union is X
(D) Is a pair U, V of non empty closed subsets of X whose intersection is X

125. Let X be a topological space. The space X is called connected—
(A) If there does not exist a separation of X
(B) If there exist a separation of X
(C) If there exist some separation of X
(D) None of these

126. Let X be a topological space. A space X is connected, iff—
(A) The only subset of X that are both open and closed in X are empty set and X itself
(B) The only subset of X that are both open but not closed in X are empty set and X itself
(C) The only subset of X that are closed in X are empty set only
(D) The only subset of X that are both open and closed in X are X itself only

127. A topological space X is totally disconnected if—
(A) Its only connected subsets are open-point set
(B) Its only connected subsets are empty set
(C) Its only connected subsets are set X itself
(D) Its only connected subsets are set X itself

128. A simple ordered set L having more than one element is called linear continum if—
(A) L has the least upper bound property
(B) If $x < y$, there exist z such that $x < z < y$
(C) Both (A) and (B)
(D) None of these

129. Given $\overline{x} \in R^n$, the norm of $\overline{x}$ is—
(A) $|\overline{x}| = (x_1^2 + x_2^2 + \ldots\ldots + x^2_n)^{1/2}$
(B) $|\overline{x}| = (x_1^2 + x_2^2 + \ldots\ldots + x^2_n)^2$
(C) $|\overline{x}| = (x_1^2 + x_2^2 + \ldots\ldots + x^2_n)$
(D) None of these

130. The Euclidean metric d on R^n is defined as—
(A) $d(\overline{x}, \overline{y}) = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots\ldots + (x_n - y_n)^2]$
(B) $d(\overline{x}, \overline{y}) = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots\ldots + (x_n - y_n)^2]^2$
(C) $d(\overline{x}, \overline{y}) = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots\ldots + (x_n - y_n)^2]^{1/2}$
(D) None of these

131. Let $\overline{x} \in R^n$, the square metric P is defined as—
(A) $p(\overline{x}, \overline{y}) = \min.\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$
(B) $p(\overline{x}, \overline{y}) = \max.\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$
(C) $p(\overline{x}, \overline{y}) = \{|x_1 - y_1|, \ldots, |x_n - y_n|\}$
(D) None of these

132. Standard bounded metric (corresponding to d) is defined as—

(A) Let X be a metric space with metric d and $\overline{d} : X \times X \to R$ such that $\overline{d}\,(x, y)$ = max. $\{d\,(x, y), 1\}$

(B) Let X be a metric space with metric d and $\overline{d} : X \times X \to R$ such that $\overline{d}\,(x, y)$ = $\{d\,(x, y), 1\}$

(C) Let X be metric space with metric d and $\overline{d} : X \times X \to R$ such that $\overline{d}\,(x, y)$ = min. $\{d\,(x, y), 1\}$

(D) None of these

133. Given an index set J, and $\overline{x}\,(x_\alpha)_{\alpha \in j}$ and $\overline{y}$ = $(y_\alpha)_{\alpha \in j}$ of R^j, $\overline{y}\,(x_\alpha, y_\alpha)$ = min. $\{d\,(x_\alpha, y_\alpha), 1\}$ the standard bounded metric on R. The uniform metric on R^j is defined as—

(A) $\overline{P}(\overline{x}, \overline{y}) = \{\overline{d}\,(x_\alpha, y_\alpha),_{\alpha \in J}\}$

(B) $\overline{P}(\overline{x}, \overline{y})$ = lub $\{\overline{d}\,(x_\alpha, y_\alpha),_{\alpha \in}\}$

(C) $\overline{P}(\overline{x}, \overline{y})$ = glb· $\{\overline{d}\,(x_\alpha, y_\alpha),_{\alpha \in j}\}$

(D) None of these

134. A sequence $\{x_m\}$ of points of X is convergence to a point $x \in X$—

(A) If for every neighbourhood U of x there exists positive integer N such that $x_n \in U$ for all $n \geq N$

(B) If for every neighbourhood U of x there exists positive integer N such that $x_n \in U$ for all $n < N$

(C) If for every neighbourhood U of x there exists positive integer N such that $x_n \in U$ for all $n = N$

(D) None of these

135. A topological space X have a countable basis at the point x—

(A) If there is a countable collection $\{U_n\}_{n \in z}$, of neighbourhood of x such that any neighborhood U of x contains atleast one of the sets U_n

(B) If there is a uncountable collection $\{U_n\}_{n \in z}$, of neighbourhood of x such that any neighbourhood U of x contains none of the sets U_n

(C) If there is a countable collection $\{U_n\}_{n \in z}$, of neighbourhood of x such that any neighbourhood of x contains more of the sets U_n

(D) None of these

136. A topological space X that has a countable basis at each of its points is said to satisfy—

(A) The first countability axiom

(B) The second countability axiom

(C) The third countability axiom

(D) The fourth countability axiom

137. Let $f : X \to Y$ also X and Y be metrizable with metrics d_x and d_y respectively. If given $x \in X$, and $\in > 0$, there exist $\delta > 0$ such that $d_x\,(x, y) < \delta \Rightarrow d_y\,[f(x), f(y)] < \in$ then—

(A) f is continuous (B) f is discontinuous

(C) f is constant (D) None of these

138. Let X and Y are topological space. A map $h : X \to Y$ is in essential map—

(A) If h is homotopic to a constant map

(B) If h is homotopic to a non constant map

(C) If h is not homotopic to a constant map

(D) None of these

139. The uniform topology on R^J is finer than the product topology and—

(A) They are same if J is infinite

(B) They are different if J is infinite

(C) They are different if J is finite

(D) None of these

140. Let $\overline{d}\,(a, b)$ = min. $\{|a - b|\}$ be the standard bounded metric on R. If x and y are two points of R^w, define

$$D(\overline{x}, \overline{y}) = \text{lub}\left\{\frac{\overline{d}\,(x_i, h_i)}{i}\right\}, \text{ then—}$$

(A) D is a not a metric that induces the product topology on R^w

(B) D is a metric that induces the product topology on R^w

(C) D is a metric that does not induces the product topology on R^w

(D) None of these

141. Let $f : X \to Y$ also X and Y be metrizable with metric d_x and d_y respectively. Then f is continuous if—

(A) To given $x \in X$ and $\in > 0$, there exist $\delta > 0$ such that $d_x(x, y) < \delta \Rightarrow d_y[f(x), (y)] < \in$

(B) To given $x \in X$ and $\in > 0$, there does not exist $\delta > 0$ such that $d_x(x, y) < \delta \Rightarrow d_y[f(x), (y)] < \in$

(C) To given $x \in X$ and $\in > 0$, there exist $\delta > 0$ such that $d_x(x, y) = \delta \Rightarrow d_y[f(x), f(y)] = \in$

(D) None of these

142. A topological space X is locally connected at x is—

(A) For every neighbourhood U of x, there is a connected neighbourhood V of x, $V \subset U$

(B) For every neighbourhood U of x, there is a connected neighbourhood V of x, $V = U$

(C) For every neighbourhood U of x, there is a disconnected neighbourhood V of x

(D) None of these

143. Topological space X is locally path connected space—

(A) If X is locally connected at each $x \in X$

(B) If X is locally connected at some $x \in X$

(C) If X is locally connected at each $x \notin X$

(D) None of these

144. A topological space X is locally path connected at x, if—

(A) For every neighbourhood U of x, there is a path connected neighbourhood V of x, $V \subset U$

(B) For every neighbourhood U of x, there is a path connected neighbourhood V of x, $V = U$

(C) For every neighbourhood U of x, there is a disconnected neighbourhood V of x

(D) None of these

145. A topological space X is connected in k leinen at x, if—

(A) For every neighbourhood U of x, there is a disconnected subset y of U that contains a neighbourhood of x

(B) For every neighbourhood U of x, there is a connected subset y of U that contains a neighbourhood of x

(C) For every neighbourhood U of x, there is a connected subset y of U that does not contains a neighbourhood of x

(D) None of these

146. If one point sets are closed in topological space X. Then X is regular, if—

(A) For each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B respectively

(B) For each pair consisting of a point x and a open set B disjoint from x, there exist disjoint closed sets containing x and B respectively

(C) For each pair consisting of a point x and a open set B disjoint from x, there exist disjoint closed sets not containing x and B respectively

(D) None of these

147. A topological space X is normal—

(A) If for each A, B, $A \cap B = \phi$ of open sets of X, there exist disjoint open sets O_1 and O_2, $O_1 \cap O_2 \neq \phi$, such that $A \subset O_1$ and $B \subset O_2$

(B) If for each pair A, B, $A \cap B \neq \phi$ of closed sets of X, there exist disjoint open sets O_1 and O_2, $O_1 \cap O_2 \neq \phi$, such that $A \subset O_1$ and $B \subset O_2$

(C) If for each pair A, B, $A \cap B = \phi$ of closed sets of X, there exist non disjoint open sets O_1 and O_2, $O_1 \cap O_2 = \phi$, such that $A \subset O_1$ and $B \subset O_2$

(D) None of these

148. A topological space X is locally connected iff—

(A) For every open set U of X, each component of U is closed in X

(B) For every closed set U of X, each component of U is open in X

(C) For every open set U of X, each component of U is open in X

(D) None of these

149. A topological space X is locally path connected iff—

(A) For every closed set U of X, each path component of U is open in X

(B) For every open set U of X, each path component of U is open in X

(C) For every open set U of X, each path component of U is closed in X

(D) None of these

150. A subspace of a completely regular space is—

(A) Normal

(B) Regular

(C) Completely regular

(D) None of these

151. Let X be a topological space given an equivalence relation on X such that $x \sim y$ if there is connected subset of X containing both x and y—

(A) Then equivalence classes are called the connected components of X

(B) Then equivalence classes are called the separation of X

(C) Then equivalence classes are called the compact of X

(D) None of these

152. The product of completely regular spaces is—

(A) Normal

(B) Regular

(C) Completely regular

(D) None of these

153. If X is completely regular, then—

(A) X cannot be imbedded in $[0, 1]^J$ for some J

(B) X can be imbedded in $[0, 1]^J$ for some J

(C) Both (A) and (B)

(D) None of these

154. Let X be a topological space X is completely regular if—

(A) X is homeomorphic to a subspace of a compact Hausdorff space

(B) X is homeomorphic to a subspace of a normal space

(C) Both (A) and (B)

(D) None of these

155. Let X be a locally compact Housdorff space which is not compact. Let Y be the one point compactification of X, then—

(A) X is subspace of Y

(B) The set Y – X consists of a single point

(C) $\overline{X} = Y$

(D) All the above

156. Let X be a topological space satisfy first countability axiom if—

(A) The point $x \in \overline{A}$, closure of $A \subset X$ iff there is a sequence of point of A converging to x

(B) The point $x \in \overline{A}$, closure of $A \subset X$ iff there is a sequence of point of A diverging to x

(C) The point $x \in \overline{A}$, closure of $A \subset X$ iff there is a sequence of points of A converging to o

(D) None of these

157. Let X be a topological space satisfy first countability axiom if—

(A) The function $f : X \to Y$ is continuous iff for every diverting sequence $\{x_n\}$ in X, converging to x the sequence $\{f(x_n)\}$ converges to $f(x)$

(B) The function $f : X \to Y$ is continuous iff for every convergent sequence $\{x_n\}$ in X, converging to x the sequence $\{f(x_n)\}$ converges to $f(x)$

(C) The function $f : X \to Y$ is continuous iff for every convergent sequence $\{x_n\}$ in X, converging to x the sequence $\{f(x_n)\}$ diverges to $f(x)$

(D) None of these

158. A subspace of a first countable space is—

(A) First countable

(B) Second countable

(C) Third countable

(D) Fourth countable

159. Every Hausdorff topological group is—

(A) Normal

(B) Regular

(C) Completely regular

(D) None of these

160. A compactification of a space X—

(A) Is Hausdorff space Y containing X such that X is dense in Y

(B) Is compact Hausdorff space Y containing X such that X is dense in Y

(C) Is compact Hausdorff space Y not containing X

(D) None of these

161. Let X be a topological space. Given an equivalence relation on X such that $x \sim y$ if there is a path in X from x to y—

(A) Then equivalence classes are called the path components of X

(B) Then equivalence classes are called the separation of X

(C) Then equivalence classes are called the compact of X

(D) None of these

162. Let X be a topological space. A collection a of subset of X are locally finite—

(A) If every point of X has a neighbourhood that intersect only finitely many elements of a

(B) If some point of X has a neighbourhood that intersect infinitely many elements of a

(C) If every point of X has a neighbourhood that intersect infinitely many elements of a

(D) None of these

163. A collection B of subsets of topological space X is countably locally finite—

(A) If B can be written as uncountable union of collections B_n, each of which is locally finite

(B) If B can be written as countable intersection of collections B_n, each of which is locally finite

(C) If B can be written as countable union of collection B_n each of which is locally finite

(D) None of these

164. Let a be a locally finite collection of subsets of X. Then—

(A) Any subcollection of a is locally finite

(B) The collection $B = \{\overline{A}\}_{A \in a}$ of closures of the elements of a is locally finite

(C) $\overline{\cup_{A\in a} A} = \cup_{A\in a}\overline{A}$

(D) All of above

165. A countable product of first countable spaces is—

(A) First countable

(B) Second countable

(C) Third countable

(D) Fourth countable

166. Nagata—Smirnov metrization theorem of G_δ set states : A subset A of topological space X is G_γ set in X—

(A) If it is equals to the union of a countable collection of open subset of X

(B) It is equal to the intersection of a countable collection of open subsets of X

(C) If it is equal to the union of a uncountable collection of open subsets of X

(D) None of these

167. Let X be a regular space with a basis B that is countably locally finite. Then—

(A) X is normal

(B) Every closed set of X is G_δ set in X

(C) Both (A) and (B)

(D) None of these

168. Let X be a regular space with basis B that is countably locally finite. Then—

(A) X is metrizable only

(B) X is normal only

(C) X is metrizable and normal both

(D) None of these

169. Let a be a collection of subsets of the space X, A collection B of subsets of X is called refinement of a (refine a)—

(A) If for some $B \in B$, there is $A \in a$ such that $B = A$

(B) If for each $B \in B$, there is $A \in a$ such that $B \subset A$

(C) If for each $B \in B$, there is $A \in a$ such that $B = \subset A$

(D) None of these

170. Sequence lemma states; Let X be a topological space and $A \subset X$—

(A) If there is a sequence of points of A converging to x, then $x \in \overline{A}$

(B) The converse holds of X is metrizable

(C) Both (A) and (B)

(D) None of these

171. If X and Y are topological space. Let f : X $\to$ Y and X be metrizable. The function f is continuous, iff—

(A) For every divergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ divergent to $f(x)$

(B) For every divergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$

(C) For every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$

(D) For every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ diverges to $f(x)$

172. The … operations are continuous function from R × R into R and quotient operation is continuous function from R × (R – {0}) into R—

(A) Addition (B) Subtraction

(C) Multiplication (D) All of these

173. If X is a topological space and if f, g : X $\to$ R are continuous function, then … are continuous functions.

(A) $f+g$ (B) $f-g$

(C) fg (D) All of the above

174. If X is a topological space and if f, g : X $\to$ R are continuous function, then—

(A) f/g is continuous

(B) If $g(x) \neq 0$ for all x, then f/g is continuous

(C) If $g(x) = 0$ for all x, then f/g is continuous

(D) None of these

175. Uniform limit theorem states. Let f_n : X $\to$ Y be a sequence of continuous functions from the topological space X into the metric space Y—

(A) If $\{f_n\}$ converges uniformly to f, then f is continuous

(B) If $\{f_n\}$ diverges uniformly to f, then f is continuous

(C) If $\{f_n\}$ converges uniformly to f, then f is discontinuous

(D) None of these

176. An m – manifold is—

(A) Hausdorff space with a uncountable basis such that some point $x \in$ X, has a neigbhourbood that is homeomorphic with an open subset of R^m

(B) Hausdorft space with a uncountable basis such that each point $x \in$ X, has a neigbhourbood that is homeomorphic with an closed subset of R^m

(C) Hausdorff space with a countable basis such that each point $x \in$ X, has a neigbhourbood that is homeomorphic with an open subset of R^m

(D) Hausdorff space with a countable basis such that some point $x \in$ X, has a neigbhourbood that is homeomorphic with an closed subset of R^m

177. An indexed collection $\{A_\alpha)$ of subsets of X is point – finite indexed family—

(A) If some $x \in$ X $\to x \in A_\alpha$ for infinitely many values of α

(B) If each $x \in$ X $\to x \in A_\alpha$ for only finitely many values of α

(C) If each $x \in$ X $\to x \in A_\alpha$ for infinitely many values of α

(D) If some $x \in$ X $\to x \in A_\alpha$ for only finitely many values of α

178. Let X be topological space. An idexed family $\{A_\alpha\}_{\alpha \in J}$ of subset of X is locally finite indexed family—

(A) If some point of X has a neighbourhood that intersects A_α for infintely many values of α

(B) If each point of X has a neighbourhood that intersects A_α for infintely many values of α

(C) If each point of X has a neighbourhood that intersects A_α for only finitely many values of α

(D) None of these

179. Following is false—

(A) Every para compact space X is normal

(B) Every closed subspace of a para compact space is para compact

(C) An arbitary subspace of a para compact space and product of para compact spaces need not be para compact

(D) Every metrizable space need not be para compact

180. Let X be a regular space. Every open covering of X has a refinement. If—
   (A) An open covering of X and countably locally finite
   (B) A covering of X and locally finite
   (C) A closed covering of X and locally finite
   (D) All the above

181. Let X be a regular space. Every open covering of X has a refinement. If—
   (A) An open covering of X and countably locally finite
   (B) An open covering of X and locally finite
   (C) A closed covering of X and locally finite
   (D) All the above

182. A space X is locally metrizable if every $x \in$ X has a neighbourhood $\cup$ that is metrizable in the subspace topology—
   (A) Locally metrizable theorem
   (B) Smirnov metrizable theorem
   (C) Special Van Kampen theorem
   (D) None of these

183. Weierstrass M–test : Let $f_n : X \to R$ be a sequence of functions from topological space X into R and

$$S_n(x) = \sum_{i=1}^{N} f_i(x).$$ The sequence $\{S_n\}$

converges uniformly to a function S if—
   (A) $|f_i(x)| \leq b_i$ for all $x \in X$ and all $i = 1, \ldots, n$
   (B) The series Σb_i is convergent
   (C) Both (A) and (B)
   (D) None of these

184. If X is a topological space, then—
   (A) Each path component of X lies in a component of X
   (B) Some path component of X lies in a component of X
   (C) Each path component of X does not lies in a component of X
   (D) None of these

185. If X is a topological space and locally path connected then—
   (A) The components and the path components of X are different
   (B) The components and the path components of X are the same
   (C) The components and the path components of X are the disjoint
   (D) None of these

186. Let X be a topological space. Given an equivalence relation on X such that $x \sim y$, if there is a connected subset of X containing both x and y—
   (A) Then equivalence classes are called the components of X
   (B) Then equivalence classes are called the separation of X
   (C) Then equivalence classes are called the compact of X
   (D) None of these

187. The components of topological space X are connected disjoint subsets of X whose—
   (A) Intersection is X such that each connected subset of X intersects only one of them
   (B) Union is X such that each connected subset of X intersects all of them
   (C) Union is X such that each connected subset of X intersects only one of them
   (D) None of these

188. A topological space X is limit point compact—
   (A) If every disjoint subset of X has a limit point
   (B) If every infinite subset of X has a limit point
   (C) If every finite subset of X has a limit point
   (D) None of these

189. A topological space X is sequentially compact—
   (A) If every sequence in a topological space has a convergent subsequence
   (B) If some sequence in a topological space has a convergent subsequence
   (C) If no sequence in a topological space has a convergent subsequence
   (D) None of these

190. A topological space X is countably compact—

(A) If every countable open covering of X contains a infinite subcollection covering X
(B) If every countable open covering of X contains a finite subcollection covering X
(C) If every countable open covering of X contains no subcollection covering X
(D) None of these

191. One of the following statement is true—
(A) Compactness implies limit point compactness
(B) Limit point compactness implies compactness
(C) Both (A) and (B)
(D) None of these

192. Uniform continuity theorem states : f is uniformly continuous on topological space X and Y—
(A) If $f : X \rightarrow Y$ is a continuous map of the compact metric space (X, d_x) to the metric space (Y, d_y)
(B) If $f : X \rightarrow Y$ is a discontinuous map of the compact metric space (X, d_x) to the metric space (Y, d_y)
(C) If $f : X \rightarrow Y$ is a continuous map of the non-compact metric space (X, d_x) to the metric space (Y, d_y)
(D) None of these

193. Let X be a metrizable space X is limit point compact, then—
(A) X is sequentially compact
(B) X is not sequentially compact
(C) Both (A) and (B)
(D) None of these

194. Let X be a locally compact Hausdorff space which is not compact. Let Y be the one point compactification of X. Then—
(A) Y is compact Hausdorff space
(B) The set Y – X consists of a single point
(C) $\overline{X} = Y$
(D) All the above

195. One of the following statement is true—
(A) If X is countably compact space then it is limit point compact space
(B) If X is countably compact space then it is not limit point compact space
(C) Both (A) and (B)
(D) None of these

196. A topological space X is locally compact at x—
(A) If there is some compact subspace. C of X that contains a neighbourhood of x
(B) If there is some compact subspace. C of X that does not contains a neighbourhood of x
(C) If there is no compact subspace, C of X that does not contains a neighbourhood of x
(D) None of these

197. Tychonoff theorem states—
(A) Any arbitrary product of compact spaces is compact in the product topology
(B) A topological space forwhich every open covering contains a countable subcovering
(C) Path connected topological space is connected but converse is not true
(D) None of these

198. A space X is completely regular if—
(A) One-point sets are closed in X
(B) For each $x_0 \in X$ and each closed set A, $x_0 \notin A$, there is a continuous function $f : X \rightarrow [0, 1]$
(C) Both (A) and (B)
(D) Either (A) or (B)

199. Compactification Y_1 and Y_2 of topological space X are equivalent—
(A) If there is a homeomorphism $h : Y_1 \rightarrow Y_2$ such that $h(x) = x$ for every $x \in X$
(B) If there is a homeomorphism $h : Y_1 \rightarrow Y_2$ such that $h(x) = x$ for some $x \in X$
(C) If there is a homeomorphism $h : Y_1 \rightarrow Y_2$ such that $h(x)$ = constant x for every $x \in X$
(D) None of these

200. Let X be completely regular, $\beta(X)$ be its Stone – Cech compactification—
(A) Then every continuous real valued function of X can be uniquely extended to real valued function of $\beta(X)$

(B) Then every bounded real valued function of X can be uniquely extended to real valued function of $\beta(X)$

(C) Then every bounded continuous real valued function of X can be uniquely extended to continuous real valued function of $\beta(X)$

(D) None of these

201. Let $A \subset X$ a topological space and $f : A \to Z$ be continuous map of A into Hausdorff space z—

(A) There is at most one extension of f to a continuous function $g : \overline{A} \to z$

(B) There is at no extension of f to a continuous function $g : \overline{A} \to z$

(C) Then every continuous real valued function of X can be uniquely extended to real valued function of β (X)

(D) Then every bounded real valued function of X can be uniquely extended to real valued function of β (X)

202. A collection of a subsets of topological space X is locally discrete—

(A) If each point X has neighbourhood that intersects at most one element of a

(B) If each point X has neighbourhood that interesects at all element of a

(C) If each point of X has neighbourhood that intersects no one element of a

(D) None of these

203. A collection B is countably locally discrete (σ – locally discrete) if—

(A) It is equal to a countable intersection of locally discrete collections

(B) It is equal to a countable union of locally discrete collections

(C) It is equal to uncountable union of locally discrete collections

(D) None of these

204. Let X be a metrizable space. If a is open covering of X, then there is a collection D of subsets of X such that—

(A) D is an open covering of X

(B) D is a refinment of a

(C) D is countably locally finite

(D) All the above

205. Let X be metrizable space. Then X has a basis that is—

(A) Countable locally finite

(B) Countable locally infinite

(C) Uncountable locally finite

(D) Uncountable locally infinite

206. Bing metrization theorem states—

(A) A space X is metrizable iff it is regular and has a basis that is countably locally discrete

(B) A space X is metrizable iff it is non-regular and has a basis that is not countably locally discrete

(C) Every para compact space X is normal

(D) Every closed subspace of a para compact space is para compact

207. X is locally compact if—

(A) It topological space X is locally compact at each $x \in X$

(B) It topological space X is locally compact at some $x \in X$

(C) It topological space X is locally compact at none $x \in X$

(D) None of these

208. The path components of topological space X are path connected disjoint subsets of X whose—

(A) Intersection is X, such that each path connected subset of X intersects only one of them

(B) Union is X, such that each path connected of X intersects all of them

(C) Union is X, such that each path connected subset of X intersects only one of them

(D) None of these

209. Let X be a locally compact Hausdarff space which is not compact. Let Y be the one point compactification of X. Then—

(A) Y is compact hausdorff space

(B) X is subspace of Y

(C) The set Y – X consists of a single point

(D) All of these

210. Let X be a hausdorff space. Then X is locally compact at x, iff—

(A) For every neighbourhood U of x, there is a neighbourhood V of x, such that $\overline{V}$ is compact and $\overline{V} \subset \overline{U}$*

(B) For some neighbourhood U of x, there is a neighbourhood V of x, such that $\overline{V}$ is compact and $\overline{V} = \overline{U}$

(C) For every neighbourhood U of x, there is a neighbourhood V of x, such that V is compact and $V \subset U$

(D) None of these

211. Let X be a locally compact Hausdorff space and Y be a subspace of X. Y is locally compact—

(A) If Y is closed in X

(B) If Y is open in X

(C) Both (A) and (B)

(D) (A) or (B)

212. A topological space X have a countable basis at x—

(A) If there is a countable collection B of neighbourhood of x, such that each neighbourhood of x contains atleast one of the elements of B

(B) If there is a collections B of neighbourhood of x, such that some neighbourhood of x contains all the elements of B

(C) If there is a countable collections B of neighbourhood of x, such that each neighbourhood of x contains all of the elements of B

(D) None of these

213. A topological space has a countable basis at each of its points is called .... countability axiom—

(A) First (B) Second

(C) Third (D) Fourth

214. A topological space X satisfies ... countability axiom if X has a countable basis for its topology—

(A) First (B) Second

(C) Third (D) Fourth

215. A subset Y of topological space X is dense in X, if—

(A) $\overline{Y} = X$ (B) $Y = X$

(C) $\overline{Y} \not\subset X$ (D) None of these

216. Separable space—

(A) A topological space having a countable dense subset

(B) A topological space having a countable non dense subset

(C) A topological space having a uncountable dense subset

(D) None of these

217. If topological space has a countable basis. Then—

(A) Every open covering of X contains a countable subcollection covering X

(B) There exist a countable subset of X which is dense in X

(C) Both (A) and (B)

(D) Either (A) or (B)

218. One of the statement is true—

(A) A subspace of a topological space having acountable dense subset need not have a countable dense subset

(B) A subspace of a topological space having a countable dense subset have a countable dense subset

(C) Both (A) and (B)

(D) None of these

219. X is locally compact Hausdorff iff—

(A) A space X is homeomorphic to an open subset of a compact Hausdorff space

(B) A space X is homeomorphic to a closed subset of a compact Hausdarff space

(C) A space X is homeomorphic to a null subset of a compact Hausdorff space

(D) None of these

220. A subset C of a topological space X is saturated (with respect to the surjective map $P : X \to Y$), if—

(A) C contains every set $P^{-1}\{Y\}$ that it intersects

(B) C contains some set $P^{-1}\{Y\}$ that it intersects

(C) C contains every set $P^{-1}\{Y\}$ that it does not intersects

(D) None of these

221. Let X and Y be topological spaces, $P : X \to Y$ be a surjective map. The map P is said to be a quotient map—

(A) Provided a subset U⊂Y is closed in Y iff P^{-1} (U) is open in X
(B) Provided a subset U⊂Y is open in Y iff P^{-1} (U) is open in X
(C) Provided a subset U⊂Y is open in Y iff P^{-1} (U) is open in X
(D) None of these

222. Following statement is true—
(A) The product of two quotient map need not be a quotient map
(B) The product of two quotient map is a quotient map
(C) Both (A) and (B)
(D) None of these

223. Let X be a topological space and $x, y \in$ X. The path in X from x to y is—
(A) A continuous map $f : [a, b] \to X$ of some open interval $[a, b]$ in the real line into X, such that $f(a) = x$ and $f(b) = y$
(B) A discontinuous map $f : [a, b] \to X$ of some closed interval $[a, b]$ in there of line into X, such that $f(a) = f(b)$
(C) A continuous map $f : [a, b] \to X$ of some closed interval $[a, b]$ in the real line into X, such that $f(a) = x$ and $f(b) = y$
(D) A continuous map $f : [a, b] \to X$ of some open interval $[a, b]$ in the real line into X, such that $f(a) = x$ and $f(b) = y$

224. A topological space X is called path connected if—
(A) Each pair of points of X can be joined by a path in X
(B) Some pair of points of X can be joined by a path in X
(C) Some pair of points of X cannot be joined by a path in X
(D) None of these

225. Following is true—
(A) If $P : X \to Y$ is a quotient map and if z is a locally compact Hausdorff space, then the map $\pi = P \times i_2 : X \times Z \to Y \times Z$ is a quotient map
(B) Let $P : A \to B$ and $q : C \to D$ be quotient maps. If B and C are locally compact Hausdorff spaces, then $P \times q : A \times C \to B \times D$ is a quotient map
(C) Both (A) and (B)
(D) None of these

226. A space X is metrizable iff it is para compact and locally metrizable—
(A) Locally metrizable theorem
(B) Smirnov metrizable theorem
(C) Special van kampen theorem
(D) None of these

227. Let X = U∪V, U and V are open in X and U∩V is path connected. Let $x_0 \in$ U∩V. If both inclusions $i : (U, x_0) \to (X, x_0)$ and $j : (V, x_0) \to (X, x_0)$ induce zero homeomorphisms of fundamental groups, then π_1 $(X, x_0) = 0$—
(A) Locally metrizable theorem
(B) Smirnov metrization theorem
(C) Special van kampen theorem
(D) None of these

228. If f and f^1 are continuous map of the space X into space Y, then f is … to f^1 $(f = f^1)$ if there is one continuous map $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f^1(x)$ for each $x \in$ X—
(A) Homotopic (B) Path homotopic
(C) Homomorphic (D) None of these

Answers

1. (C)	2. (B)	3. (A)	4. (A)	5. (A)
6. (B)	7. (A)	8. (A)	9. (B)	10. (D)
11. (B)	12. (A)	13. (A)	14. (A)	15. (A)
16. (A)	17. (A)	18. (A)	19. (A)	20. (B)
21. (B)	22. (A)	23. (B)	24. (A)	25. (A)
26. (B)	27. (A)	28. (A)	29. (A)	30. (B)
31. (B)	32. (B)	33. (A)	34. (C)	35. (C)
36. (A)	37. (A)	38. (B)	39. (A)	40. (B)
41. (A)	42. (A)	43. (A)	44. (A)	45. (C)
46. (C)	47. (A)	48. (A)	49. (A)	50. (B)
51. (A)	52. (A)	53. (A)	54. (A)	55. (A)
56. (A)	57. (C)	58. (A)	59. (A)	60. (A)
61. (A)	62. (A)	63. (A)	64. (A)	65. (B)
66. (A)	67. (C)	68. (A)	69. (A)	70. (A)
71. (A)	72. (C)	73. (A)	74. (A)	75. (A)
76. (B)	77. (A)	78. (B)	79. (A)	80. (B)
81. (B)	82. (A)	83. (A)	84. (B)	85. (B)
86. (B)	87. (A)	88. (A)	89. (B)	90. (A)

91. (C)	92. (A)	93. (A)	94. (C	95. (A)
96. (A)	97. (B)	98. (A)	99. (B)	100. (A)
101. (A)	102. (A)	103. (A)	104. (A)	105. (A)
106. (C)	107. (A)	108. (A)	109. (A	110. (B)
111. (B)	112. (C)	113. (A)	114. (B)	115. (B)
116. (B)	117. (C)	118. (A)	119. (A)	120. (B)
121. (A)	122. (A)	123. (C)	124. (A)	125. (A)
126. (A)	127. (A)	128. (A)	129. (A)	130. (C)
131. (B)	132. (C)	133. (B)	134. (A)	135. (A)
136. (A)	137. (A)	138. (A)	139. (B)	140. (A)
141. (A)	142. (A)	143. (A)	144. (A)	145. (B)
146. (A)	147. (B)	148. (C)	149. (B)	150. (C)
151. (A)	152. (C)	153. (B)	154. (B)	155. (D)
156. (A)	157. (B)	158. (A)	159. (C)	160. (B)
161. (A)	162. (A)	163. (C)	164. (D)	165. (A)
166. (B)	167. (B)	168. (C)	169. (B)	170. (C)
171. (C)	172. (D)	173. (D)	174. (B)	175. (A)
176. (C)	177. (B)	178. (C)	179. (D)	180. (D)
181. (D)	182. (A)	183. (C)	184. (C)	185. (B)
186. (A)	187. (C)	188. (B)	189. (A)	190. (B)
191. (A)	192. (A)	193. (A)	194. (D)	195. (A)
196. (A)	197. (A)	198. (C)	199. (A)	200. (C)
201. (A)	202. (A)	203. (B)	204. (D)	205. (A)
206. (A)	207. (A)	208. (C)	209. (D)	210. (A)
211. (D)	212. (A)	213. (A)	214. (B)	215. (A)
216. (A)	217. (C)	218. (A)	219. (A)	220. (A)
221. (C)	222. (A)	223. (C)	224. (A)	225. (C)
226. (B)	227. (C)	228. (A)		

●●

Probability & Statistics

Probability is a concept of Mathematics which measures the degree of certainty or uncertainty of the occurence of events. If any event can happen in m ways and fails in n ways, and each of the $(m + n)$ ways are equally likely to occur, then probability of happening of the events is defined as the ratio, $\frac{m}{m+n}$ and that of its failing as $\frac{n}{m+n}$. If the probability of the happening is denoted by P and not happening by q, then $P + q = 1$, If the event is certain to happen, its probability is unity. If the happening is impossible, then the probability is zero.

Definition : If there are n exhaustive, mutually exclusive and equally likely cases and m of them are favourable to the happening of an event E, then

$$P(E) = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}} = \frac{m}{n}$$

Addition Law of Probability or Theroem of Total Probability

If the probability of an event A happening a result of a trial is P(A) and the probability of a mutually exclusive event B happening is P(B). then the probability of either of the events happening as a result of the trial is

$$P(A + B) \text{ or } P(A \cup B) = P(A) + P(B)$$

Thus for a number of mutually exclusive events $A_1, A_2 \ldots\ldots A_n$ we have

$$\begin{aligned} P(A_1 + A_2 + \ldots\ldots A_n) &= P(A_1 \cup A_2 \cup A_3 \ldots\ldots \cup A_n) \\ &= P(A_1) + P(A_2) + \ldots\ldots + P(A_n) \end{aligned}$$

Conditional Probability—The probability of the happening of an event A when another event B is known to have already happened is called 'Conditional Probability' and is denoted by P(A/B).

Mutually Independent Events—An event A is said to be independent of an event B, if

$$P(A/B) = P(A)$$

i.e., if the probability of happening of A is independent of the happening of B.

Multiplicative Law of Probability—The probability of simultaneous occurence of two events is equal to the probability of the events multiplied by conditional probability of the other *i.e.*, for two events A and B,

$$P(A \cap B) = P(A) \times P(B/A)$$

Where P(B/A) represents the conditonal probability of occurence of B when the event A has already happened.

Generally if $A_1, A_2 \ldots\ldots A_n$ are n independent events, then

$$\begin{aligned} P(A_1, A_2, \ldots\ldots A_n) \\ &= P(A_1)\, P(A_2) \ldots\ldots P(A_n) \end{aligned}$$

Baye's Theorem—

An event A corresponds to a number of exhaustive events $B_1, B_2 \ldots\ldots B_n$. If $P(B_i)$ and $P(A/B_i)$ are given, then

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i)\, P(A/B_i)}{\Sigma P(B_i)\, P(A/B_i)}$$

Proof : By the multiplication law of probability

$$P(AB_i) = P(A)\, P(B_i/A) = P(B_i)\, P(A/B_i)$$

$$\therefore\ P\left(\frac{B_i}{A}\right) = \frac{P(B_i)\, P(A/B_i)}{P(A)}$$

Since the event A corresponds to $B_1, B_2 \ldots\ldots B_n$, we have by the addition law of probability

$$\begin{aligned} P(A) &= P(AB_1) + P(AB_2) + P(AB_3) + \ldots\ldots + P(AB_n) \\ &= \Sigma P(AB_i) \\ &= \Sigma P(B_i)\, P(A/B_i) \end{aligned}$$

$$P\left(\frac{B_i}{A}\right) = \frac{P(B_i)\, P(A/B_i)}{\Sigma P(B_i)\, P(A/B_i)}$$

which is know as the theorem of inverse probability.

Example : There are three bags : First containing 1 white, 2 red, 3 green balls. Second 2 white, 3 red and 1 green balls and third 3 white, 1 red and 2 green balls. Two balls are drawn from a bag chosen at random. These are found to be one white and another red. What is the probability that the balls are come from the second bag ?

Solution : Let B_1, B_2, B_3 pertain to the 1st, 2nd and 3rd bag chosen and A : The two balls are red and white

$$P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$$

$P(A/B_1)$ = P (a white and a red balls are drawn from frist bag)

$$P(A/B_2) = \frac{2}{5}$$

and $P(A/B_3) = \frac{1}{5}$

By Baye's Theorem,

$$P\left(\frac{B_2}{A}\right) = \frac{P(B_2)\,P(A/B_2)}{P(B_1)\,P(A/B_1) + P(B_2)\,P(A/B_2) + P(B_3)\,P(A/B_3)}$$

$$= \frac{6}{11}$$

Corr : If P_1, P_2 be the probabilities of happening of the two events, then—

(i) the probability that the first event happens and second fails is $P_1(1 - P_2)$

(ii) the probability that atleast one of the events happens is $1 - (1 - p_1)(1 - p_2)$

(iii) the probability that both the events happens is P_1P_2.

Random Variable—If a real variable X be associated with the outcome of random experiment, then since the values variable for instance, if a random experiment E cosists of tossing a pair of dice, the sum X of the two number which turn up have the value 2, 3, 4, …… 12 depending on chance.

If a ranodm variable takes a fintie set of values, it is called a **discrete variate**. On the other hand, if it assumes infinite number of uncountable values, it is called a continuous variate.

Discrete Probability Distribution—Suppose a discrete variate X is the outcome of some experiment. If the probability that X takes the values x_i is P_1, then

$$P(X = x_i) = P_i, \quad \text{for } i = 1, 2, \ldots\ldots,$$

and $$\sum_{i=0}^{n} P_i = 1$$

Distribution Function—The distribution $F(x)$ of the discrete variate X is defined by

$$F(x) = P(X = x)$$
$$= \sum_{i=1}^{n} P(x_i),$$

where x is any integer.

Example : The probability density function of a variate X is—

X :	0	1	2	3	4	5	6
P(X) :	K	3K	5K	7K	9K	11K	13K

(i) Find $P(X < 4)$, $P(X \geq 5)$, $P(3 < X \leq 6)$.

(ii) What will be the minimum value of K so that $P(X \leq 2) > 3$

Solution : (i) Since

$$\sum_{i=0}^{6} P(x_i) = 1,$$

therefore

$$K + 3K + 5K + 7K + 9K + 11K + 13K = 1$$

$$\Rightarrow \quad K = \frac{1}{49}$$

$$\therefore \quad P(X < 4) = K + 3K + 5K + 7K = 16K$$
$$= \frac{16}{49}$$

$$P(X \geq 5) = 11K + 13K = 24K$$
$$= \frac{24}{49}$$

$$P(3 < X \leq 6) = 9K + 11K + 13K = 33K$$
$$= \frac{33}{49}$$

(ii) $P(X \leq 2) = K + 3K + 5K = 9K > 3$

$$\therefore \quad K > \frac{1}{3}$$

Thus, min value of

$$K = \frac{1}{3}.$$

Continuous Probability Function—When a variate X takes every value in an inerval, it gives rise to continuous distribution defined by the variate like heights or weights are continuous distributions.

The probability of a continuous variate X is defined by a function $f(x)$ such that the probability of the variate x failing in the small interval $x - \frac{1}{2}$ dx to $x + \frac{1}{2} dx$ is $f(x)\, dx$.

Symbolically it can be expressed as

$$P\left[x - \frac{1}{2} dx \le x \le x + \frac{1}{2} dx\right] = f(x)\, dx$$

Then $F(x)$ is called the probability density function and continuous curve,

$$y = f(x) \text{ is}$$

the density function,

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

Distribution Function—

If $\quad F(x) = P(X \le x)$

$$= \int_{-\infty}^{x} f(x)\, dx,$$

then $F(x)$ is defined as the cummulative distribution function or simply the distribution function of the continuous variate X. It is the probability that the value of the variate X will be less than x.

Properties of Distribution Function $F(x)$:

(i) $F'(x) = F(x) \ge 0$ non decreasing function

(ii) $F(\infty)$

(iii) $F(\infty) = 1$

(iv) $P(a \le x \le b) = \int_a^b f(x)\, dx$

$$= \int_{-\infty}^{b} f(x)\, dx - \int_{-\infty}^{a} f(x)\, dx$$

Example : If the function defined as follows a density function ?

(i) $\quad f(x) = e^{-x} \quad x \ge 0$

$\quad\quad = 0 \quad x < 0$

(ii) If so, determine the probability that the variate having this density will fall in the interval (1, 2).

(iii) Also find the cummulative distribution function F(2).

Solution : (i) Here $f(x) \ge 0$ for all x in (1, 2) and

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{\infty} f(x)\, dx = 0 + \int_0^{\infty} e^{-x}\, dx$$

$$= -e^{-x} \Big]_0^{\infty} = 1$$

Hence the function satisfy the requirements for a density function.

(ii) Required Probability

$$= P(1 \le x \le 2)$$

$$= \int_1^2 e^{-x}\, dx = e^{-1} - e^{-2}$$

$$= 0{\cdot}368 - 0{\cdot}135$$

$$= 0{\cdot}233$$

(iii) Cummulative Probability function,

$$F(2) = \int_{-\infty}^{2} f(x)\, dx$$

$$= \int_{-\infty}^{0} 0 \cdot dx + \int_0^2 e^{-x}\, dx$$

$$= 1 - e^{-2}$$

$$= 0{\cdot}865$$

Binomial Distribution

It is concerned with trials of repetitive nature in which only the occurence or non occurence, success or failure, acceptance or rejection, yes or no of a particular event is of interest.

If we perform a series of independent trials such that for each trial, p is the probability of a success and q that of failure, the probability of r success in series of n trials is given by,

$${}^nC_r \cdot p^r q^{n-r} \text{ and } \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r} = 1$$

Example : The probability that a pen manufactured by a company, will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, find the probability that—

(a) exactly two will be defective.

(b) atleast two will be defective.

(c) none will be defective.

Solution : Probability of defective pen

$$= \frac{1}{10} = 0{\cdot}1$$

$\therefore$ Probability of a non defective pen

$$= 1 - 0{\cdot}1 = 0{\cdot}9$$

(a) Probability that exactly two will be defective $= {}^{12}C_2\, (0{\cdot}1)^2\, (0{\cdot}9)^{11}$

$$= 0{\cdot}2301$$

(b) Probability that at least two will be defective $= 1 -$ [Probability that either no or, one will be defective]

$$= 1 - {}^{12}C_0\,(0{\cdot}9)^{12} - {}^{12}C_1\,(0{\cdot}1)\,(0{\cdot}9)^{11}$$
$$= 0{\cdot}3412$$

(c) Probability that none will be defective

$$= {}^{12}C_{12}\,(0{\cdot}9)^{12} = 0{\cdot}233$$

Mean and Variance of the Binomial Distribution

Mean, $u = \sum\limits_{r=1}^{n} r\,p(r) = np;$

Variance, $\sigma^2 = \sum\limits_{r=1}^{n} [r^2\,p(r) - u^2] = npq$

Poisson Distribution

The Poisson distribution is a limiting case of Binomial distribution. It is derived when of success p is very small. The probability of r success in n trials be

$$p(X = r) = \frac{m^r\,e^{-m}}{r\,!}$$

Where, $m = np \rightarrow$ mean of binomial distribution.

Example : If the probability of a bad reaction from a certain injection is 0·001. Determine the chance that out of 2000 individuals more than two will get a bad reactions.

Solution : $m = np$

$$= 2000\,(0{\cdot}001)$$
$$= 2$$

Probability that more than 2 will get a bad reaction

= 1 – [Probability that no one gets a bad reacton + Probability that one gets a bad reaction + Probability that two gets bad reaction]

$$= 1 - \left[e^{-m} + \frac{m^1 e^{-m}}{1\,!} + \frac{m^2 e^{-m}}{2\,!}\right]$$
$$= 1 - \left[\frac{1}{e^2} + \frac{2}{e^2} + \frac{2}{e^2}\right] = 1 - \frac{5}{e^2}$$
$$= 0{\cdot}32$$

Comparison of Frequency Distribution

1. **Measures of Central Tendency—**

(i) **Mean**—If $x_1, x_2 \ldots\ldots x_n$ be n items, then their arithmetic mean denoted by $\bar{x}$ is given by

$$\bar{x} = \frac{x_1 + x_2 + \ldots\ldots x_n}{n}$$
$$= \frac{1}{n}\sum_{i=1}^{n} x_i$$

In case the data is arranged in a frequency distribution in which $x_1, x_2, x_3 \ldots\ldots x_n$ are the values of the variable and $f_1, f_2, f_3 \ldots\ldots f_n$ are the corresponding frequencies, then

Arithmetic mean,

$$\bar{x} = \frac{f_1x_1 + f_2x_2 + \ldots\ldots + f_nx_n}{f_1 + f_2 + \ldots\ldots + f_n}$$
$$= \frac{\Sigma f_ix_i}{f_1 + f_2 + \ldots\ldots f_n}$$
$$= \frac{\Sigma f_ix_i}{\Sigma f_i}$$

Calculation of Mean Short-Cut Methods

When the magnitude of the frequencies and the variable are large, their products turn out to be large, then to reduce labour following method is useful.

Methods of Assumed Mean—Take arbitrary number n, called the assumed mean or the provsional mean, somewhere in the middle of the highest and the lowest values of the variable in the given data,

Shifting the origin to the point, the formula

$$\bar{x} = \frac{\Sigma f_ix_i}{\Sigma f_i}$$

can be written as

$$\bar{x} - a = \frac{\Sigma f_i(x_i - a)}{\Sigma f_i}$$
$$= a + \frac{\Sigma f_id_i}{\Sigma f_i}$$

Where $d_i = a$ is called the diversion of variate x from the assume mean A.

(ii) **Median**—If the items of a series are arranged in ascending or descending order of magnitude, then value of middle item is called the **MEDIAN**. Thus the median has the same number of items above it as below it.

If n is the number of items, then the median is the $\left(\frac{n+1}{2}\right)^{\text{th}}$ item in case n is odd.

If n is even, then the median is average of the two middle items.

In case of grouped data, Median (M_d)

$$= l + \frac{\left(\frac{n}{2} - C\right)}{f} \times i$$

where l = lower limit of the class in which the median lies

f = frequency of this class

C = cummulative frequency of the class preceding the median class

i = class interval and n is the total frequency

(iii) **Mode**—The mode is defined as that value of the variable which occurs more frequently *i.e.*, the value of the maximum frequency.

For a grouped distribution, Mode

$$= l + \frac{f_m + f_1}{(f_m - f_1)\,(f_m - f_2)} \times i$$

where l = lower limit of the modal class (*i.e.*, class having the maximum frequency)

f_m = maximum frequency

f_1 and f_2 = frequencies of the classes preceeding and following the modal class.

In symmetrical distribution, the mean, median and mode coincide but in other distribution they are connected as

Mean – Mode = 3 (Mean – Median)

Measures of Dispersion

In a frequency distribution it is equally important to know the variates are cluttered around or scattered away from the point of central tendency. Such variation is called the DISPERSION. Some commonly used measures of dispersion—

(i) Range

(ii) Quartile deviation

(iii) Average or mean deviation

(iv) Standard deviation

(i) **Range**—It is difference between the greatest and least values of the variable *i.e.*, Range = Greatest value – Least value

(ii) **Quartile Deviation**—Quartile are the sizes of the items which divide the series into four equal parts. The second quartile is the MEDIAN. The first and third quartiles also known as LOWER and UPPER QUARTILES a mathematically represented by the formulas

$$Q_1 = 1 + \frac{\left(\frac{n}{4} - C\right)i}{f}$$

$$Q_2 = \frac{l + \left(\frac{3n}{4} - C\right)i}{f}$$

where symbols have their usual meaning.

The difference between upper and lower quantities *i.e.*, $Q_3 - Q_4$ is called the QUARTILE RANGE and heat of this range is known as QUARTILE DEVIATION. It is a better measure than range as it accounting 50% of the data.

(iii) **Average Deviation or Mean Deviation**—It is defined as the arithmetic average of the absolute values of the deviations *i.e.*, all deviation tape pointer, from the mean mode as median.

$$\text{Mathematically A·D}_m = \frac{\Sigma \mid f(x - \bar{x}) \mid}{\Sigma f}$$

Where $A.D_m$ = Average deviation from the mean $\bar{x}$.

(iv) **Standard Deviation**—It is the square root of the mean of square of the differences of the various from the mean Mathematically,

$$\sigma = \sqrt{\left[\frac{\Sigma \mid f_i\,(x_i - \bar{x})^2}{N}\right]}$$

Where, N = total frequency = Σf_i

If the deviations are measured from any other value instead of mean, it is called the root mean square deviation and its square is known as the variance.

Co-efficient of Variation

The ratio of the standard deviation to the mean, *i.e.*, $(\sigma/\bar{x})$ is known as the co-efficient of variation. This is a ratio having no dimension and is used for comparing. The variations between the two groups with different means.

Relation Between Measures of Dispersion

(i) Quartile Deviation = $\frac{2}{3}$ (Standard deviation)

(ii) Mean Deviation—$\frac{1}{5}$ (Standard deviation)

Correlation—In a divariate distribution, if the classes in one variable are associated by

classes in the other, then variables are called **CORRELATED**. Correlation is called positive otherwise negative. If the ratio of two variable deviation is constant, then correlation is said to be perfect.

Co-efficient of Correlation—The numerical measures of correlation is called the co-efficient of correlation and is defined as—

$$r = \frac{\Sigma XY}{n\sigma_x\sigma_y}$$

$$= \frac{\Sigma xy}{\sqrt{\Sigma x^2 \Sigma y^2}}$$

$$\left[\because \ \sigma_x^2 = \frac{\Sigma x^2}{n}; \sigma_y^2 = \frac{\Sigma y^2}{n}\right]$$

where, X = deviation from mean, $\bar{x} = x - \bar{x}$

Y = deviation from mean, $\bar{y} = y = \bar{y}$

σ_x = S.D. of x series

σ_y = S.D. of y series

n = number of values of the two variables

Methods of Calculation

(a) **Direct Method**—Substituting the value of σ_x and σ_y in the above formula, we get

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2 \Sigma Y^2}}$$

$$\text{or, } \frac{n\Sigma xy - \Sigma x \Sigma y}{\sqrt{[n\Sigma x^2 - (\Sigma x)^2] \times [n\Sigma y^2 - \Sigma x^2\}]}}$$

REGRESSION—Regression is the estimation or production of unknown values of one variable from known value of another variable. It measrues the nature and extent correlation.

Line of Regression—If the reactor diagram indicates some relationship between variables X and Y, then the dots of the reactor diagram will be concentrated round a curve. This curve is called the curve of regression. When the curve is straight line, then it is called line of regression.

Equation of the line of regressin is,

$$y - \bar{y} = \frac{r\sigma_y}{\sigma_x}(x - \bar{x})$$

and, equation of the line of regression of x on y is

$$x - \bar{x} = \frac{r\sigma_x}{\sigma_y}(y - \bar{x})$$

Regression co-efficent

Regression co-efficient is defined as the slope of lines of regression, *i.e.,* regression coefficient of line x on $y = r\frac{\sigma_x}{\sigma_y}$

Solved problems

1. X is a Poisson variable and it is found that the probability that $x = 2$ is two thirds of the probability that $x = 1$. Find the probability that $x = 0$ and the probability that $x = 3$. What is the probability that exceeds ?
2. If a random variable has Poisson distribution such that P(1) = P(2), find
   (i) mean of the distribution.
   (ii) P(4)
3. Fit a Poisson distribution to the set of observations :

x:	0	1	2	3	4
f:	122	60	15	2	1

4. If in a lot of 500 solenoids, 25 are defective; find the probability of 0, 1, 2, 3 defective solenoids in random sample of 20 solenoids.
5. A product of 0·5% defective and is packed in cartons of 100. What percentage contains not more than 3 defectives ?
6. If 10 per cent of the rivets produced by a machine are defective, then find the probability that out of 5 rivets chosen at random.
   (i) none will be defective.
   (ii) one will be defective.
   (iii) at least two will be defective.
7. The probability that on entering student will graduate is 0·4. Determine the probability that out of 5 students,
   (i) None, (ii) one, and (iii) at least one will graduate.
8. Frequency of accidents per shift in a factory is as shown in the following table :

Accident per shift	0	1	2	3	4
Frequency	180	92	24	3	1

   Calculate, the mean number of accidents per shifts.
9. The incidence of occupational disease in an industry is such that the workmen have a 10% chance of suffering from it. What is probability that in a group of 7, 5 or more will suffer from it ?

10. A car hire firm has two cars which it hires out day by day. The numbers of demands for a car on each day is distributed as Poisson distribution with mean 1·5. Calculate the proportion of days.

    (i) On which there is no demand

    (ii) On which demand is refused ($e^{-1\cdot5} = 0\cdot2231$).

11. A manufacturers knows that the condensers he makes contain on the average 1% defectives. He packs them in boxes of 100. What is the probability that a box picked at random will contain 3 or more faulty condensers ?

12. A certain screw making machine produces an average of 2 defective screws out of 100, and packs them in boxes of 500. Find the probability that a box contains 15 defective screws.

13. A sortie of 20 aeroplanes is sent on an operational flight. The chances that an aeroplane fails to return is 5%. Find the probability that—

    (i) One plane does not return

    (ii) at most 5 planes do not return.

14. The probability that a bomb dropped from a plane will strike the target is $\frac{1}{5}$. If six bombs are dropped, then find the probability that—

    (i) exactly two will strike the target.

    (ii) at least two will strike the target.

15. If an average 1 vessel in every 10 is wreeked; find the probability that out of 5 vessels expected to arrive, at least 4 will arrive safely.

16. If the probability that a new born child is a male is 0·6, then find the probability that in a family of 5 children, there are exactly 3 boys.

17. If the chance that one of the ten telephone lines is busy at an instant is 0·2, then—

    (i) What is the chance that 5 of the lines are busy ?

    (ii) What is the probability that all the lines are busy ?

18. If $(x) = \begin{cases} \frac{1}{2}(x+1), & 1 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$

    represents the density of a random variabel X, find $\Sigma(x)$ and var. (x).

19. Find the standard deviation for the following discrete distribution—

x :	8	12	16	20	24
$p(x)$:	1/8	1/6	3/8	1/4	1/12

20. A random variable x has the following probability function—

Values of x :	–2	–1	0	1	2	3
$p(x)$:	0·1	k	0·2	$2k$	0·3	k

Find the value of k of and calculate mean and variance.

EXPLANATIONS

1. Since

$$p(2) = \frac{2}{3}p(1)$$

$$\therefore \quad \frac{m^2}{2!}e^{-m} = \frac{2}{3}me^{-m}$$

$$\Rightarrow \quad m = \frac{4}{3}$$

$$p(0) = e^{-m} = e^{-4/3} = 0\cdot2636$$

$$p(3) = \frac{m^3e^{-m}}{3!} = \frac{\left(\frac{4}{3}\right)^3}{6} \times e^{-4/3} = 0\cdot1041$$

$$p(x > 3) = 1 - p(0) - p(1) - p(2) - p(3)$$

$$= 1 - 0\cdot2636 - \frac{4}{3} \times 0\cdot2626 - \frac{4}{3} \times \frac{2}{3} \times 0\cdot2636 - 0\cdot1041$$

$$= 0\cdot0465$$

2. (i) Probability of r success in Poisson distribution $= \frac{m^r}{r!}e^{-m}$

where, m = mean = $n.p.$

$$\frac{m^1}{1!}e^{-m} = \frac{m^2}{2!}e^{-m}$$

$$me^{-m} = \frac{m^2}{2!}e^{-m}$$

$$m = 2$$

(ii)

$$p(4) = \frac{m^r}{r!}e^{-m} = \frac{2^4}{4!}e^{-2} = \frac{16}{24}e^{-2} = \frac{2}{3}e^{-2}$$

3.

$$\text{Mean} = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{60 + 36 + 6 + 4}{200} = 0\cdot5$$

Mean of Poisson distribution,

$$m = 0.5$$

$\therefore$ Theoretical frequency for r success

$$= \frac{Ne^{-m}(m)^r}{r!}$$

$$= \frac{200\, e^{-0.5}(0.5)^r}{r!}$$

where, $r = 0, 1, 2, 3, 4$

Theoretical frquencies are

x:	0	1	2	3	4 ($e^{-s} = 0.61$)
f:	121	61	15	2	0

4. Probability of defective

$$= \frac{25}{500}$$

$$= 0.05$$

Probability of non defective

$$= 1 - 0.05$$

$$= 0.95$$

Since 20 sample are randomly chosen, therefore

Probability 20 sample are round only chosen

Probability of no defective $= {}^{20}C_0 \times (0.95)^{20} = 0.3585$

Probability of 1 defective $= {}^{20}C_1 \times (0.05) \times (0.95)^{19} = 0.3773$

Probability of 2 defective $= {}^{20}C_2 \times (0.05)^2 \times (0.95)^{18} = 0.1887$

Probability of 3 defective $= {}^{20}C_3 \times (0.05)^3 \times (0.95)^{17} = 0.0596$

5. Probability of defective $= \frac{5}{100} = 0.005$

Probability of non-defective $= 1 - 0.005 = 0.995$

Since, there are 100 product, hence required probability for not more than 3 defective

$= {}^{100}C_0 \times (0.995)^{100} + {}^{100}C_1 \times (0.005) \times (0.995)^{99} + {}^{100}C_2 \times (0.005)^2 \times (0.995)^{98} + {}^{100}C_3 \times (0.005)^3 \times (0.995)^{97}$

$= 0.9983$

Required percentage = 99.83

6. Probability of defective

$$= \frac{10}{100} = 0.1$$

Probability of non-defective

$$= 0.9$$

Since 5 rivets chosen at random, therefore

(i) In case of none will be defective

Required probability

$$= {}^5C_0 \times (0.1)^0 \times (0.9)^5$$

$$= 0.59049$$

(ii) Probability of one will be defective

$$= {}^5C_1 \times (0.1)^1 \times (0.9)^4$$

$$= 0.32805$$

(iii) Probability of at least 2 will be defective

$$= 1 - {}^5C_0 \times (0.9)^5 - {}^5C_1 \times (0.1) \times (0.9)^4$$

$$= 0.08146$$

7. Probability of engineering graduate student 0.4

So probability of non graduate engineering student $= 0.6$

Since there are 5 students, hence

(i) Probability of none graduate student

$$= {}^5C_0 \times (0.4)^0 \times (0.6)^5$$

$$= 0.08$$

(ii) Probability of one graduate student

$$= {}^5C_1 \times (0.4)^1 \times (0.6)^4$$

$$= 0.26$$

(iii) Probability of at least one graduate student $= 1 - {}^5C_0 \times (0.4)^0 \times (0.6)^5$

$$= 1 - 0.08 = 0.32$$

8. Mean $= \frac{\Sigma f_i x_i}{\Sigma f_i}$

$$= \frac{0 \times 180 + 1 \times 92 + 2 \times 24 + 3 \times 3 + 4 \times 1}{180 + 92 + 24 + 3 + 1}$$

$$= \frac{153}{300}$$

$$= 0.51$$

9. Probability of suffering diseases

$$= \frac{10}{100} = 0.1$$

Here, $n = 7$

$\therefore$ Mean, $m = np$

$$= 7 \times 0.1$$

$$= 0.7$$

$$p(x \geq 5) = p(5) + p(6) + p(7)$$

$$= \frac{m^5}{5!}e^{-m} + \frac{m^6}{6!}e^{-m} + \frac{m^7}{7!}e^{-m}$$

$$= e^{-0.7}\left[\frac{(0.7)^5}{5!}+\frac{(0.7)^6}{6!}+\frac{(0.7)^7}{7!}\right]$$
$$= 0.0008$$

10. Mean; $m = 1.5.\ e^{-1.5}$
$= 0.2231$

(i) Probability on which there is no demand,
$$p(0) = e^{-m}$$
$$= e^{-1.5}$$
$$= 0.2231$$

(ii) Since firm has 2 cars, therefore probability on which there is no demand
$$= 1 - p(0) - p(1) - p(2)$$
$$= 1 - 1.2231 - 1.5 \times e^{-1.5} - \frac{1}{2}\times 1.5 \times e^{1.5}$$
$$= 0.1913$$

11. Probability of defective codenser,
$$p = \frac{1}{100}$$
$$= 0.01$$
Mean, $m = np$
$$= 100 \times \frac{1}{100}$$
$$= 1$$
$$p(x \geq 3) = 1 - p(0) - p(1) - p(2)$$
$$= 1 - e^{-m} - me^{-m} - \frac{m^2 e^{-m}}{2!}$$
$$= 1 - e^{-1} - e^{-1} - \frac{1}{2}e^{-1}$$
$$= 0.08$$

12. Probability of defective screws
$$= \frac{2}{100} = \frac{1}{50}$$
Since, $n = 500$,
therefore
Mean, $m = np$
$$= 500 \times \frac{1}{50}$$
$$= 10$$
Probability of 15 defective out of 500
$$= \frac{mre^{-m}}{r!}$$
$$= (10)^{15} \times \frac{e^{-10}}{15!}$$
$$= 0.347 \qquad [\because\ n = 100]$$

13. Probability that an aeroplane fails to returen
$$= \frac{5}{100} = \frac{1}{20}$$
Probability that an aeroplane returns
$$= 1 - \frac{1}{20} = \frac{19}{20}$$
Since 20 planes are sended, hence

(i) Probability that one plane does not return
$$= {}^{20}C_1\left(\frac{1}{20}\right)\left(\frac{19}{20}\right)^{19}$$

(ii) Probability that at most 5 planes does not return
$$= {}^{20}C_0\left(\frac{1}{20}\right)^0\left(\frac{19}{20}\right)^{20} + {}^{20}C_1\left(\frac{1}{20}\right)^1\left(\frac{19}{20}\right)^{19} + {}^{20}C_2\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^{18}$$
$$+ {}^{20}C_3\left(\frac{1}{20}\right)^3\left(\frac{19}{20}\right)^{17} + {}^{20}C_4\left(\frac{1}{20}\right)^4\left(\frac{19}{20}\right)^{16}$$
$$+ {}^{20}C_5\left(\frac{1}{20}\right)^5\left(\frac{19}{20}\right)^{15}$$
$$= \sum_{r=1}^{5} {}^{20}C_r\left(\frac{1}{20}\right)^r\left(\frac{19}{20}\right)^{20-r}$$

14. Probability of non striking the target by bomb $= \frac{4}{5} = 0.8$

Probability of striking the target by bomb
$$= \frac{1}{5} = 0.2$$
Since 6 bombs are dropped, hence

(i) Required probability when at least 2 will strike the target
$$= 1 - {}^6C_0\,(0.8)^6 - {}^6C_1\,(0.2)^1\,(0.8)^5$$
$$= 0.345$$

(ii) Probability of exactly 2 bombs striking the target
$$= {}^6C_2 \times (0.2)^2 \times (0.8)^4$$
$$= 0.245$$

15. Probability of one vessel wreeked out of $10 = \frac{1}{10}$

Probability of no vessel wreeked out of $10 = \frac{9}{10}$

Required probability when at least 4 arrive safely out of 5
$$= {}^5C_4\left(\frac{9}{10}\right)^4\left(\frac{1}{10}\right)^1 + {}^5C_5\left(\frac{9}{10}\right)^5$$

$= \frac{5 \times 9^4}{10^5} + \frac{9^5}{10^5}$

$= \frac{45927}{50000}$

16. Probability that new born child is male

$= 0{\cdot}6$

Probability that new born child is female

$= 0{\cdot}4$

$\therefore$ Probability of 3 boys out of 5

$= {}^5C_3 \times (0{\cdot}6)^3 \times (0{\cdot}4)^2$

$= 0{\cdot}3456$

17. Probability of one telephone busy out of ten

$= 0{\cdot}2$

(i) Required probability

$= {}^{10}C_5 \times (0{\cdot}2)^5 \times (0{\cdot}8)^5$

$= 0{\cdot}026$

(ii) Probability that all lines are busy

$= {}^{10}C_{10} \times (0{\cdot}2)^{10} \times (0{\cdot}8)^0$

$= 1{\cdot}004 \times 10^{-7}$

18. $\Sigma(x) = \int_{-1}^{+1} x f(x)\, dx$

$= \int_{-1}^{+1} x \times \frac{1}{2}(x+1)\, dx$

$= \frac{1}{2}\int_{-1}^{+1} (x^2 + x)\, dx$

$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{+1}$

$= \frac{1}{2}\left[\frac{1}{3} + \frac{1}{2} + \frac{1}{3} - \frac{1}{2}\right]$

$= \frac{1}{3}$

$\text{Var}(x) = \int_{-1}^{+1} (x - \upsilon)^2 f(x)\, dx$

$= \int_{-1}^{+1} \left(x - \frac{1}{3}\right)^2 \frac{1}{2}(x+1)\, dx$

$\frac{1}{18}\int_{-1}^{+1} (9x^2 - 6x + 1)(x+1)\, dx$

$= \frac{1}{18}\int_{-1}^{+1} (9x^3 + 3x^2 - 5x + 1)\, dx$

$= \frac{1}{18}\left[\frac{9x^4}{4} + x^3 - \frac{5x^2}{2} + x\right]_{-1}^{+1}$

$= \frac{1}{18}\left[\frac{9}{4} + 1 - \frac{5}{2} + 1 - \frac{9}{4} + 1 + \frac{5}{2} + 1\right]$

$= \frac{4}{18}$

$= \frac{2}{9}$

19. Mean, $\upsilon = \Sigma x \cdot p(x)$

$= 8 \times \frac{1}{8} + 12 \times \frac{1}{6} + 16 \times \frac{3}{8} + 20 \times \frac{1}{4} + 24 \times \frac{1}{12}$

$= 16$

$\therefore \quad \sigma^2 = \Sigma(xu)^2, p(x)$

$= (8-16)^2 \times \frac{1}{8} + (12-16)^2 \times \frac{1}{6} +$

$(16-10)^2 \times \frac{3}{8} + (20-16)^2 \times \frac{1}{4} + (24-16)^2 \times \frac{1}{12}$

$= 8 + \frac{16}{6} + 4 + \frac{32}{6}$

$= 20$

$\therefore$ Standard deviation

$= \sigma$

$= (20)^{1/2}$

$= 2\sqrt{5}$

20. For the value of k,

$\Sigma p(x) = 1$

$\therefore 0{\cdot}1 + k + 0{\cdot}2 + 2k + 0{\cdot}3 + k$

$= 1$

$\Rightarrow \quad 4k = 0{\cdot}4$

$\Rightarrow \quad k = 0{\cdot}1$

$\therefore$ Mean $= \Sigma p(x)$

$= -2 \times 0{\cdot}1 - 1 \times 0{\cdot}1 + 0 \times 0{\cdot}2 + 1 \times 2 \times 0{\cdot}1 + 2 \times 0{\cdot}3 + 3 \times 0{\cdot}1$ [taken $k = 0{\cdot}1$]

$= -0{\cdot}2 - 0{\cdot}1 + 0 + 0{\cdot}2 + 0{\cdot}6 + 0{\cdot}3$

$= 0{\cdot}8$

$\sigma^2 = \Sigma(x-u)^2 p(x)$

$= (-2-0{\cdot}8)^2 \times 0{\cdot}1 + (-1-0{\cdot}8)^2 \times 0{\cdot}1 + (0-0{\cdot}8)^2 \times 0{\cdot}2 + (1-0{\cdot}8)^2 \times 0{\cdot}2 + (2-0{\cdot}8)^2 \times 0{\cdot}3 + (3-0{\cdot}8)^2 \times 0{\cdot}1$

$= 0{\cdot}1[(-2{\cdot}8)^2 + (-1{\cdot}8)^2 + (-0{\cdot}8)^2 \times 2 + (0{\cdot}2)^2 \times 2 + (1{\cdot}2)^2 \times 3 + (2{\cdot}8)^2]$

$= 0{\cdot}1[7{\cdot}84 + 3{\cdot}24 + 1{\cdot}28 + 0{\cdot}8 + 4{\cdot}31 + 7{\cdot}84]$

$= 2{\cdot}23$

Solved Problems (Statistics)

1. Find the mode from the following data—

Age	Frequency
0 – 6	6
6 – 12	11
12 – 18	25
18 – 24	35
24 – 30	18
30 – 36	12
36 – 42	6

2. Find the standard deviation of the following two series which one shows greater variaton :

Series A	Series B
192	83
288	87
236	93
229	109
184	124
260	126
348	126
291	101
330	102
243	108

3. Calculate the median of following distribution giving marks obtained by students in a certain examination :

Marks	No. of Students
40 – 50	2
30 – 40	7
20 – 30	12
10 – 20	9
0 – 10	1

4. The following data related to sizes of shoes at a store during a given week. Find the median size of the shoes. Also calculate the quartile; 7th dectile and 8th percentile.

Size	Frequency
2	2
5·5	5
6	15
6·5	30
7	60
7·5	40
8	23
8·5	11
9	4
9·5	1

5. Calculate the median and quartiles of the following items :

14, 8, 6, 12, 15, 9, 18.

6. Psychological tests of intelligence and engineering ability were applied to 10 students. Here is a record of ungrouped data showing intelligence ratio (I.R.) and engineering ratio (E.R.), calculate the co-efficient of correlation.

Student	I.R.	E.R.
A	105	101
B	104	103
C	102	100
D	101	98
E	100	95
F	99	96
G	98	104
H	96	92
I	93	97
J	92	94

7. The number examined, the mean weight and S.D. in each group of examination by three medical examiners are given below. Find the mean weight and S.D. of the entire data when grouped together.

Mean Exam	Number Examined	Mean wt. (lbs)	SD (lbs)
A	50	113	6
B	60	120	7
C	90	115	8

8. Find the mean deviation of the following frequency distribution :

Class	Frequency
0 – 6	8
6 – 12	10
12 – 18	12
18 – 24	9
24 – 30	5

9. The daily earning of 60 workers in rupees are given below, calculate the average income per worker.

Rupees, x	Number of workers, f	Product
2	10	20
3	16	48
4	11	44
5	8	40
6	6	36
7	4	28
8	3	24
9	2	18
$\Sigma f = 60$	$\Sigma fx = 35$	258

10. Short cut method : The following figures are giving the heights in cms of 7 kids chosen at random. Calculate the simple arithmetic mean by direct method and shortcut method.

S. No.	Height, x
1	64
2	59
3	67
4	69
5	65
6	70
7	68

11. In a partially destroyed laboratory record; only the lines of regression of y on x and x on y are available as $4x - 5y + 33 = 0$ and $20x - 9y = 107$ respectively. Calculate $\bar{x}$, $\bar{y}$ and the coefficient of correlation between x and y.

12. The two regression equations of the variables x and y are $x = 19{\cdot}13 - 0{\cdot}87y$ and $y = 11{\cdot}64 - 0{\cdot}50x$. Find—

(i) Mean of x's

(ii) Mean of y's

(iii) Correlation co-efficient between x and y

EXPLANATIONS

1.

Age	Frequency	Cummulative Frequency
0 – 6	6	6
6 – 12	11	17
12 – 18	$25 = f_{-1}$	42
18 – 24	$35 = f$	77
24 – 30	$18 = f_1$	95
30 – 36	12	107
36 – 42	6	113

$$\therefore \text{Mode} = l + \frac{f - f_{-1}}{2f - f_{-1} - f_1} \times i$$

$$= 18 + \frac{35 - 25}{70 - 25 - 18} \times 6$$

$$= 20{\cdot}22$$

2.

Series A			Series B		
Size x	Dev. from ass. avg. 260	Sq. of dev. dx^2	Size y	Dev from 105 (dy)	Sq. of dev. dy^2
192	–68	4624	83	–22	484
288	28	784	87	–18	324
236	–24	576	93	–12	144
229	–81	961	109	4	16
184	–76	5776	124	19	361
260	0	0	126	21	441
348	88	7744	126	21	441
291	31	961	101	–4	16
330	70	4900	102	–3	9
243	–17	289	108	3	9
2601		$\Sigma dx^2 = 26615$	1059		2245

For A, Arithmetic mean,

$$\bar{x} = \frac{\Sigma x}{n} = \frac{2601}{10} = 260{\cdot}1$$

Standard deviation,

$$\sigma = \sqrt{\frac{\Sigma dx^2}{n} - (\bar{x} - 0)^2}$$

$$= \sqrt{\frac{26615 - 10(0{\cdot}1)^2}{10}} = \sqrt{2661{\cdot}49}$$

$$= 57{\cdot}6 \text{ (Approx)}$$

Coefficient of variation

$$= \frac{\sigma}{x} \times 100 = \frac{51{\cdot}6}{260{\cdot}1} \times 100 = 19{\cdot}8$$

For B, Arithmetic mean

$$= \frac{1059}{10} = 105{\cdot}9$$

Standard deviation

$$= \sigma = \sqrt{\frac{2245}{10} - (105{\cdot}9 - 105)^2}$$

$$= 14{\cdot}96$$

Co-efficient of variation

$$= \frac{14{\cdot}96}{105{\cdot}9} \times 10 = 14{\cdot}1$$

Since the co-efficient of variation for series A is greater than that of B, hence series A show greater variation.

3.

Marks	Frequency	Cummulative Frequency
40 – 50	2	2
30 – 40	7	9
20 – 30	12	21
10 – 20	9	30
0 – 10	1	31

Median = size of $\left(\frac{n+1}{2}\right)^{\text{th}}$ *i.e.,* 16th item which lies in 20 – 30

Now,

$$\text{Median} = l_1 + \frac{l_2 - l_1}{f}(m - c')$$

Where, c' = total frequency
= cummulative frequency of the median group.

$$\therefore \quad \text{Median} = 20 + \frac{30-20}{12}[16 - (31 - 21)]$$

$$= 20 + \frac{10}{12} \times 6 = 26 \text{ marks.}$$

4. Cummulative

2 7 22 52 112 152 175 186 190 191

$$\text{Median} = \text{Size of } \left(\frac{n+1}{2}\right)^{th}$$

or $\left(\frac{191+1}{2}\right)^{th}$ *i.e.* 96^{th} pair = 7

$$\text{Lower quartile} = \text{size of } \left(\frac{n+1}{4}\right)^{th}$$

or $\left(\frac{191+1}{4}\right)^{th}$ *i.e.,* 48^{th} pair = 6·5

$$\text{Upper quartile} = \text{size of } \left[\frac{3(n+1)}{4}\right]^{th}$$

or 144^{th} pair = 7·5

$$2^{nd} \text{ qurtile} = \text{size of } \left[\frac{2(n+1)}{5}\right]^{th}$$

or $76{\cdot}8^{th}$ pair = 7

$$7^{th} \text{ dectile} = \text{size of } \left[\frac{7(n+1)}{10}\right]^{th}$$

or $134{\cdot}5^{th}$ pair = 8

$$85^{th} \text{ percentile} = \text{size of } \left[\frac{85(n+1)^{th}}{100}\right]$$

or $163{\cdot}2^{th}$ pair = 8

5. Arranging them in the ascending order of magnitude.

S. No.	Size in items
1	6
2	8
3	9
4	12
5	14
6	15
7	18

$$\text{Median} = \text{size of } \left(\frac{n+1}{2}\right)^{th}$$

or, 4^{th} item = 12

$$\text{Lower quartile} = \text{size of } \left(\frac{n+1}{4}\right)^{th}$$

or $\left(\frac{7+1}{2}\right)^{th}$ or 2^{nd} item = 8

$$\text{Upper quartile} = \text{size of } \frac{3}{4}(n+1)^{th}$$

or $\left(\frac{3 \times 8}{4}\right)^{th}$ or 6^{th} item = 15

6.

Student	I. x	R. $x - \bar{x} = X$	E. y	R $y - \bar{y} = Y$	X^2	Y^2	XY
A	105	6	101	3	36	9	18
B	104	5	103	5	25	25	25
C	102	3	100	2	9	4	6
D	101	2	98	0	4	0	0
E	100	1	95	–3	1	9	–3
F	99	0	96	–2	0	4	0
G	98	–1	104	6	1	36	–6
H	96	–3	92	–6	9	36	18
I	93	–6	97	–1	36	1	6
J	92	–7	94	–4	49	16	28
Total	990	0	980	0	170	140	92

Mean of x, $\bar{x} = \dfrac{990}{10}$

$= 99$

and Mean of y, $\bar{y} = \dfrac{980}{10}$

$= 98$

$\Sigma X^2 = 170$

$\Sigma Y^2 = 140$

and $\Sigma XY = 92$

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2 \Sigma Y^2}}$$

$$= \frac{92}{\sqrt{170 \times 140}}$$

$$= 0{\cdot}59$$

7. Here $n_1 = 50,$

$\bar{x}_1 = 113,$

$\sigma_1 = 6$

$n_2 = 60,$

$\bar{x}_2 = 120,$

$\sigma_2 = 7$

$n_3 = 90,$

$\bar{x}_3 = 115,$

$\sigma_3 = 8$

$\therefore$ Mean of entire data,

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1 + n_2 + n_3}$$

$$= \frac{50 \times 113 + 60 \times 120 + 90 \times 115}{50 + 60 + 90}$$

$$= 116$$

If σ is the S.D. of the entire data, then

$$N\sigma^2 = n_1\sigma_1{}^2 + n_2\sigma_2{}^2 + n_3\sigma_3{}^2 + n_1D_1{}^2 + n_2D_2{}^2 + n_3D^2{}_3$$

Where, $N = n_1 + n_2 + n_3$

$= 200$

$D_1 = \bar{x}_1 - \bar{x}$

$= -3,$

$D_2 = \bar{x}_2 - \bar{x}$

$= 4,$

$D_3 = \bar{x}_3 - \bar{x}$

$= -1$

$$\therefore\ 200\,\sigma^2 = 56 \times 36 + 60 \times 49 + 90 \times 64 + 50 \times 9 + 60 \times 16 + 90 \times 1$$

$$\Rightarrow \quad \sigma^2 = \frac{12000}{200}$$

$$= 60$$

$$\Rightarrow \quad \sigma = \sqrt{60}$$

$$= 7{\cdot}7461$$

8.

Class	Mid Value (x)	Frequency (f)	$d = x - a$	fd	$\|x - 14\|$	$f\|x - 14\|$
0 – 6	3	8	–12	–96	11	88
6 – 12	9	10	–6	–60	5	50
12 – 18	15	12	0	0	1	12
18 – 24	21	9	+ 6	54	7	63
24 – 30	27	5	+ 12	60	13	65
Total		44		–42		278

$\therefore$ Mean $= a + \dfrac{\Sigma fd}{\Sigma f}$

$= 15 - \dfrac{42}{44}$

$= 14$ (nearly)

and average deviation

$= \dfrac{\Sigma f|x - \bar{x}|}{\Sigma f}$

$= \dfrac{278}{44}$

$= 6{\cdot}3$

9. We know,

$\bar{x} = \dfrac{\Sigma fx}{\Sigma f}$

Arithmetic mean

$= \dfrac{258}{60}$

$=$ Rs. 4·30

Arithmetic mean

$= a + \dfrac{\Sigma fd}{\Sigma f}$

$= 5 + \dfrac{-42}{60} = 5 - \dfrac{7}{10}$

$=$ Rs. 4·30

Here, $d = x - 5$

$\Sigma fd = -42$

10. Here number of children,

$n = 7$

and $\Sigma x = 462$

$\therefore$ Arithmetic mean

$= \dfrac{\Sigma x}{n}$

$= \dfrac{462}{7}$

$= 66$ cm

Short cut method : Deviations from assumed mean 65 are

$-1, -6, 2, 4, 0, 5, 3$

$\therefore$ Arithmetic mean

$= 65 + \dfrac{1}{n}\Sigma d$

$= 65 + \dfrac{1}{7}(7)$

$= 66$ cm

11. Since the regression lines pass through $(\bar{x}, \bar{y})$, therefore

$4\bar{x} - 5\bar{y} + 33 = 0,$

and $20\bar{x} - 9\bar{y} = 107$

Solving, we get $\bar{x} = 13,$

and $\bar{y} = 17$

Rewriting the line of regression of 4 on x,

$y = \dfrac{4}{5}x + \dfrac{33}{5};$

$\sigma_{yx} = r\dfrac{\sigma_y}{\sigma_x}$

$= \dfrac{4}{5}$

Rewriting the line of regression of x and y as

$x = \dfrac{9}{20}y + \dfrac{107}{9};$

$\sigma_{xy} = r\dfrac{\sigma_x}{\sigma_y}$

$= \dfrac{9}{20}$

$\therefore$ $r^2 = \dfrac{4}{5} \times \dfrac{9}{20}$

$= 0{\cdot}36$

or, $r = 0{\cdot}6$

12. Since the mean of x's and the mean of y's lie on the two regression lines, therefore

$$\bar{x} = 19{\cdot}13 - 0{\cdot}87\,\bar{y} \quad \ldots(i)$$

$$\bar{y} = 11{\cdot}64 - 0{\cdot}50\,\bar{x} \quad \ldots(ii)$$

Multiplying equation (ii) by 0·87 and subtracting from equation (i), we get

$$[1 - (0{\cdot}87)(0{\cdot}50)]\,\bar{x} = 19{\cdot}13 - (11{\cdot}64)(0{\cdot}87)$$

$$\therefore \quad \bar{x} = 15{\cdot}79$$

and $\quad \bar{y} = 11{\cdot}64 - (0{\cdot}50)(15{\cdot}79)$

$$= 3{\cdot}74$$

Thus regression co-efficient of y on x is $-0{\cdot}504$ and that of x on y is $-0{\cdot}87$.

Now since the co-efficient of correlation is the geometric mean between the two regression co-efficient, therefore

$$r = \sqrt{(-0{\cdot}50)(-0{\cdot}87)}$$

$$= \sqrt{0{\cdot}43}$$

$$= -0{\cdot}66$$

[**Note** : –ve sign is taken since both the regression co-efficient are –ve]

OBJECTIVE TYPE QUESTIONS

1. The mean and standard deviation of a binomial distribution are 10 and 2 respectively. The value of P is—
   (A) 1·0 (B) 0·8
   (C) 0·6 (D) 0·4

2. If the probability for A to fail in an examination is 0·2 and that for B is 0·3, then the probability that either A fails or B fails is—
   (A) 0·5 (B) 0·44
   (C) 0·06 (D) None of these

3. The probability that a teacher will give an unannounced test during any class is 1/5. If a student is absent twice then probability that misses atleast one test is—
   (A) $\frac{2}{3}$ (B) $\frac{4}{5}$
   (C) $\frac{7}{25}$ (D) $\frac{9}{25}$

4. The probability that atleast one of the events A and B occurs is 0·6. If A and B occur simultaneously with probability 0·2, then $P(\bar{A}) + P(\bar{B})$ is—
   (A) 0·4 (B) 0·8
   (C) 1·2 (D) 1·4

5. If M and N are two events, then probability that exactly one of them occurs is—
   (A) $P(M) + P(N) + 2P(M \cap N)$
   (B) $P(M) + P(N) - 2P(M \cap N)$
   (C) $P(\overline{M}) + P(\overline{N}) + 2P(M \cap N)$
   (D) $P(M \cap \overline{N}) + P(\overline{M} \cap N)$

6. If A and B are two mutually exclusive events, then P(A + B) is equal to—
   (A) P(A) (B)
   (B) P(A) + P(B)
   (C) $P(A)P(B') + P(A')P(B)$
   (D) $P(A)P(B') - P(A')P(B)$

7. The binomial distribution with mean 20 and standard deviation 4 is—
   (A) $\left(\frac{1}{5}+\frac{4}{5}\right)^{100}$ (B) $\left(\frac{4}{5}+\frac{1}{5}\right)^{100}$
   (C) $\left(\frac{4}{5}+\frac{1}{5}\right)^{50}$ (D) None of these

8. A purse contains 4 copper coins and 3 silver coins. The second purse contains 6 copper coins and 2 silver coins. A coin is taken out from any purse, the probability that it is a copper coin is—
   (A) $\frac{11}{4}$ (B) $\frac{37}{36}$
   (C) $\frac{3}{7}$ (D) $\frac{3}{4}$

9. In a Binomial distribution, if the mean is 9 and S.D. is $\sqrt{6}$, then values of n and p respectively are—
   (A) $27, \frac{1}{3}$ (B) $81, \frac{1}{9}$
   (C) $36, \frac{1}{4}$ (D) $18, \frac{1}{2}$

10. $P(A \cap B) = 0{\cdot}15$, $P(B') = 0{\cdot}10$, then P(A/B) is—

(A) $\frac{1}{3}$ (B) $\frac{1}{4}$

(C) $\frac{1}{5}$ (D) $\frac{1}{6}$

11. A and B toss 3 coins. The probability that they both obtain the same number of head is—

(A) $\frac{1}{9}$ (B) $\frac{3}{16}$

(C) $\frac{5}{16}$ (D) $\frac{3}{8}$

12. The probability of having atleast one tail in five throws with a coin is—

(A) $\frac{31}{32}$ (B) $\frac{1}{32}$

(C) $\frac{1}{5}$ (D) 1

13. India plays two P matches each with West Indies and Australia. In any match the probabilities of India getting points 0, 1 and 2 are 0·45, 0·05, and 0·5 respectively (Assuming that outcomes are independent) the probability of India getting atleast 7 points is—

(A) 0·8750 (B) 0·0875

(C) 0·0625 (D) 0·0250

14. Two events A and B have probabilities 0·25 and 0·50 respectively, the probability that both A and B occur simultaneously is 0·14. Then the probability that neither A nor B occur is—

(A) 0·39 (B) 0·375

(C) 0·89 (D) 0·86

15. If three disc are thrown simultaneously, then the probability of getting score of 5 is—

(A) $\frac{5}{216}$ (B) $\frac{1}{6}$

(C) $\frac{1}{36}$ (D) $\frac{1}{72}$

16. Let A and B be two independent events. The probability that both A and B occur is $\frac{1}{12}$ and the probability that neither A nor B occurs is $\frac{1}{2}$. The respective probabilities of A and B are—

(A) $\frac{1}{6}$ and $\frac{1}{2}$

(B) $\frac{1}{2}$ and $\frac{1}{6}$

(C) $\frac{1}{3}$ and $\frac{1}{4}$ or $\frac{1}{4}$ and $\frac{1}{3}$

(D) None of these

17. A box contains 100 tickets numbered 1, 2, 3, …… 100. Two tickets are chosen at random. If it is given that the maximum number on the two chosen tickets is not more than 10, then the probability that the minimum number on them is not less than 5 is—

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$

(C) $\frac{152}{153}$ (D) None of these

18. A box contains 5 black and 5 red balls. Two balls are randomly picked one after another from the box, without replacement. The probability for both balls being red is—

(A) $\frac{1}{90}$ (B) $\frac{1}{5}$

(C) $\frac{19}{90}$ (D) $\frac{2}{9}$

19. Two dice are thrown. What is the probability that the sum of the numbers on the two dice is eight ?

(A) $\frac{5}{36}$ (B) $\frac{5}{18}$

(C) $\frac{1}{4}$ (D) $\frac{1}{3}$

20. An unbiased coin is tossed three times. The probability that the head turns up in exactly two cases is—

(A) $\frac{1}{9}$ (B) $\frac{1}{8}$

(C) $\frac{2}{3}$ (D) $\frac{3}{8}$

21. The probability that two friends share the same birth month is—

(A) $\frac{1}{6}$ (B) $\frac{1}{12}$

(C) $\frac{1}{144}$ (D) $\frac{1}{24}$

22. A box contains 5 black balls and 3 red balls. A total of three balls are picked from the box one after another, without replacing themback. The brobability of getting two black balls and one red ball is—

(A) $\frac{3}{8}$ (B) $\frac{2}{15}$

(C) $\frac{15}{28}$ (D) $\frac{1}{2}$

23. If 20 per cent managers are technocrats, the probability that a random committee of 5 managers consists of exactly 2 technocrats is—

(A) 0·2048 (B) 0·4000

(C) 0·4096 (D) 0·9421

24. A box contains three blue balls and four red balls. Another identical box contains two blue balls and five red balls. One ball is picked at random from one of the boxes and it is red. The probability that it came from the first box is—

(A) $\frac{2}{3}$ (B) $\frac{4}{9}$

(C) $\frac{4}{7}$ (D) $\frac{2}{7}$

25. The equilibrium data of component A in the two phases B and C are give below—

X (moles of A / moles of B)	Y (moles of A / moles of B)
1	0·5
2	4·125

The estimate of y for $x = 4$ by fitting a quadratic expression of a form $y = mx^2$ for the above data is—

(A) 15·5 (B) 16

(C) 16·5 (D) 17

26. Three identical dice are rolled the probability that the same number will appear on each of them is—

(A) $\frac{1}{6}$ (B) $\frac{1}{36}$

(C) $\frac{1}{18}$ (D) $\frac{3}{28}$

27. A box contains 6 red balls and 4 green balls, one ball is randomly picked and then a second ball is picked withut replacement of the first ball. The probability that both the balls are green is—

(A) $\frac{1}{15}$ (B) $\frac{2}{25}$

(C) $\frac{2}{15}$ (D) $\frac{4}{25}$

28. A pair of fair dice is rolled simultaneously. The probability that the sum of the numbers from the dice equals six is—

(A) $\frac{1}{6}$ (B) $\frac{7}{36}$

(C) $\frac{5}{36}$ (D) $\frac{1}{12}$

29. A box contains 8 balls, 2 of which are defective. The probability that none of the balls drawn are defective when two are drawn at random without replacement is—

(A) $\frac{15}{28}$ (B) $\frac{9}{16}$

(C) $\frac{7}{16}$ (D) $\frac{1}{8}$

30. Four fair coins are tossed simultaneously. The probability that at least one head turns up—

(A) $\frac{1}{16}$ (B) $\frac{15}{16}$

(C) $\frac{7}{8}$ (D) $\frac{1}{8}$

31. From a pack of regular playing cards, two cards are drawn at random. What is the probability that both cards will be kings, if the first card is not replaced ?

(A) $\frac{1}{26}$ (B) $\frac{1}{52}$

(C) $\frac{1}{169}$ (D) $\frac{1}{221}$

32. The following data about the flow of liquid was observed in a continuous chemical process plant—

Flow rate	Frequency
7·5 to 7·7	1
7·7 to 7·9	5
7·9 to 8·1	35
8·1 to 8·3	17
8·3 to 8·5	12
8·5 to 8·7	10

Mean flow rate of the liquid is—

(A) 8·00 litres / sec (B) 8·06 litres / sec
(C) 8·16 litres / sec (D) 8·26 litres / sec

33. A regression model is used to express a variable Y as a function of another variable X. This implies that—
(A) There is a casual relationship between Y and X
(B) A value of X may be used to estimate a value of Y
(C) Values of X exactly determine values of Y
(D) There is no casual relationship between Y and X

34. Manish has to travel from A to D changing bases at stops B and C enroute. The maximum waiting time at either stop can be 8 minutes each, but any time of waiting up to 8 minutes is equally likely at both places. He can afford up to 13 minutes of total waiting time if he is to arrive at D on time. What is the probability the Manish will arrive late at D ?
(A) $\frac{8}{13}$ (B) $\frac{13}{14}$
(C) $\frac{119}{128}$ (D) $\frac{9}{128}$

35. Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between successive arrivals. The length of a phone call is distributed exponentially with mean 3 minutes. The probability that an arrival does not have to wait before service is—
(A) 0·3 (B) 0·5
(C) 0·7 (D) 0·9

Answer with Explanation

1. (C) Given $np = 10$

and $npq = 2^2 = 4$

$\Rightarrow q = \frac{4}{10} = \frac{2}{5}$

$\Rightarrow p = 1 - q = \frac{3}{5}$

2. (B) Required Probability

$= 1 -$ (neither A nor B fails)

$= 1 - p$(Both A & B are successful)

$= 1 - (1 - 0·2)(1 - 0·3)$

$= 0·44$

3. (D) Required Probability

$= 1 - p$(student does not miss any test)

$= 1 - p$(the teacher does not given the test on both days)

$= 1 - \left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{5}\right)$

$= 1 - \frac{16}{25}$

$= \frac{9}{25}$

4. (C) $P(\overline{A}) + P(\overline{B})$

$= 2 - (P(A) + P(B))$

$= 2 - P(A \cup B) - P(A \cap B)$

$= 2 - 0·6 - 0·2$

$= 1·2$

5. (D) Required Probability

= P(only one M and N occurs)

= P(M occurs but not N) + P(N occurs but not M)

$= P(M \cap \overline{N}) + P(\overline{M}) \cap N)$

(Note—that $M \cap \overline{N}$ and $N \cap \overline{M}$ *i.e.*, $M - N$ and $N - M$ are ME)

6. (B) Given $A \cap \phi = P(A \cap B) = 0$

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)$

7. (B) Given Mean $= 20$

and S.D. $= 4$

$\therefore np = 20$

and $npq = (\text{S.D.})^2 = 16$

From here, we get

$q = \frac{4}{5}$,

$p = \frac{1}{5}$

and $n = 100$

So the binomial distribution is ($p = q$)

$$n = \left(\frac{4}{5}+\frac{1}{5}\right)^{100}$$

8. (B)Required probability

$$= \frac{1}{2}\times\frac{4}{7}+\frac{1}{2}\times\frac{6}{8}$$
$$= \frac{2}{7}+\frac{3}{8}$$
$$= \frac{16+21}{56}$$
$$= \frac{37}{56}$$

9. (A) Given : $np = 9$,

and $\sqrt{npq} = \sqrt{6}$,

$\Rightarrow$ $npq = 6$

$\Rightarrow$ $q = \frac{6}{9}$

$= \frac{2}{3}$

$\Rightarrow$ $p = 1-q$

$= 1-\frac{2}{3}$

$= \frac{1}{3}$

and $np = 9$

$\Rightarrow$ $n = \frac{9}{p}$

$= 27$

10. (D) $P(A/B) = \frac{P(A\cap B)}{P(B)}$

$$= \frac{P(A\cap B)}{1-P(B')}$$
$$= \frac{0{\cdot}15}{1-0{\cdot}10}$$
$$= \frac{15}{90}$$
$$= \frac{1}{6}$$

11. (C) Required Probability

$$= \frac{1}{8}\times\frac{1}{8}+\frac{3}{8}\times\frac{3}{8}+\frac{3}{8}\times\frac{3}{8}+\frac{1}{8}\times\frac{1}{8}$$
$$= \frac{1\times1+3\times3+3\times3+1\times1}{8\times8}$$

12. (A) It is a case of Bernoullian trials where success is "a trial comes up"

Here, $p = \frac{1}{2}$

and $q = \frac{1}{2}$

$\therefore$ Required Probability

$$= 1-p(0)$$
$$= 1-{}^5C_0\left(\frac{1}{2}\right)^5$$

13. (B) p(India gets 7 points)

= p(India wins 3 matches and drews exactly one of the four matches)

$$= {}^4C_3\,(0{\cdot}5)^3\,(0{\cdot}05)$$
$$= 4\times\left(\frac{1}{2}\right)^3\times\left(\frac{1}{20}\right)$$
$$= 0{\cdot}0250$$

p (India gets 8 points)

= p(India coins all the four matches)

$$= (0{\cdot}5)^4$$
$$= 0{\cdot}0625$$

$\therefore$ p(India gets atleast 7 points)

$$= 0{\cdot}0250+0{\cdot}0625$$
$$= 0{\cdot}0875$$

14. (A) p(neither A nor B)

$= 1 -$ P(either A or B)

$= 1 - P(A\cup B)$

$= 1 - \{(P(A)+P(B)-P(A\cap B)\}$

15. (C) Total number of equally likely events is 216 and favourable outcomes are :

(2, 2, 1), (2, 1, 2), (1, 2, 2), (3, 1, 1), (1, 3, 1), (1, 1, 3)

$\therefore$ Required Probability

$$= \frac{6}{216}$$
$$= \frac{1}{36}$$

16. (C) If $P(A) = x$

and $P(B) = y$

then $P(A\cap B) = \frac{1}{12}$

$\Rightarrow$ (neither A nor B occurs)

$$= P(A')\,P(B')$$
$$= \frac{1}{2}$$

$\Rightarrow \quad P(A') P(B') = \frac{1}{2}$

or, $(1-x)(1-y) = \frac{1}{2}$

17. (A) Let A be the event "maximum number is not more than 10" and B. "minimum number is not less than 5"

$\Rightarrow \quad P(A) = \frac{^{10}C_2}{^{100}C_2}$

and $\quad P(A \cap B) = p$(number on selected ticketsis from 5 to 10)

$= \frac{^{6}C_2}{^{100}C_2}$

(since there are six tickets with such numbers)

$\therefore$ Required Probability

$= P(B/A)$

$= \frac{P(A \cap B)}{P(A)}$

18. (D) The probability of drawing a red ball

$= \frac{5}{10}$

If the ball is not replaced, the box will have a ball,

So probability of drawing the red ball in next chance $= \frac{4}{9}$

Hence probability of drawing 2 balls

$= \frac{5}{10} \times \frac{4}{9}$

$= \frac{2}{9}$

19. (A) 20. (D)

21. (B) Probability that first friend is born in any month $= 100\%$

$= 1$

Probability that second friend is born in the same month as that of first friend

$= 1 \times \frac{1}{12}$

$= \frac{1}{12}$

22. (C) Let B refer to black balls and R to red balls consider the combinations BBR, BRB, RBB

Probability of BBR $= \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6}$

$= \frac{5}{28}$

Probability of BRB $= \frac{5}{8} \times \frac{3}{7} \times \frac{4}{6}$

$= \frac{5}{28}$

Probability of RBB $= \frac{3}{8} \times \frac{5}{7} \times \frac{4}{6}$

$= \frac{5}{28}$

$\therefore$ Total probability of getting two black balls and one red ball $= \frac{5}{28} + \frac{5}{28} + \frac{5}{28}$

$= \frac{15}{28}$

23. (A) Probability of technocrat manager

$= \frac{20}{100} = \frac{1}{5} = p$

Probability of non-tecnocrat manager

$= \frac{80}{100} = \frac{4}{5} = q$

Probability of a random committee of 5 with exactly 2 technocrats

$= {}^5C_2\, p^2 q^3$

$= \frac{5 \times 4}{1 \times 2} \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^3$

$= 0{\cdot}2048$

24. (D) Number of boxes

$= 2$

Probability of picking ball from one box

$= \frac{1}{2}$

Total number of balls

$=$ Blue balls + Red balls

$= 3 + 4$

$= 7$

Probability of red ball

$= \frac{4}{7}$

Thus probability of getting red ball from one box $= \frac{1}{2} \times \frac{4}{7}$

$= \frac{2}{7}$

25. (D)

26. (B) As (1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6) are only favourable outcomes.

$\therefore$ Required probability

$$= \frac{6}{216}$$

27. (D) $\frac{^4C_1}{^{10}C_1} \times \frac{^3C_1}{^9C_1} = \frac{4}{10} \times \frac{3}{9}$

$$= \frac{2}{15}$$

28. (C) 29. (A) 30. (B)

31. (D) Probability of both cards being kings

$$= \frac{4}{52} \times \frac{3}{51}$$

$$= \frac{1}{221}$$

32. (C)

Flow rate (litre/sec)	Mean value of flow rate (x)	Frequency (f)	fx
7·5 – 7·7	7·6	1	7·8
7·7 – 7·9	7·8	5	36
7·9 – 8·1	8·0	35	280
8·1 – 8·3	8·2	17	139·4
8·3 – 8·5	8·4	12	100·8
8·5 – 8·7	8·6	10	86
		$\Sigma f = 80$	$\Sigma fx = 652{\cdot}8$

$\therefore$ Mean flow rate

$$= \frac{\Sigma fx}{\Sigma f}$$

$$= \frac{652{\cdot}8}{80}$$

$$= 8{\cdot}16$$

33. (B) 34. (A) 35. (A)

Linear Programming

Convex-Set and n-Dimensional Space

Euclidean Space, E^n—An n-dimensional Euclidean space (Euclidean Vector Space) is a collection of all vector (points)

$$\vec{a} = [a_1, a_2, \ldots\ldots, a_n]$$

(1) Vector Multiplication (by scalar) and Vectors Addition—

$$\lambda\vec{a} = [\lambda a_1, \lambda a_2, \ldots\ldots, \lambda a_n]$$

and $\vec{a} + \vec{b} = [a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n]$

were $\vec{a} = [a_1, a_2, \ldots, a_n]$

and $\vec{b} = [b_1, b_2, \ldots, b_n]$

(2) Distance between Two Vectors—

$$|\vec{a} - \vec{b}| = \sqrt{\sum_{i=1}^{n} (a_1 - b_1)^2}$$

(3) Point Set—$X = \{\bar{x} : P(\vec{x})$, where $P(\vec{x})$ is a property defined on $\vec{x}\}$.

(4) Hypersphere—A hypersphere in E^n with centre at a and radius $\varepsilon > 0$ is the set of points

$$X = \left\{\frac{\vec{x}}{|\vec{x} - \vec{a}|} = \in\right\}$$

(5) Inside—The inside of a hypersphere with centre at a and radius $\varepsilon > 0$, is the set of points

$$X = \left\{\frac{\vec{x}}{|\vec{x} - \vec{a}|} < \in\right\}$$

(6) $\in$-Neighbourhood—An $\in$-neighbourhood of the point $\vec{a}$, is the set of points inside the hypersphere with centre at a and radius $\varepsilon > 0$,

$$X = \left\{\frac{\vec{x}}{|\vec{x} - \vec{a}|} < \in\right\}$$

(7) Interior Point—A point $\vec{a}$ is an interior of set A if these exists an $\in$-neighbourhood about $\vec{a}$ which contains only points of set A.

(8) Boundary Points—A point is a boundary point of a set A every $\in$-neighbourhood about $\vec{a}$ contains points which are in the set and points which are not in the set.

(9) Open Set—A set is an open set if it contains only interior points.

(10) Closed Set—A set is closed if it contains all its boundary points.

(11) Complement—The complement of any set A in E^n, A^c is the set of all points in E^n but not in A.

(12) Strictly Bounded—A set A is strictly bounded if there exists a positive number r such that for every $\vec{a} \in A, |\vec{a}| < \vec{r}$.

(13) Bounded from Above—Set A is bounded from above if there exists $\vec{r}$, with each component finite such that for all $\vec{a} \in A, \vec{a} \le \vec{r}$.

(14) Bounded from Below—Set A is bounded from below if there exists $\vec{r}$, with each component finite, such that for each

$$\vec{a} \in A, \vec{a} \le \vec{r}.$$

(15) Line—The line passing through $\vec{x_1}$ and $\vec{x_2}$ $(\vec{x_1} \ne \vec{x_2})$ in E^n, is the set of the points

$$X = \{\vec{x} / \vec{x} = \lambda\vec{x_2} + (1 - \lambda)\vec{x_1}, \lambda \text{ real number.}\}$$

(16) Polygonal Path—A path joining such that $x_1 = y_1, y_2, \ldots\ldots, y_R = x_2$ where y_1 joins y_2, y_2 joins y_3, …… and y_R joins x_2.

(17) Line Segment—The line segment joining points $\vec{x_1}$ and $\vec{x_2}$ in E^n, is the set of points

$$X = \{\vec{x} / \vec{x} = \lambda\vec{x_2} + (1 - \lambda)\vec{x_1}, 0 \le \lambda \le 1\}$$

(18) Connected Set—An open set is connected if any two points in the set can be joined by a polygonal path lying entirely within the set.

(19) Region—A region in E^n is connected set of points in E^n.

(20) Normal—Given the hyper-plane $\overrightarrow{cx} = z$ in E^n ($\overrightarrow{c} \neq 0$), then $\overrightarrow{c}$ is a vector normal to the hyper-plane.

(21) Parallel Hyper Planes—Two hyper-planes are parallel if they have the same unit normal.

(22) Open Half-Spaces—The set $X_1 = (\overrightarrow{x}/\overrightarrow{cx} < z)$ and $X_2 = (\overrightarrow{x}/\overrightarrow{cx} > z)$ are open half-spaces.

(23) Closed Half-Spaces—The set

$$X_3 = (\overrightarrow{x}/\overrightarrow{c}\ \overrightarrow{x}) \leq z)$$

and $$X_4 = (\overrightarrow{x}/\overrightarrow{c}\ \overrightarrow{x} \leq z)$$

are closed half spaces.

(24) Convex Sets—A set is convex if for any points $\overrightarrow{x_1}$, $\overrightarrow{x_2}$ in the set, the line segment joining these points is also in the set.

(25) Convex Combination—A convex combination of a finite number of points $\overrightarrow{x_1}$, $\overrightarrow{x_2}$, ..., $\overrightarrow{x_m}$ is a point $\overrightarrow{x} = \sum_{i=1}^{m} \mu_i = 1$

(26) Extreme Points—A points $\overrightarrow{x} \in X$ is an extreme point of a convex set X iff there do not exist point $\overrightarrow{x_1}$, $\overrightarrow{x_2}$ ($\overrightarrow{x_1} \neq \overrightarrow{x_2}$) in the set such that

$$\overrightarrow{x} = (1-\lambda)\overrightarrow{x_1} + \lambda\overrightarrow{x_2}, 0 < \lambda < 1.$$

(27) Convex Polyhedron—The convex hull of finite number of points is a convex polyhedron spanned by these points.

(28) Convex Hull—The convex hull of a set is the intersection of all convex sets which contain A.

(29) Simplex—The convex hull of any set of $n + 1$ points from E^n which do not lie on a hyper-plane in E^n is simplex.

(30) Supporting Hyper-Planes—Given a boundary points $\overrightarrow{w}$ of a convex set X : then $\overrightarrow{cx} = z$ is a supporting hyperplane at $\overrightarrow{w}$ if $\overrightarrow{c}\ \overrightarrow{w} = z$ iff of X lies in one closed half-space produced by the hyperplane, *i.e.*, $\overrightarrow{cu} \geq z$ for all $\overrightarrow{u} \in X$ or $\overrightarrow{cu} \leq z$ for all $\overrightarrow{u} \in X$.

(31) Edge—Let $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ be distinct extreme points of the convex set X. The line segment joining them in edge of convex set if it is the intersection of X with supporting hyperplane. If $\overrightarrow{x_1}$ is an extreme point of X, and if there exists another point $\overrightarrow{x} \in X$ such that $\overrightarrow{x} = \overrightarrow{x_1} + \lambda(\overrightarrow{x'} - \overrightarrow{x_1'})$ is in X for every $\lambda \geq 0$ and if in addition, the set $L = \{\overrightarrow{x}/\overrightarrow{x}) = \overrightarrow{x}_i + \lambda(\overrightarrow{x'} - \overrightarrow{x}_1{}^*)$, $\lambda \geq 0\}$ is the Intersection of X with supporting hyper plane, then the set L is edge of X which extends to infinity.

(32) Adjacent extreme points—Two distinct extreme points $\overrightarrow{x}_i$ and $\overrightarrow{x}_2$ of the convex set X is adjecent if the line segment joining them is an edge of the convex set.

(33) Cone—A cone c is a set of points such that if $\overrightarrow{x}$ is in set then $\mu\overrightarrow{x}$, $\mu \geq 0$ is also in set *i.e.* the cone generated by the set $X = \{\overrightarrow{x}\}$ is the set $c = \{\overrightarrow{y}/\overrightarrow{y} = \mu\overrightarrow{x}, \mu \geq 0 \text{ and } \overrightarrow{x} \in X\}$.

(34) Vertex—The point $\overrightarrow{O} = (0, 0, \ldots, 0)$ is an element of any cone and is called vertex of the cone.

(35) Negative—The negative C of the cone is $C = (\overrightarrow{y})$ is the set $C = (-\overrightarrow{y})$.

(36) Sum—The sum of two cones

$$C_1 = (\overrightarrow{y}), C_2 = (\overrightarrow{z}), \text{is}$$

$$C = \{\overrightarrow{y} + \overrightarrow{z}/\overrightarrow{y} \in C_1 \text{ and } \overrightarrow{z} \in C_2\}$$

(37) Polar Cone—If $c = \{\overrightarrow{y}\}$ is a cone, then C^+, the cone polar to c, is the set

$$C^+ = \{\overrightarrow{v}/\overrightarrow{v}, \overrightarrow{u} \geq 0, \overrightarrow{u} \in c\}$$

(38) Convex Cone—A cone is convex cone if it is a convex set.

(39) Half Line—Given a single point $\overrightarrow{a} \neq 0$, a half line (ray) is a set $L = \{\overrightarrow{y} : \overrightarrow{y} = u\overrightarrow{a}, u \geq 0\}$.

(40) Containing Half-Space—Given a cone C. Let $\vec{C} \neq 0$ be an element of C^+. Then containing half-space is the set

$$H_s = \{x : a' \vec{x} \geq 0\}$$

(41) Orthogonal Cone—Given a cone C in E^n. The orthogonal cone $C^\perp$ is the set of all vectors E^n such that $\vec{u}$ is orthogonal to every $D \in C$ *i.e.*

$$C^\perp = \{\vec{u} : \vec{uv} = 0, \vec{v} \in C\}.$$

(42) Dimension of a Cone—The dimension of a cone C is the maximum number of linearly independent vectors in C.

(43) Convex Polyhedral—A convex polyhedral cone C is the sum of a finite number of half lines, *i.e.* $C = \sum_{i=1}^{r} L_i$.

(44) Extreme Supporting Half-Spaces—The set of points in E^n such that $H_F = \{\vec{v} : \vec{cv} \geq 0\}$ is an extreme supporting half sapce for then-dimensional convex polyhedral cone C generated by the points $\vec{a_1} \ldots \vec{a_r}$ if C lies in the half space H_F and $(n - 1)$ linearly independent points from the set $\vec{a_1}, \vec{a_2}, \ldots \vec{a_n}$ lies on the hyperplane $\vec{cv} = 0$.

(45) Extreme Supporting Hyperplane—The hyperplane $\vec{cv} = 0$ which form the boundary of the extreme supporting half-space is called on extreme supporting hyperplane for the convex polyhedral cone C.

Some Important Theorems and Results

(I) The set E^n is both open and closed.

(II) Strictly bounded set lies inside a hyperplane of radius r, with its centre at the origin.

(III) Extreme point is the boundary point of a set, converse is not true.

(IV) Hyper-plane is a convex set.

(V) Closed half space is a convex set.

(VI) Intersection of two convex sets is also a convex set.

(VII) Intersection of two closed sets is a closed set.

(VIII) Intersection of finite number of hyperplanes, half-spaces or both of is a convex set.

(IX) Intersection of finite number of hyperplanes or closed half spaces or of both is a closed convex set.

(X) The set of feasible solutions to a linear programming problem is a closed convex set.

(XI) The set of all convex combinations of a finite number of points $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$ is a convex set.

(XII) The convex hull of a finite number of points $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_m$ is the set of all convex combinations of $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_m$.

(XIII) **Theorems on Separating Hyper-Planes**

(a) Given any closed convex set X, a point $\vec{y} \in X$ or there exist a hyperplane which contains $\vec{y}$ such that all of X is contained in one open half space produced by that hyper-plane.

(b) If $\vec{w}$ is a boundary point of a closed convex set, then there is at least one supporting hyper-plane at $\vec{w}$.

(c) A closed convex set which is bounded from below has extreme points in every supporting hyperplane.

(d) If a closed, strictly bounded convex set X has a finite number of extreme points, any point in the set can be written as a convex combination of the extreme points, *i.e.* set X is a convex hull of its extreme points.

(XIV) A set of points is a convex cone iff the sum $\vec{v}_1 + \vec{v}_2$ is in the set, when $\vec{v}_1$ and $\vec{v}_2$ are and if $u\vec{v}$ is in the set when $\vec{v}$ for any $u \geq 0$.

(XV) The sum $c_1 + c_2$ of two convex cones is also convex.

(XVI) The cone generated by a convex set is a convex cone.

(XVII) The cone C generated by a convex polyhedral is a convex polyhedral cone.

Linear Programming Problems (BASIC)

General Linear Programming Problem—Given a set of m-linear inequalities or equations in

r-variables, we wish to find non-negative values of these variables which satisfy the constraints and maximize or minimize some linear functions of the variables.

Optimize $z = c_1x_1 + c_2x_2 + \ldots + c_rx_r$ ...(1)

Subject to

$a_{i1}X_1 + a_{i2}X_2 + \ldots + a_{ir}X_r \{\leq, =, \geq\} b_i,$ ...(2)

$i = 1, 2, \ldots\ldots, m$

with restrictions (non-negativity)

$x_j \geq 0, j = 1, 2, \ldots\ldots r$

Here a_{ij}, b_i, c_j are known constants and one of the sign $\leq, =, \geq$, holds for each constraints and vary from constraints to constraint.

Determine the value of x_j ($j = 1, 2, \ldots r$) (Decision variables) ?

Decision Variables—x_j's are decision variables.

Objective Function—A linear function $z = \Sigma c_j x_j$, which is to be maximize or minimize.

Constraints—Inequalities $\sum_j a_{ij} x_j \{\leq, =, \geq\} b_i$ ($i = 1, 2, \ldots\ldots m$)

Non-negative Restrictions—$x_j \geq 0$ ($j = 1, 2, \ldots r$).

Solution—The values of the variables x_j's ($j = 1, 2, \ldots m$) satisfying constraints(2).

Feasible Soltuion—A solution satisfying non-negative restriction, $x_j \geq 0$ ($j = 1, 2, \ldots r$)

Optimal Solution—A feasible solution which optimizes the objective function (1).

Conversion of Inequality into Equality

Slack variable—Given $\sum_{j=1}^{r} a_{kj} x_j \leq b_j$, then $\sum_{j=1} a_{hj} x_j - x_{r+k} = b_j$, where $x_{r+k} = 0$ is slack variable

Surplus variable—Given $\sum_{j=1}^{r} a_{kj} x_j \geq b_j$, then $\sum_{j=1}^{r} a_{kj} x_j - x_{r+k} = b_h$, where $x_{r+k} \geq 0$ is surplus variable.

Conversion of $-b_j$ to b_j—Given $\sum_{j=1}^{r} a_{hj} x_j \{\leq, =, \geq\} - b_j$, multiply both side by (-1) gives $\sum_{j=1}^{r} (-a_{nj}) x_j \{\geq, =, \leq\} b_j$.

Conversion of minimization to maximization—Given min $z = \Sigma c_j x_j$ is equivalent to max $(-z) = \Sigma (c_j)x_j$ *i.e.* min (z) = max $(-z)$ and max (z) = min $(-z)$.

Cost (price) Associated with Slack and Surplus Variables

The cost associated with slack and surplus variables is assumed to be zero.

i.e. if
$$z = \sum_{j=1}^{r} c_j x_j + \sum_{h=1}^{u} c_n x_{r+h} + \sum_{k=1}^{v} c_k x_{k+r}$$

where c_h and c_k are cost associated with slack x_{r+h} and surplus x_{k+r} variables respectively and $c_h = c_k = 0$, which gives

$$z = \sum_{j=1}^{r} c_j x_j + \sum_{h=1}^{u} 0.x_{r+h} + \sum_{k=1}^{v} 0.\, x_{k+r} = \sum_{j=1}^{r} c_j x_j$$

Standard linear Programming Problem

$$\max z = \sum_{j=1}^{r} c_j x_j$$

Subject to

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, (i = 1, 2, \ldots m)$$

with $x_j \geq 0, (j = 1, 2, \ldots\ldots n)$

where $b_i \geq 0$ ($i = 1, \ldots\ldots m$)

Conversion of General Linear Programming Problem to Standard linear Programming Problem

By introducing slack and surplus veriables, making all $b_j \geq 0$ and maximize objective function z, we can convert general linear programming to standard linear programming problem, which gives

$$\max z = \sum_{j=1}^{r} c_j x_j$$

Subject to

$$\sum_{j=1}^{r} a_{hj} x_j + x_{r+n} = b_h, (h = 1, \ldots\ldots u)$$

$$\sum_{j=1}^{r} a_{kj} x_j - x_{r+h} = b_k, (k = u + 1, \ldots v)$$

$$\sum_{j=1}^{r} a_{pj} x_j = b_p, (p = v+1, \ldots m)$$

with $x_j \geq 0,$

this gives max $z = \overrightarrow{cx}$

subject to $\overrightarrow{\mathrm{A}x} = \vec{b}$ with $\vec{x} \geq 0$

where $\vec{c} = \{c_1, c_2, \ldots, c_n\}$

$\vec{x} = \{x_1, x_2, \ldots x_n\}$

$\vec{\mathrm{A}} = [a_{ij}]_{m \times m}$ matrix

and $\vec{b} = \{b_1, b_2, \ldots b_m\}$.

Simultaneous Linear Equations and Basic Solution

Given m-simultaneous linear equations is n known $(m < n)$

Given $\overrightarrow{\mathrm{A}x} = \vec{b}$

i.e. $\sum_{j=1}^{n} a_{ij} x_j = b_i, (i = 1, 2, \ldots m)$

where $\vec{\mathrm{A}} = [a_{ij}]_{m \times n}$,

$\vec{b} = [b_i, \ldots\ldots b_m]$

and $\vec{x} = [x_1, x_2, \ldots\ldots x_n]$

Basic Solution—For a given system of m-simultaneous linear equations in n unknowns,

$$\vec{\mathrm{A}}\,\vec{x} = \vec{b}\ (m < n)$$

and rank $(\vec{\mathrm{A}}) = m$

If $\vec{\mathrm{B}}$ is any $m \times m$ non singular matrix, chosen from $\vec{\mathrm{A}}$ and if all the $n \times m$ variables not associated with the columns of this matrix $\vec{\mathrm{B}}$ are set equal to zero, the solution of the resulting system of equation is called a basic solution. The m variables associated with matrix B are called basic variables *i.e.* $\vec{\mathrm{B}}$ is a $m \times m$ non singular matrix from $\vec{\mathrm{A}}$, then basic solution is $\vec{x}_{\mathrm{B}} = \mathrm{B}^{-1}b$, where all the variables not associated with columns from A in B are set equal to zero.

Degeneracy—A basic solution to $\overrightarrow{\mathrm{A}x} = \vec{b}$ is degenerate if one or more of the basic variables vanish.

Some Important Results

(1) A necessary and sufficient condition for the existence and non-degeneracy of all possible basic solutions of $\overrightarrow{\mathrm{A}x} = \vec{b}$ is the linear independence for every set of m columns from the argumented matrix $\vec{\mathrm{A}}_b = (\vec{\mathrm{A}}, \vec{b})$.

(2) A necessary and sufficient condition for any given basic solution $\vec{x}_{\mathrm{B}} = \vec{\mathrm{B}}^{-1}\vec{b}$ to be non-degenerated is the linear independence of $\vec{b}$ and every set of $(m-1)$ columns from $\vec{\mathrm{B}}$.

Basic Feasible Solution—A basic solution which is feasible, *i.e.* $\vec{x}_{\mathrm{B}} \geq \vec{0}$.

Associate Cost Vector—If $\vec{x}_{\mathrm{B}}$ is a basic feasible solution to max $z = \vec{c}\ \vec{x}$, subject to $\vec{\mathrm{A}}\ \vec{x} = \vec{b}, \vec{x} \geq \vec{o}$, then the vector $\vec{c}_{\mathrm{B}} = (c_{\mathrm{B}1}, \ldots c_{\mathrm{B}m})$, where c_{B} is the cost associated with basic varibale B, is called cost associated vector with basic feasible solution $\vec{x}_{\mathrm{B}}$

Improved basic feasible solution—If $\vec{x}_{\mathrm{B}}$ and $\vec{x}_{\mathrm{B}}$ are two basic feasible solutions to the standard linear programming problem. Then $\vec{x}_{\mathrm{B}}$ in the improved basic feasible solution in comparision to $\vec{x}_{\mathrm{B}}$, if $\vec{c}_{\mathrm{B}}\ \vec{x}_{\mathrm{B}} \geq \vec{c}_{\mathrm{B}}\ \vec{c}_{\mathrm{B}}$, where $\vec{x}_{\mathrm{B}}$ and $\vec{c}_{\mathrm{B}}$ are associated cost vectors with basic feasible solution on $\vec{x}_{\mathrm{B}}$ and $\vec{x}_{\mathrm{B}}$.

Optimal Basic Feasible Solution

A basic feasible solution $\vec{x}_{\mathrm{B}}$ is optimal basic feasible solution to max $z = \vec{c}\ \vec{x}$, subject to $\vec{\mathrm{A}}\ \vec{x} = b, c = \vec{\mathrm{O}}$ if $z^* = \vec{c}_{\mathrm{B}}, \vec{x}_{\mathrm{B}}$ for every basic feasible solution $\vec{x}_{\mathrm{B}}$ and its associated cost vector $\vec{c}_{\mathrm{B}}$.

Net Evaluation

If $\vec{c}_{\mathrm{B}}$ is a basic feasible solution to max $z = \vec{c}\ \vec{x}$, subject to $\vec{\mathrm{A}}\ \vec{x} = \vec{b}, \vec{x} \geq 0$ and $\vec{c}_{\mathrm{B}}$ is the cost associated with $\vec{x}_{\mathrm{B}}$. For each column

vector $\vec{a_j}$ in $\vec{A}$, which is not column vector of $\vec{B}$, let $a_j = \sum_{i=1}^{m} y_{ij}\ b_i$.

Then $z_j = \sum_{i=1}^{m} \vec{c}_{bi}\ y_{ij}$ is called evaluation corresponding $\vec{a_j}$ and the number $z_j - c_j$ is called net evaluation corresponding to $\vec{a_j}$.

Some Important Theorems

(1) If a linear programming problem has a feasible solution then it also has a basic feasible solution.

(2) There exists finite number of basic feasible solutions to linear programming problem.

(3) If a linear programming problem has a basic feasible solution and we drop one of the basic vector and introduce a non-basic vector in the basis set, then the new solution obtained is also a basic feasible solution.

(4) If $\vec{x}_B$ is a basic feasible solution to, max $z = \vec{c}\ \vec{x}$, subject to $\vec{A}\ \vec{x} = \vec{b}$, $\vec{x} = 0$.

If $\vec{x}_B$ is another basic feasible solution, obtained by replacing one of the basic vector of B by non-basic vector a_j, for which net evaluation $z_j - c_j < 0$. Then $\vec{x}_B$ is an improve basic feasible solution *i.e.*

$$\vec{c}_B\vec{x}_B \geq \vec{c}_B\vec{x}_B$$

(5) If there exist a basic feasible solution to a given linear programming problem and if for at least one j, has which $z_j - c_j < 0$ and $y_{ij} \leq 0$ ($i = 1, \ldots\ldots m$). Then there does not exist any optimum solution to linear programming problem.

(6) A sufficient condition for a basic feasible solution to an linear programming problem to be an optimum (maximum) is that $z_j - c_j$ $\frac{\notin \vec{A}x = \vec{b}}{|\underline{n}|\underline{m}|\underline{n-m}}$ for all j for which the column vector $\vec{a_j} \geq \in \vec{A}$ is not in the basis B. *i.e.* $\vec{a}$ $\notin \vec{B}$

(7) A necessary and sufficient condition for a basic feasible solution to a linear programming problem to be an optimum (max) is that $z_j - c_j \geq$ for all j for which $\vec{a} \notin \vec{B}$.

(8) Any convex combination of k-different optimum solution to a linear programming problem is again on optimum solution to the problem.

Extreme Point and Basic Feasible Solution

(1) Set of feasible solutions to $\vec{A}x = \vec{b}$ is a convex set.

(2) Every basic feasible solution is an extreme point of the convex set of the feasible solution.

(3) Every extreme point is a basic feasible solution to the set of contraints.

(4) The convex set of feasible solutions has a finite number of the extreme point.

(5) The number of extreme points is equal to the number of different basic feasible solutions and the number of basic feasible solutions (extreme points) can not be greater, than $|\underline{n}|$ $\underline{m}|\underline{n-m}$.

(6) In the absence of degeneracy, there is a one-to-one correspondence between the extreme points and the basic feasible solution.

Geometric Interpretation of L.P.P.

Whenever the feasible solution of linear programming problem exists, the region of feasible solution is a convex set and there also exist extreme proints. If the optimal solution exists one of the extreme point is optimal.

Whenever the optimal value of objective function z is finite, at least one extreme point of the region of feasible solution has an optimal solution.

If optimal solution is not unique, there are points other than extreme points that were optimal, but in any coase one extreme point is optimal.

Simplex Method and its Geometric Interpretation

Simplex method is an algebric iterative procedue for solving linear programming problem in finite number of steps (and states for unbounded solution, if exist).

If there is an optimal solution, one of the extreme point is optimal and there are finite number of extreme points. By simplex method we move step by step from a given extreme point to an optimal extreme point.

The simplex method moves along an 'edge' of the region of feasible solutions from one extreme point to adjacent one. Of all the adjacent extreme points, one chosen is that which gives greatest increase (or greatest decrease) in objective function z.

At which extreme point, by simplex method we know that, whether that extreme point is optimal if not, what the next extreme point will be.

Whenever by simplex method, on extreme point which, has on edge leading to infinite exist and the objective function can be increased (or decreased) by moving along that edge, the simplex method states for unbounded solution.

Imp—If origin is a feasible solution, it will be an extreme point because of all variables are non-negative.

Graphic Method of Solution

The graphic solution procedure is one method of solving two variables linear programming problems and involves the following steps :

(1) Formulate the problem in terms of a series of mathematical constraints and an objective function.

(2) Plot each of the constraints as follow. Each inequality in the constraints equation be written as equality. Give any arbitrary value to one variable and get the value of other variable by solving the equation. Similarly, give another arbitrary value to the variable and final the corresponding value of the other variable. Now plot these two sets of values. Connect these points by a straight line. This exercise is to be carried out for each of the constraints equations. Thus, there will be as many straight lines as there are equations, each straight lines representing one constraints.

(3) Identify the feasible region (or solution space), *i.e.* the area which satisfy all the contraints simultaneously. For 'greater than' constraints, the feasible region will be the area which lies above the constraints, lines. For 'greater than or equal to' or 'less than or equal to' constraints, the feasible region includes the points on the constraints lines also.

(4) **Corner point method**

(a) Identify the each of the corner (or extreme points) of the feasible region either by visual inspection or the method of simultaneous equations.

(b) Compute the profit/cost at each corner point by substituting the co-ordinates of that point into the objective function.

(c) Identify the optimal solution at that corner point which shows highest profit (In a maximization problem) or lowest cost (In a minimization problem).

Linear Programming (Advanced)

The linear programming model includes

(a) Decision variables (continuous), we seek to determine.

(b) Objective (goal), we aim to optimize (linear function).

(c) Constraints, that need to be satisfied (linear function).

Non-negativity Restriction—$x_i \geq 0$.

Feasible Solution—Any solution that satisfies all the constraints.

Optimum Feasible Solution—Any feasible solution, that optimize the objective.

Graphical L.P. Solutions

Optimum solution is associated with a corner points of the solution space, where two lines intersects.

The Simplex Method

In graphical method, the optimum LP solution is always associated with corner points (extreme points) of the solution space.

The simplex method is a mathematical treatment to obtain and identify these extreme points algebraically.

To obtain optimal solution, we first convert the LP model to standard LP model by using slack and surplus variable, to convert inequality constraints into equality. After it we try to find basic solutions of the simultaneous linear equations. These (algebraic) basic solutions completly define all the (geometric) extreme points of the solution space. The simplex

algorithm located the optimum among these basic solutions.

Standard Form and its Basic Solutions

Standard Form

(a) All the constraints with non-negativity restriction on variables are equations with non-negative right hand side.
(b) All variables are non-negative.
(c) Objective function is maximization, type.

Conversion of Inequality into Equality

(a) ($\leq$ type) – by slack variable.
(b) ($\geq$ type) – by surplus variable.
(c) The negative right hand side is converted to non-negative right hand side by multiplying – 1 to both sides.
(d) ($\leq$ type) inequality converts to ($\geq$ – type) inequality by multiplying – 1.

Conversion of Unrestricted Variables

Replace the unrestricted variable x_j by $x^{*}_j - x^{-}_j$ wher $x^{+}_j, x^{-}_j \geq 0$.

i.e. $\quad x_j = x^{+}_j - x^{-}_j$

Convesion of Maximization to Minimization

$$\text{Max } f(x_1, \ldots x_n) = \min(-f(x_1, \ldots x_n)$$
$$\text{Min } f(x_1, \ldots x_n) = \max(-f(x_1, \ldots x_n)$$

Determination of Basic Solutions

The standard LP form includes m-simultaneous linear equations and n-unknowns (variables), ($m < n$).

(a) Assign n–m variables. Zero value.
(b) Remaining m-variables are determined by solving resulting m-simultaneous equations. If m-equations yield a unique solution, then the associated m-variables are called basic variables and remaining n–m variables are referred to non-basic variables.

The unique solution (basic variables) and non-basic variables comprises a basic solution. If all variables are non-negative then basic solution is feasible solution otherwise it is infeasible.

The maximum number of possible basic solution for m equations and n unknown is $\binom{n}{m}$.

Unrestricted Variables and Basic Solutions

For unrestricted variables

$$x_j = x^{+}_j - x^{-}_j;\ x^{+}_j, x^{-}_j \geq 0.$$

Since x^{+}_j and x^{-}_j are dependent variables, only one can be the basic variable.

i.e., if x^{+}_j is basic variable, then x^{-}_j is non-basic variable.

If x^{-}_j is basic variable, then x^{+}_j is non-basic variable.

The Simplex Algorithm

The simplex algorithm is designed to locate the optimum by conectrating on a selected number of the basic feasible solutions of the problem.

The simplex method always starts with a basic feasible solution and then attempts to find other basic feasible solution that will impreve the objective value. This is possible only if an increase in a current zero (non-basic) feasibles can lead to an improvement in the objective value. However for a current zero variable to become positive, one of the current basic variables must be removed to guarantee that the new solution will include exactly m basic varaibles. The selected zero variable is the entering variable and the removing basic variable is the leaving variable.

Basic	z	$x_1 x_2 \ldots x_n$	**Solution**
z			z-row
x_1 x_2 . . . x_n			

Optimality Conditions

The entering variable in a maximization (minimization) problem is the non-basic variable having the most negative (positive) coefficient in the row.

Maximization—z-row coefficient-most negative.

Minimization—z-row coefficient-most positive.

The optimality reaches, at the iteration where all the z-row cofficients of non-basic variables are non-negative for maximization and non-positive for minimization.

Feasibility Condition

For both the maximization and minimization problem, the leaving variable is the basic variable associated with the smallest non-negative ratio.

Steps for Simplex Method

Step 1. Determine the starting basic variables.

Step 2. Select the entering variable using the optimality condition, stop if no entering variables.

Step 3. Select the leaving variable using feasibility conditions.

Step 4. Determine the new basic solution.

The simplex method computations are iterative, because next tableau is produced from the current tableau.

Artificial Starting Solution

For the LP's in which all the constraints are of ($\leq$) type, with non-negative right hand side, the slack offers a convenient starting basic feasible solution. For ($\geq$) type and (=) type constraints, new variables are introduced, known as artificial variable to produce initially basic feasible solution for simplex tableau.

Basic Solution

Given m-simultaneous linear equations in n-variable ($m < n$).

$$\vec{O}^{-1}\vec{x}_B = \vec{c}\ d_x^{\Delta}\ x$$

where $\quad \vec{A} = [a_{ij}]_{m \times n}$

and $\quad \text{rank } A = m$

If $\vec{B}$ is a sub matrix of $\vec{A}$, formed by m-linear independent columns of $\vec{A}$. Then solution obtained by setting n–m variables not associated with the columns of $\vec{B}$ equal to zero and solving resultant equation is called basic solution.

The m-variables associated of $\vec{B}$ are called basic variables and (n–m) remaining variables that are equal to zero are called non-basic variables.

The solution $\vec{x}_B = \vec{B}^{-1}\ \vec{b}$ is called basic solution.

Basic Feasible Solution—$\vec{x}_B \geq \vec{O}$

Associated Cost Vectors

If $\vec{x}_B$ is a basic feasible solution to standard L.P.P., then

$$\vec{c}_B = [c_{B_1}, \ldots\ldots\ c_{B_m}]$$

where c_{Bi} are components of $\vec{c}$ associated with the basic variables called cost vector associated with $\vec{x}_B$.

Improved Basic Feasible Solution

If $\vec{x}_B$ and $\overset{\Delta}{x}_B$ are two basic feasible solutions to standard L.P.P. $\overset{\Delta}{x}_B$ is improved basic feasible solution, compared to

$\vec{a}_j \in \hat{\vec{c}}_B\ \hat{\vec{x}}_B \geq \vec{B} \notin \rightarrow B$

if $\hat{\vec{c}}_B\ \hat{\vec{x}}_B > \vec{c}_B\ \vec{x}_B$ (maximization)

$\hat{\vec{c}}_B\ \hat{\vec{x}}_B < \vec{c}_B\ \vec{x}_B$ (minimization)

Fundamental Properties of Solutions

(1) If linear programming problem has a feasible solution, then it also has a basic feasible solution.

(2) There exist only finite number of basic feasible solutions to a linear programming problem.

(3) If linear programming problem have a basic feasible solution. Then if we drop one of the basic variable and introduce a non-basic variable to basic solution, then new solution is also a basic feasible solution.

(4) If $\vec{x}_B$ be a basic feasible solution, $\hat{\vec{x}}_B$ is a basic feasible solution obtained by admitting a non-basic column vector $\vec{a}_j$ in the basis for which net evaluation.
$z_j - c_j < 0$ (maximization),
$z_j - c_j > 0$ (minimization).

(5) If $z_j - c_j = 0$ for at least one j for which $y_{ij} > 0$, $i = 1, 2, \ldots\ldots, m$; then the another basic feasible solution is obtained which gives an unchanged value of the objective function.

(6) A sufficient condition for a basic feasible solution to be optimum is that $z_j - c_j \geq 0$ for all j for which the column vector $\vec{a}_j \in \vec{A}$ is not in the basis $\vec{B}$.

(7) A basis feasible solution is an optimum (maximum) is that $z_j - c_j \geq 0$ for all j for which $\vec{a_j} \notin \vec{B}$.

(8) Any convex combination of k different optimum solutions, is again an optimum solution.

The Simplex Tableau and Simplex Method

Given Max Z $= 6x_1 + 4x_2$

Subject to,

$$x_1 + 2x_2 \leq 720$$
$$2x_1 + x_2 \leq 780$$
$$x_1 \leq 320$$
$$x_1, x_2 \geq 0$$

Standard L.P. Problem :

$$\text{Max } z = 6x_1 + 4x_2 + Os_1 + Os_2 + Os_3$$

Subject to,

$$x_1 + 2x_2 + s_1 = 720$$
$$2x_1 + x_2 + s_2 = 780$$
$$x_1 + + s_3 = 320$$

Non-negativity restriction,

$x_1, x_2, s_1, s_2, s_3 \geq 0$

Here z-row is—$z - 6x_1 - 4x_2$ and slack variables s_1, s_2, s_3 have provided initial basic feasible solution.

Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	1	2	1	0	0	720	720
s_2	2	1	0	1	0	780	340
s_3	1	0	0	0	1	320	320
z	−6	−4	0	0	0		

↑ Entering variable (pivot column)

Here the entering variable is spotted as the non-basic variable having most negative cofficient in z-row.

The leaving variables is spotted as the minimum non-negative ratio.

Gauss-Jordan Computation (for new basic solution)

Pivot column—A column associated with entering variable.

Pivot row—A row associated with leaving variable.

Pivot element—An intersection of pivot row and pivot column provides pivot element.

Also, New pivot row = curmet pivot row + pivot element.

New row = current row – (its pivot coefficient)

New Pivot Row

Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	0	2	1	0	–1	400	200
s_2	0	1	0	1	–2	140	140 →
s_3	1	0	0	0	1	320	:
z	0	−4	0	0	6		

↑ Entering variable

Here s_3 is pivot row (current) and x_1 – row is the new pivot row (1 0 0 0 1/320) + 1 = (1 0 0 0 1/320)

s_2– row (new)

$= (2\ 1\ 0\ 1\ 0/780) - 2\ (1\ 0\ 0\ 0\ 1/320)$
$= (0\ 1\ 0\ 1 - 2/140)$

s_1–row (new)

$= (1\ 2\ 1\ 0\ 0/720) - 1\ (1\ 0\ 0\ 0\ 1/320)$
$= (0\ 2\ 1\ 0 - 1/400)$

z-row (new)

$= (-6\ -4\ 0\ 0\ 0/0) + 6(1\ 0\ 0\ 0\ 1/320)$
$= (0 - 4\ 0\ 0\ 6/320)$

Entering variable (most negative coefficient in z-row)

$= x_2$

Leaving variable (minimum non-negative ratio) = s_2

Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	0	0	1	2	3^*	120	40 →
s_2	0	1	0	1	–2	140	:
s_3	1	0	0	0	1	320	320
z	0	0	0	4	–2	2480	

↑ Entering variable

s_2-row (pivot row) current and x_2-row (pivot row) new.

(0 1 0 1 – 2/140) + 1 = (0 1 0 1 – 2/140)

s_1-row (new)

$= (0\ 2\ 1\ 0 - 1/400) - 2(0\ 1\ 0\ 1 - 2/140)$
$= (0\ 0\ 1\ -2\ 3/120)$

x_1-row (new)

$= (1\ 0\ 0\ 0\ 1/320) - 0(0\ 1\ 0\ 1 - 2/140)$
$= (1\ 0\ 0\ 0\ 1/320)$

z-row (new)

$= (0 - 4\ 0\ 0\ 6/320) + 4(0\ 1\ 0\ 1\ -2/140)$

$= (0\ 0\ 0\ 4 - 2/2480)$

Basic	x_1	x_2	s_1	s_2	s_3	Solution	Ratio
s_1	0	0	$\frac{1}{3}$	$\frac{2}{3}$	1	40	
s_2	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	220	
s_3	1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	280	
z	0	0	$\frac{2}{3}$	$\frac{8}{3}$	0	2560	

x_1-row pivot row (current) and x_2-row pivot row (new)

$$(0\ 0\ 1\ 2\ 3/120) + 3 = \left(0\ 0 - \frac{1}{3}\ \frac{2}{3}\ 1/140\right)$$

x_2-row (new)

$$= (0\ 1\ 0\ 1 - 2/140) + 2\left(0\ 0 - \frac{1}{3}\ \frac{2}{3}\ 1/140\right)$$

$$= \left(0\ 1\frac{2}{3}\ -\frac{1}{3}\ 0/220\right)$$

x_1-row (new)

$$= (1\ 0\ 0\ 0\ 1/320) - \left(0\ 0\ -\frac{1}{3}\ \frac{2}{3}\ 1/140\right)$$

$$= \left(1\ 0\ -\frac{1}{3}\ \frac{2}{3}\ 0/280\right)$$

z-row (new)

$$= (0\ 0\ 0\ 4 - 2/2480)$$

$$+ 2\left(0\ 0 - \frac{1}{3}\ \frac{2}{3}\ 1/140\right)$$

$$= \left(0\ 0\ \frac{2}{3}\ \frac{8}{3}\ 0/2560\right)$$

$$= (\geq)$$

$$\sum_{i=1}^{m} a_{ij}\, y_i$$

∴ **Decision variables**	**Optimum values**
x_1	220
x_2	280
z	2560

Some Important Consideration

Optimality Conditions (entering variable)

The entering variable in a maximization (minimization) problem is the non-basic variable having the most negative (positive) coefficient in z-row. The optimum is reached at the iteration where all the z-row coefficients of the non-basic variables are non-negative (non-positive).

Feasibility Condition (leaving variable)

For both the maximization and minimization problem, leaving variable is the basic variable associated with the smallest non-negative ratio.

For simplex method—Determine the starting basic feasible solution select entering variable using optimality condition select, leaving variable using feasibility condition determine new basic solution by Gauss-Jordan computation.

Artificial Starting Solution

For the linear programming problems in which all the constraints one of ($\leq$) type (with non-negative right hand side), the slack gives a starting basic feasible solution. When the model involves ($\geq$) and (=) constraints, then we introduce variable that are not the part of solution, known as artificial variables to get initial basic feasible solutions. The two known methods are

The Big-M method and Two phase method.

Big-M Method

It starts with linear programming problem in standard form. For any equation i that does not have a slack variable, we argument an artificial variable Ri, which become a part of initial basic solution and assign penalty in the objective function to force then to zero level at further iteration of the simplex algorithm. In objective function we use MRi in case of maximization an –MRi in case of minimization.

Here new z-row = old z-row M Σ(Ri – row) (maximization)

Now z-row = old z-row – M Σ(Ri – row) (minimization)

Standard L.P. Problem

(1) Minimize $\quad z = 4x_1 + x_2$

(2) Subject to $\quad 3x_1 + x_2 = 3$

$$4x_1 + 3x_2 - x_3 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

(3) Using the stepping-stone path, determine the maximum number of items that can be allocated to the route selected in setp 2 and adjust the allocation appropriately.

(4) Return to Setp 1.

Modi Method

Use the current solution and operations (a) and (b) below to the marginal cost of sending material over each of the unoccupied routes.

(a) Set $u_1 = 0$. Find row indices $u_2, \ldots, u_m$ and column indices $v_1, \ldots, v_n$ such that $c_{ij} = u_i + v_i$ for every occupied cell.

(b) Let $ij = c_{ij} - (u_i + v_j)$ for every unused cell : ij is the mariginal cost of introducing cell i, j into the solution.

Setp 2 through 4 are the same and in the stepping-stone method.

Assignment Problems

It is a particular case of transportation problem in which a number of operations are to be assigned to an equal number of operators, where each operator performs only one operation. Where objective is to maximize overall profit or minimize overall cost for a given assignment schedule.

Let n-jobs be assigned to n-operators. Let c_{ij} be the cost in currend in assigning i-th job to j-th operator and let

$$x_{ij} \begin{cases} 1, \text{ if } i\text{-th job is assigned to } j\text{-th operator} \\ 0, \text{ otherwise} \end{cases}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}$$

W

$$\begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then the assignment problem is the linear programming problem,

$$\text{Min. } z = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij}\, c_{ij}$$

Subject to the Constraints

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, \ldots\ldots, n$$

$$\sum_{j=1}^{m} x_{ij} = 1, i = 1, 2, \ldots\ldots, n$$

with $x_{ij} = 0$ or 1

(1) The optimum assignment schedule remains unaltered if a constant is add or substract to/from all the elements of the row or column of the assignment cost matrix.

(2) If for an assignment problem and $c_{ij} \geq 0$, then the assignment schedule (x_{ij}) which satisfies. $x_{ij}\, c_{ij} = 0$ is optimal.

(3) **Konig-Egervary theorem**—If P is a non-empty subset of the points of a matrix $A = [a_{ij}]$, then the maximum number of independent points that can be selected in P is equal the minimum number of lies covering all the elements of P.

Assignment Alogrithm (Hungarian Assignment Method)

(1) The cost matrix is square, if not make it square by adding suitable number of dummy rows (or columns) with zero cost element.

(2) Locate the smallest cost elements in each row of the cost matrix subtract this smallest element from each element of that row the resultant is that there is at lest one zero in each row of the reduced cost matrix.

(3) In the reduced cost matrix obtained, locate the smallest element in each column in it. Subtract the smallest value from every other entry in the column, the resulttant is that there is at least one zero in each of the rows and columns of the second reduced cost matrix.

(4) In second reduced cost matrix, search an optimum assignment :

(a) Examine the rows successively untiall a row with exactly one zero is found. Enterchanging this zero as an cross cut all O's in its column. Proceed in similar manner untill all the rows have been examined. If there are more than one zero in any row than do not touch that ow and pass on to the next row.

(b) Repeat the procedure for the columns of the reduced cost matrix. If there is no single zero in any row or column of the reduced matrix, then arbitrarily choose a row or column having the minimum number of O's. Arbitrarily select and enterchangel any one O in the row or column thus chosen and cross all other O's in its row and column. Repeat steps (a) and (b) until all the zero's have been either assigned or crossed.

(c) If each row and each column of the reduced matrix has one and only one assigned O, the optimum assignment is made in the cells of an-rectangled O's. Otherwise go to the next step.

(5) Draw the minimum number of horizontal and/or vertical lines through all the O's as follows:

(a) Mark (.) the rows in which assignment has not been made.

(b) Mark (.) columns which have zeros in the marked rows.

(c) Mark (.) rows (not already marked) which have assignments in marked columns.

(d) Repeat (b) and (c) untill the chain of marking is completed.

(e) Draw straight lines through all unmarked rows and marked columns.

(6) If the minimum number of lines passing through all the zero is equal to the number of rows or columns, the optimum solution is attained by an arbitrary allocation in the positions of the zeros not crosed in (3). Otherwise go to next.

(7) Revise the costs matrix os follows—

(a) Find the smallest element not covered by any of the lines of (4).

(b) Subtract this from all the uncrosed elements and add the same at the point of inter-section of the two lines.

(c) Other elements crossed by the lines remain unchanged.

(d) Go to 4 and repeat the procedure till an optimum solution is attained.

(8) **A basic Solution to the system is degenerated if :**

(A) Some basic variables are non-zero.

(B) Some basic variables are equal to zero.

(C) Some basic variable are positive.

(D) Some basic variable are negative.

OBJECTIVE TYPE QUESTIONS

1. If A = $[a_{ij}]$ is the pay off matrix, then saddle point exist when—

(A) $\min_j \max_i a_{ij} \leq \max \min_j a_{ij}$

(B) $\min_j \max_i a_{ij} \geq \max_i \min_j a_{ij}$

(C) $\min_j \max_i a_{ij} = \max_i \min_j a_{ij}$

(D) None of these

2. In two person zero sum game—

(A) Gain by one player is equal to loss by the other player

(B) Two players gains

(C) Gain by one player is more than the loss by other player

(D) None of these

3. Game is a situation where—

(A) Players have same objectives

(B) Players have conflicting objectives

(C) Players have no objectives

(D) None of these

4. If a variable x_j is unrestricted in sign in the primal linear programming problem, then the corresponding (*i.e.* the *j*th) dual constraint is—

(A) With ≤ sign if the primal is a minimization problem.

(B) With ≥ sign if the primal in a maximization problem

(C) With equality sign

(D) None of these

5. If the dual of the problem has infeasible solution, then the value of objective function is—

(A) Unbounded (B) Bounded

(C) No solution (D) None of these

6. An assignment problem is—

(A) A special case of transportation problem

(B) Dual problem

(C) Non-linear programming problem

(D) None of these

7. A basic solution to the system is non-degenerated, if—

(A) One or more basic variables vanishes

(B) No basic variable vanishes

(C) Basic variables exist

(D) None of these

8. An hyper-plane is—

(A) Convex set

(B) Not a convex set

(C) Disk with hole

(D) None of these

9. Given a system of linear equations, if B is one of the basic solution for given basic B, then—
   (A) $\vec{x}_B = \neq$
   (B) $\vec{x}_B\ \vec{b} = \vec{B}^{-1}$
   (C) $\vec{x}_B = \vec{B}^{-1}\ b$
   (D) None of these

10. Set $A \subset E^n$ is bounded below if $J\vec{r}$ for every $\vec{a} \notin A$, is—
   (A) $\vec{r} \le \vec{a}$ (B) $\vec{r} \ge \vec{a}$
   (C) $\vec{r} = \vec{a}$ (D) $\vec{r} > \vec{a}$

11. Set $A \subset E^n$ is bounded above if $J\ \vec{r}$ for every $\vec{a} \notin A$, is—
   (A) $\vec{r} \le \vec{a}$ (B) $\vec{r} \ge \vec{a}$
   (C) $\vec{r} = \vec{a}$ (D) None of these

12. If either of the dual has a finite solution then, objective functions of two problems are—
   (A) Equal
   (B) Unequal
   (C) Equal or unequal
   (D) None of these

13. The neighbourhood about a point is the set of points—
   (A) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} > \in\right\}$ (B) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} < \in\right\}$
   (C) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} = \in\right\}$ (D) None of these

14. The inside of a hypersphere with centre at and radius > 0, is the set of points—
   (A) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} < \in\right\}$ (B) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} > \in\right\}$
   (C) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} = \in\right\}$ (D) None of these

15. Let E^n be an Euclidean space, a hypersphere in E^n with centre at and radius > 0 is a set of points—
   (A) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} > \in\right\}$ (B) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} < \in\right\}$
   (C) $\left\{\dfrac{\vec{x}}{|\vec{x} - \vec{a}|} = \in\right\}$ (D) None of these

16. The equation of line passing through points $\vec{x_1}$ and $(\vec{x_1} \neq \vec{x_2})$ in E^n is—
   (A) $\lambda\vec{x_2} + (1 - \lambda)\vec{x_1} = \vec{x} \in E^n.\lambda$ any real number
   (B) $\lambda\vec{x_2} + (1 - \lambda)\vec{x_1} \in E^n, \lambda < 0$
   (C) $\lambda\vec{x_2} + (1 - \lambda)\vec{x_1} = \vec{x_1}\ E^n, 1 < x < 0$
   (D) None of these

17. Set $\left\{\dfrac{(x_1, x_2)}{(x_2 - 1)^2 + (x_2 - 3)} \ge 2\right\}$ is—
   (A) Strictly bounded
   (B) Bounded above
   (C) Bounded below
   (D) None of these

18. The set $A \subset E^n$ is bounded below, if there exist—
   (A) Upper limit
   (B) Lower limit
   (C) Upper and lower both limits
   (D) None of these

19. Given $\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ the one of the base is—
   (A) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$
   (C) $\begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 6 \\ 3 & 4 \end{bmatrix}$

20. The basic to $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ is—
   (A) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$
   (C) $\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 5 \end{bmatrix}$

21. The assignment problem—
    (A) Is a special case of transportation problem
    (B) Can be solved by simplex alogrithm
    (C) Both A and B are true
    (D) Both A and B are false

22. In cost matrix of assignment problem, from each row we select the … element, which can give atleast one zero elements in each row—
    (A) Smallest (B) Biggest
    (C) Any (D) None of these

23. The cost matrix in assignment problem is a—
    (A) Square matrix (B) Rectangle matrix
    (C) Diagonal matrix (D) None of these

24. Every basic feasible solution in convex set of solution is—
    (A) Extreme point
    (B) Boundary point
    (C) Non-extreme point
    (D) Non-boundary point

25. Set feasible solutions to $\vec{A}\ \vec{x} = \vec{b}$ is—
    (A) Convex sets
    (B) Non-convex set
    (C) Disconnected set
    (D) None of these

26. Two hyper-planes are parallel if they have—
    (A) Same unit normal
    (B) Distinct unit normal
    (C) Same normal
    (D) None of these

27. Simplex method is the—
    (A) Algebraic procedure for solving L.P.P.
    (B) Geometric procedure for solving L.P.P.
    (C) Both (A) and (B)
    (D) None of these

28. If a feasible solution of linear programming problem exist, the region of feasible solution is—
    (A) Closed convex set
    (B) Connected set
    (C) None convex set
    (D) None of these

29. If the dual of linear programming problem have a finite optimal solution, then primal process—
    (A) Finite optimal solution
    (B) Unbounded solution
    (C) No solution
    (D) None of these

30. The dual of a dual linear programming problem is—
    (A) Primal (B) Dual
    (C) Dual of a dual (D) None of these

31. The Euclidean space E^n is—
    (A) Open set
    (B) Closed set
    (C) Both open and closed set
    (D) Neither open nor closed

32. Let E^2 is Euclidean plane, a equation of circle with centre (0, 0) and radius $\in$ is—
    (A) $|\vec{x}| < \in$ (B) $|\vec{x}| > \in$
    (C) $|\vec{x}| = \in$ (D) None of these

33. The intersection of finite number of convex sets is—
    (A) Convex set (B) Open set
    (C) Closed set (D) None of these

34. The complement of a closed set is—
    (A) Open set
    (B) Closed set
    (C) Both open and closed
    (D) None of these

35. The complement of an open set is—
    (A) Open set
    (B) Closed set
    (C) Neither open nor closed
    (D) None of these

36. In a balanced transportation problem with m origins and n destinations the number of linearly independent constraints is—
    (A) $m + n$ (B) $m + n + 1$
    (C) $m + n - 1$ (D) None of these

37. A set A E^n is closed set, then—
    (A) It contains all its boundary points
    (B) It contains all its interior points
    (C) Both (A) and (B) are true
    (D) Both (A) and (B) are false

38. A set A E^n is open set, then—
    (A) It contains all its boundary points
    (B) It contains all its interior points
    (C) Both (A) and (B) are true
    (D) Both (A) and (B) are false

39. The optimal solution of the region are—
    (A) (0, 0) (0, 200) (0, 300) and (400, 0)
    (B) (0, 0) (0, 200) (300, 0) and (200, 100)
    (C) (0, 0) (0, 200) (0, 300) and (200, 100)
    (D) (0, 0) (0, 300) (400, 0) and (200, 100)

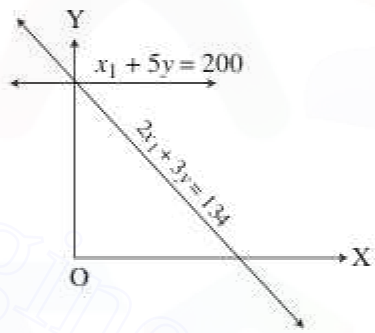
40. A company manufactures two types of telephone sets A and B. The A type telephone set requires 2 hrs and B type requires 4 hours to make. The company has 800 work hours per day 300 telephone can pack in a day. The selling prices of A and B types telephones are Rs. 300 and Rs. 400 respectively. For maximum profits company produces x telephones of A type and y telephones of B-types. Then except $x \geq 0$, and $y \geq 0$, linear constraints are—
    (A) $x + 2y \leq 400; x + y \geq 300$
    Max $z = 300x + 400y$
    (B) $2x + y \leq 400; x + y \geq 300$
    Max $z = 400x + 300y$
    (C) $2x + y \geq 400; x + y \geq 300$
    Max $z = 300x + 400y$
    (D) $x + 2y \leq 400; x + y \geq 300$
    Max $z = 300x + 400y$

41. The probable region of the linear programming problem in the above question is of the type—
    (A) Bounded (B) Unbounded
    (C) Parallelogram (D) Square

42. A whole sale merchant wants to start the business of cereal with Rs. 24,000. Wheat is Rs. 400 per quintal and rice is Rs. 600 per quintal. He has capacity for store 200 quintal cereal. He earns the profit Rs. 25 per quintal on wheat and Rs. 40 per quintal on rice. If he stores x quintal rice and y quintal wheat, then for max profit the objective function is—
    (A) $25x + 40y$ (B) $40x + 25y$
    (C) $400x + 600y$ (D) $\frac{400}{40}x + \frac{600}{25}y$

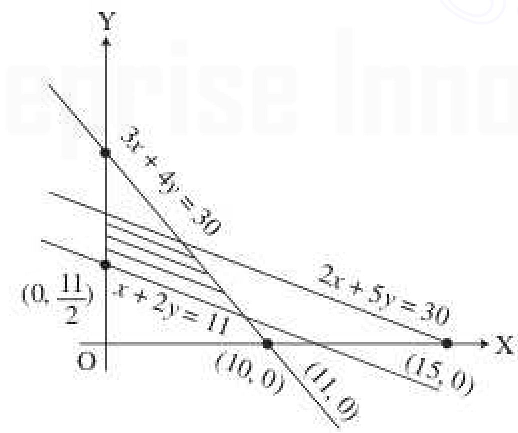
43. A firm makes pent and shirts. A shirt take 2 hours on machine and 3 hours of man labour while a pent takes 3 hours on machine and 2 hour of man labour. In a week there are 70 hours machine and 75 hours of man labour available. If the firm determine to make x shirts and y pents a week then for this linear constraints are—
    (A) $x \geq 0, y \geq 0,$ $2x + 3y \geq 70,$ $3x + 2y \geq 75$
    (B) $x \geq 0, y \geq 0,$ $2x + 3y \leq 70,$ $3x + 2y \geq 75$
    (C) $x \geq 0, y \geq 0,$ $2x + 3y \geq 70.$ $3x + 2y \leq 75$
    (D) $x \geq 0, y \geq 0,$ $2x + 3y \leq 70,$ $3x + 2y \leq 75$

44. The minimum value of objective function $c = 2x + 2y$ in the given feasible region, is—



    (A) 134 (B) 40
    (C) 38 (D) 80

45. The solution of set of constraints $x + 2y \geq 11$, $3x + 4y \leq 30$, $2x + 5y \leq 30$, $x \geq 0$, $y \geq 0$ includes the point



    (A) (2, 3) (B) (3, 2)
    (C) (3, 4) (D) (4, 3)

46. For the L.P. problem Min $z = 2x_1 + 3x_2$ such that $-x_1 + 2x_2 \leq 4$, $x_1 + x_2 \leq 6$, $x_1 + 3x_2 \geq 9$, and $x_1, x_2 \geq 0$ —
    (A) $x_1 = 1{\cdot}2, x_2 = 2{\cdot}4, z = 10{\cdot}4$
    (B) $x_1 = 2{\cdot}4, x_2 = 1{\cdot}2, z = 10{\cdot}2$
    (C) $x_1 = 1{\cdot}3, x_2 = 2{\cdot}5, z = 10{\cdot}6$
    (D) $x_1 = 1{\cdot}2, x_2 = 2{\cdot}6, z = 10{\cdot}2$

47. For the L.P. problem Min $z = x_1 + x_2$ such that $5x_1 + 10x_2 \leq 0$, $x_1 + x_2 \geq 1$, $x_2 \leq 4$ and $x_1, x_2 \geq 0$—
    (A) There is a bounded solution
    (B) There is a no solution
    (C) There is a infinite solution
    (D) None of these

48. For the L.P. problem Max $z = 3x_1 + 2x_2$ such that $2x_1 - x_2 \geq 2$, $x_1 + 2x_2 \leq 8$ and $x_1, x_2 \geq 0$, z = —
    (A) 12 (B) 24
    (C) 36 (D) 40

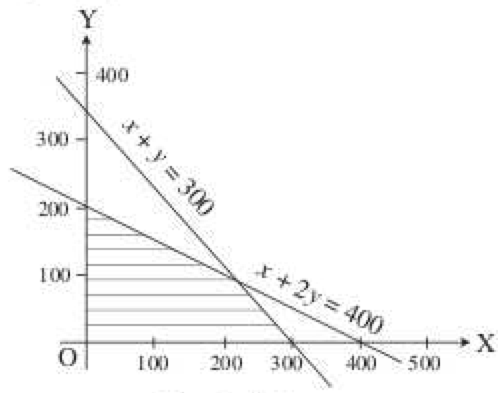
49. In Hungarian alogrithm the selected smallest elements of each row of cost matrix in assignment problem given atleast …… in each row of the reduced cost matrix—
    (A) One zero element
    (B) One positive element
    (C) One negative element
    (D) None of these

50. An assignment problem consists of—
    (A) A set of n-jobs
    (B) A set of n-operators
    (C) Cost associated with each jobs
    (D) All of these

Answer with Explanation

1. (C)	2. (A)	3. (B)	4. (C)	5. (A)
6. (A)	7. (A)	8. (A)	9. (C)	10. (A)
11. (B)	12. (A)	13. (B)	14. (A)	15. (C)
16. (A)	17. (C)	18. (B)	19. (A)	20. (A)
21. (A)	22. (A)	23. (A)	24. (A)	25. (A)
26. (A)	27. (A)	28. (A)	29. (A)	30. (A)
31. (C)	32. (C)	33. (A)	34. (A)	35. (B)
36. (C)	37. (C)	38. (B)	39. (D)	40. (B)
41. (A)	42. (B)	43. (D)	44. (D)	45. (C)
46. (D)	47. (C)	48. (B)	49. (A)	50. (D)

(Q. 39) to (41) The linear constraints are



$$x + 2y \leq 400$$
$$x + y \leq 300$$
and $$x, y \geq 0$$

Hence vertices of feasible region are (300, 0), (0, 200), (200, 100).

The optical solution in most at (200, 100)

42. Maximum profit
$$z = 40x + 25y$$

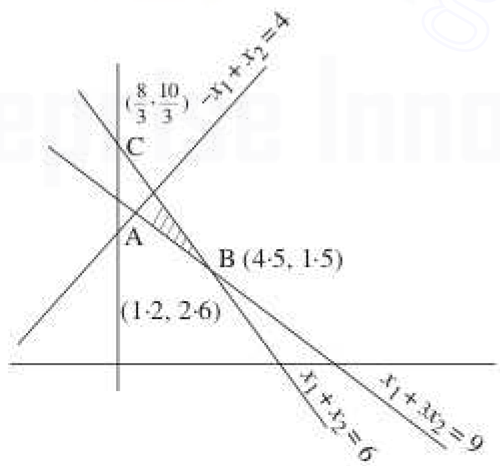
43.

	Machine	Man
Shirts : x	2 hrs	3 hrs
Pents : y	3 hrs	2 hrs

Linear constraints are
$$2x + 3y \leq 70,$$
$$3x + 2y \leq 75,$$
and $$x, y \geq 0$$

44. Min $z = 2(0) + 2(40) = 80$.

46. The graph of linear programming problem is as given below.



Hence the required feasible region given by the graph whose vertices are A(1·2, 2·6),

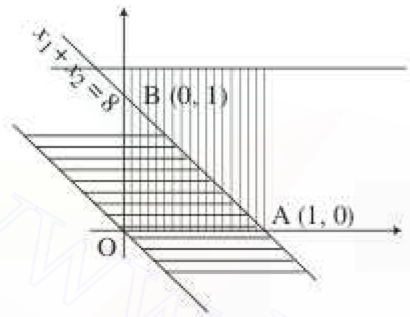
B(4·5, 1·5) and C$\left(\frac{8}{3}, \frac{10}{3}\right)$.

Thus objective function is minimum at A(1·2, 2·6).

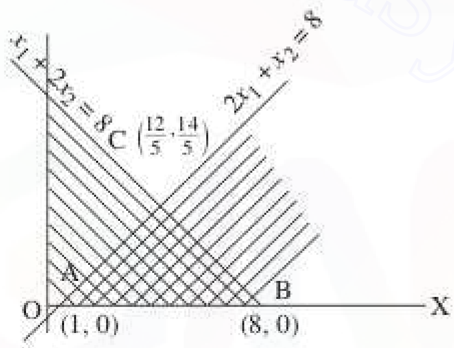
So $\quad x_1 = 1·2, x_2 = 2·6$

and $\quad z = 2 \times 1·2 + 3 \times 2·6 = 10·2.$

47. As there may be infinite x_1 and x_2's on line $x_1 + x_2 = 1$.



48. Change the inequalities into equations and draw the graph of lines, thus we get the required feasible retion.



Hence it is bounded by the vertices A(1, 0), B(8, 0) and C$\left(\frac{12}{5}, \frac{14}{5}\right)$Now by evaluation of the objective function for the vertices of feasible region it is found to be maximum at (8, 0).

Hence solution is,

$$z = 3 \times 8 + 0 \times 2 = 24.$$

Common Data Questions

Directions (Q. 1 to 4)—A departmental head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimate, of the time each man would take the perform each task, is given in the matrix below:

Tasks	Men			
	I	II	III	IV
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

1. The following is the—
   (A) Balanced problem
   (B) Unbalanced problem
   (C) Both (A) and (B) are true
   (D) None of these

2. The optimal man-hours is—
   (A) 59 (B) 49
   (C) 39 (D) 29

3. The optimal assignment is—
   (A) A — III, B — IV
   (B) A — III, C — II
   (C) B — I, C — II
   (D) A — IV, D — I

4. The following is a—
   (A) Transportation problem
   (B) Assignment problem
   (C) Game problem
   (D) None of these

Solution :

Step 1. Subtracting the smallest element of each row from every element of the corresponding row, we get the reduced matrix :

$$\begin{bmatrix} 7 & 15 & 6 & 0 \\ 0 & 15 & 1 & 13 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{bmatrix}$$

Step 2. Subtracting the smallest element of each column of the reduced matrix from every element of the corresponding column, we get the following reduced matrix :

$$\begin{bmatrix} 7 & 11 & 5 & 0 \\ 0 & 11 & 0 & 13 \\ 23 & 0 & 2 & 0 \\ 9 & 12 & 13 & 0 \end{bmatrix}$$

Step 3. Starting with row 1, we entectangle — (*i.e.*, make assignment) a single zero, if any, and cross (×) all other zeros in the column so marked. Thus we get

$$\begin{bmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & 0 & 13 \\ 23 & \boxed{0} & 2 & 0 \\ 9 & 12 & 13 & 0 \end{bmatrix}$$

In the above matrix, we arbitrarily enrectangled a zero in column 1, because row 2 had two zeros.

Step 4.

$$\begin{bmatrix} 2 & 6 & 0 & 0 \\ 0 & 11 & 0 & 18 \\ 23 & 0 & 2 & 5 \\ 4 & 7 & 8 & 0 \end{bmatrix}$$

Step 5.

$$\begin{bmatrix} 2 & 6 & 0 & 0 \\ 0 & 11 & 0 & 18 \\ 23 & 0 & 2 & 5 \\ 4 & 7 & 8 & 0 \end{bmatrix}$$

Directions (Q. 5 to 9)—

Given Min $10x_1 + 5x_2 + 5x_3$,

Subject to

$5x_1 - 5x_2 - 3x_3 \leftarrow 1$

$-x_1 + x_2 \leftarrow -3$

$x_1 - x_3 \leftarrow -7$

$-4x_1 + 4x_2 + x_3 \leftarrow 5$

x_1 ? 0

5. In the dual of this problem we have—
   (A) Max $y_1 - 3y_2 - 7y_3 - 5y_4$
   (B) Max $10x_1 + 5x_2 + 5x_3$
   (C) Max $5x_1 - 5x_2 - 3x_3 - 1$
   (D) None of these
6. The problem have—
   (A) Possible solution
   (B) No-feasible solution
   (C) Bounded solution
   (D) None of these
7. The problem have—
   (A) Bounded solution
   (B) Unbounded solution
   (C) Feasible solution
   (D) None of these
8. The dual of the problem have—
   (A) Non-negative variables
   (B) Negative variables
   (C) Unrestricted variables
   (D) None of these
9. In the dual, we have—
   (A) Max $y_1 - 3y_2 - 7y_3 + 5y_4$
   and $y_i \square \leftarrow \leftarrow 0, i = 1, 2, 3, 4$
   (B) Min $y_1 - 3y_2 - 7y_3 + 5y_4$
   $y_i \leftarrow 0, i = 1, 2, 3, 4$
   (C) Both (A) and (B) are true
   (D) None of these

Solution :

The dual is

Max $y_1 - 3y_2 - 7y_3 + 5y_4$

Subject to

$5y_1 - y_2 + y_3 - 4y_4 \leftarrow 10,$

$-5y_1 + y_2 + 4y_4 \leftarrow 5,$

$-3y_1 - y_3 + y_4 \leftarrow 5,$

$y_i = 0.$ (I = 1, 2, 3, 4)

The iterations are given as follows. The solution is unbounded.

Hence there is no feasible solution of the given L.P.

Table

Basic	y_1	y_2	y_3	y_4	s_1	s_2	s_3	Sol.
y_0	−1	3	7	−5	0	0	0	0
s_1	5	−1	1	−4	1	0	0	10
s_2	−5	1	0	4	0	1	0	5
s_3	−3	0	−1	1	0	0	1	5
	$\frac{-29}{4}$	$\frac{17}{4}$	7	0	0	$\frac{5}{4}$	0	$\frac{25}{4}$
s_1	0	0	1	0	1	1	0	15
y_4	$\frac{-5}{4}$	$\frac{1}{4}$	0	1	0	$\frac{1}{4}$	0	$\frac{5}{4}$
s_3	$\frac{-7}{4}$	$\frac{-1}{4}$	−1	0	0	$\frac{-1}{4}$	1	$\frac{15}{4}$

The optimal assignment

A — III, B — I, C — II and D — IV.

The minimum total time is 17 + 13 + 19 + 10 = 59 man hours.

10. Max $z = 5x_1 + 12x_2 + 4x_3$,
    Subject to $x_1 + 2x_2 + x_3 \leftarrow 5$,

$$2x_1 - x_2 + 3x_3 = 2,$$
$$x_1, x_2, x_3 \geq 0.$$

We change the equality into inequalities and bring the LPP in canoncial form.

Solution :

Max $z = 5x_1 + 12x_2 + 4x_3$,

Subject to $x_1 + 2x_2 + x_3 \leq 5$, y_1

$$2x_1 - x_2 + 3x_3 \leq 2 \; y_2^+$$
$$-2x_1 + x_2 - 3x_3 \leq -2, \; y_3^-$$
$$x_1, x_2, x_3 \geq 0$$

The dual of above is

Min $w = 5y_1 + 2(y_2^+ - y_2^-)$

Subject to

$$y_1 + 2(y_2^+ - y_2^-) \geq 5,$$
$$2y_1 - (y_2^+ - y_2^-) \geq 12,$$
$$y_1 + 3(y_2^+ - y_2^-) \geq 4,$$
$$y_1, y_2^+, y_2^- \geq 0$$

The above is same as

Min $w = 5y_1 + 2y_2$

Subject to
$$y_1 + 2y_2 \geq 5$$
$$2y_1 - y_2 \geq 12,$$
$$y_1 + 3y_3 \geq 4,$$
$$y_1 \geq 0 \text{ and } y_2$$

unrestricted in sign.

Calculus of Variations and Integral Equations

(1) Calculus of Variations

The First Variation

Admissible Arc—It is the arc of integration such that the integral $I = \int_a^b F\left(x, y, \frac{dy}{dx}\right) dx$, can be determined, where y is a function of x and $F = F(x, y, y')$ process a continuous partial derivatives.

Weak Variation—(1) Let $y = S(x)$ and $y = S(x) + \varepsilon t(x)$ be two admissible curves of integral, $I = \int_a^b F\left(x, y, \frac{dy}{dx}\right) dx$, where ε is an arbitrary constant independent of x and y, $t(x)$ is an arbitrary function of x, independent of ε. With this restriction on $t(x)$, the ordinate y is said to be a weak variation.

(2) The variation y is said to be of weak variation if $t(x)$ and $t'(x)$ are of same order of smallness.

First Variation—The integral $I_1 = \varepsilon \int_a^b \left(t\frac{\partial F}{\partial S} - \frac{dt}{dx}\frac{\partial F}{\partial S}\right) dx$, where $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$.

Second Variation—The integral $I_2 = \frac{\varepsilon}{2!} \int_a^b \left(t^2\frac{\partial^2 F}{\partial S^2} - 2tt'\frac{\partial^2 F}{\partial S \partial S'} + t'^2 \frac{\partial^2 F}{\partial S'^2}\right) dx$ where $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$.

Euler's Characteristic Equation—Given the function $F = F(x, y, y')$, the differential equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$, is referred as Euler's characteristic equation. The Euler's characteristic equation is also known as Euler-Lagrange equation.

Extremals—It is solution to Euler's characteristic equation.

Discontinuous Solutions—Discontinuous solutions are obtained by joining several continuous arcs, each of which satisfies the Eulerian characteristic equation. The corners occur at the points of junctions.

Some Important Theorems

(1) The integral $I = \int_a^b F(x, y, y')\, dx$, whose end points are fixed, is stationary for weak variations, if y satisfies Euler's characteristic equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$.

(2) The integral $\int_a^b F(y, y')\, dx$, whose ends are fixed is stationary for weak variation if y satisfies the differential equation $F - y'\frac{\partial F}{\partial y'} = C$, where C is an arbitrary constant.

(3) The integral $\int_a^b F(x, y')\, dx$, whose ends are fixed, is stationary for weak variations if y satisfies the differential equation $\frac{\partial F}{\partial y'} = C$, where C is an arbitrary constant.

(4) **Legender's test**—If (i) Euler's characteristic equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ is satisfied, (ii) the range of integration (a, b) is sufficiently small and (iii) the sign of $\frac{\partial^2 F}{\partial y'^2}$ is constant throughout this range. Then stationary value $I_s = \int_a^b F(x, S, S')\, dx$ of integral $I = \int_a^b F(x, y, y')\, dx$, is a maximum if $\frac{\partial^2 F}{\partial y'^2} < 0$ and minimum if $\frac{\partial^2 F}{\partial y'^2} > 0$.

(5) If Euler's characteristic equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$ vanishes identically, then the indefinite integral $\int F(x, y, y')\, dx$ can be evaluated as a function of x and y. The integral has then a value which is independent of the path of integration and which is a function of the end positions only.

The Second Variation

Second Variation—The integral $I_2 = \frac{\varepsilon}{2!} \int_a^b (t^2F_{00} + 2tt'F_{01} + t'^2F_{11})dx$.

Jacobi's Accessory Equation—The accessory equation $\left\{F_{00} - \frac{d}{dx}(F_{01})\right\} u - \frac{d}{dx}\left\{F_{11}\frac{du}{dx}\right\} = 0$, where $\frac{\partial^2 F}{\partial S^2} = F_{00}$, $\frac{\partial^2 F}{\partial S \partial S'} = F_{01}$ and $\frac{\partial^2 F}{\partial S'^2} = F_{11}$.

Conjugate Points (Kinetic foci)—If (i) $u(x)$ is the solution of the accessory equation, (ii) a is the abscissa of the point A, and (iii) $u(a) = 0$, then the roots of $u(x) = 0$ are the abscissae of the point on the curve $y = S(x)$ conjugate to A.

Some Important Theorems

1. If $t(a) = t(b) = 0$

$$\int_a^b (t^2F_{00} + 2tt'F_{01} + t'^2F_{11})dx$$

$$= \int_a^b \left\{t^2F_{00} - t^2\frac{d}{dx}(F_{01}) - t\frac{d}{dx}(t'F_{11})\right\} dx.$$

2. If $t(a) = t(b) = 0$ and u is a solution of $\left\{F_{00} - \frac{d}{dx}(F_{01})\right\} u - \frac{d}{dx}\left\{F_{11}\frac{du}{dx}\right\} = 0$, then

$$I_2 = \frac{\varepsilon}{2!}\int_a^b F_{11}\left(t' - t\frac{u'}{u}\right)^2 dx.$$

3. Let $y = S(x)$ be the equation of the extremal through the fixed points A and B for which the integral $I = \int_a^b F(x, y, y')dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, then for weak variation I is a maximum when $\frac{\partial^2 F}{\partial S'^2}$ is negative and a minimum when it is positive.

- If (i) a is the abscissa of the point A, (ii) $u_1(x)$ and $u_2(x)$ are independent solutions of the accessory equation, then the equation for the abscissa of the points conjugate to A is, $\frac{u_1(x)}{u_2(x)} = \frac{u_1(a)}{u_2(a)}$.
- If $y = S(x, c_1, c_2)$ is the general solution of Euler's characteristic equation, then the equation for abscissa of points conjugate to A is $\frac{\partial y}{\partial c_1} / \frac{\partial y}{\partial c_2} = \left(\frac{\partial y}{\partial c_1} / \frac{\partial y}{\partial c_2}\right)_{x=a}$

4. If the extremals, which pass through the point A have an envelope E, then the points of contact of E and the extremal AB are the points conjugate to A.

5. If $u(x)$ a solution of the accessory equation, it cannot have double zero.

6. If $u_1(x)$ and $u_2(x)$ are independent solutions of the accessory equation, then the ratio $u_1(x)/u_2(x)$ steadily increases or steadily decreases as x increases.

The Generalized Results

1. Let the values of t_0 and t_1 and the functional form of F be given, then the integral $\int_{t_0}^{t_1} F(q_1, q_2 \ldots, q_n : \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n ; t)\, dt$, where the q_i, $i = 1, 2, \ldots, n$ are arbitrary functions of t, is stationary for weak variations when the q_i, $i = 1, 2, \ldots n$ satisfy the n equations $\frac{\partial F}{\partial q_m} - \frac{\partial}{dt}\left(\frac{\partial F}{\partial \dot{q}_m}\right) = 0$ $(m = 1, 2, \ldots, n)$.

2. Let $\int_{t_0}^{t_1} F(q_1, q_2 \ldots, q_n : \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n ; t)\, dt$, where the q_i, $i = 1, 2, \ldots n$ satisfy the n second order partial differential equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_m}\right) = 0$ $(m = 1, 2, \ldots n)$. If at every point of a sufficiently small range t_0 to t_1 (i) $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ $(r = 1, 2, \ldots, n)$ all have constant sign, (ii) $\left(\frac{\partial^2 F}{\partial \dot{q}^2_r}\right)\left(\frac{\partial^2 F}{\partial \dot{q}^2_s}\right) > \left(\frac{\partial^2 F}{\partial \dot{q}_r \partial \dot{q}_s}\right)$, $\left(\begin{matrix} r = 1, 2, \ldots, n \\ s = 1, 2, \ldots, n \end{matrix}, r \neq s\right)$. Then I is a maximum if the sign of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ are all negative and minimum if they are all positive [evidently (ii) ensures that they all have the same sign].

3. Let $\frac{d^m y}{dx^m}$ be denoted by y_m and let the values of $y, y_1, y_2, \ldots, y_{n-1}$ be given for both $x = a$ and $x = b$. Also let the functional form of F be given : then the integral $I = \int_a^b F(x, y, y_1, y_2, \ldots y_n)\, dx$ is stationary when y satisfies the equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_1}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y_2}\right) - \ldots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y_n}\right) = 0.$

4. Let F be a given functional form, let $p_r = \frac{\partial z}{\partial x_r}$, where z is a function of $x_1, x_2, \ldots, x_n$, and let the integral $I = \iiint \ldots \int F(z, x_1, x_2, \ldots, x_n : p_1, p_2, \ldots, p_n)\, dx_1\, dx_2 \ldots dx_n$ be taken through an n-dimensional region bounded by given fixed boundaries of dimension $n - 1$.

Then the integral I stationary when z is a solution of the second order partial differential equation $\frac{\partial F}{\partial z} - \frac{\partial}{dx_1}\left(\frac{\partial F}{\partial p_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial p_2}\right) - \frac{\partial}{\partial x_n}\left(\frac{\partial F}{\partial p_n}\right) = 0$.

Relative Maxima and Minima

1. By suitable choice of the constants λ, a solution of second order equation $\frac{\partial}{\partial y}(F - \lambda\phi) - \frac{d}{dx}\left\{\frac{d}{\partial y_1}(F - \lambda\phi)\right\} = 0$ can be found which renders the integral I of equation $I = \int_a^b F(x, y, y_1)\, dx$ stationary, which passes through the end points of the range of integration of $\int_a^b F(x, y, y_1)$ and $\int_a^b \phi(x, y, y_1)\, dx$.

Isoperimetrical Problem

1. By suitable choice of arbitrary constant and of the constant λ, it is possible to find solution of $\frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) - \lambda\left\{\frac{\partial \psi}{\partial x} - \frac{d}{dt}\left(\frac{\partial \psi}{\partial \dot{x}}\right)\right\} = 0$ and $\frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) - \lambda\left\{\frac{\partial \psi}{\partial y} - \frac{d}{dt}\left(\frac{\partial \psi}{\partial \dot{y}}\right)\right\} = 0$ which makes the integral of equations $I = \rho g \int_A^B y\, ds$ stationary and at same time satisfy condition $l = \int_A^B ds$.

2. Solution of the simultaneous equations $\frac{\partial G}{\partial x} - \frac{d}{dx}\left(\frac{\partial G}{\partial \dot{x}}\right) - \left\{\mu\frac{\partial S}{\partial x} - \frac{d}{dt}\left(\mu\frac{\partial S}{\partial \dot{x}}\right)\right\} = 0$ and $\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial \dot{y}}\right) - \left\{\mu\frac{\partial s}{\partial y} - \frac{d}{dt}\left(\mu\frac{\partial s}{\partial \dot{y}}\right)\right\} = 0$ can be found which satisfies equation $S(x, y, t, \dot{x}, \dot{y}) = 0$ and which make the integral $I = \int_{t_1}^{t_2} G(x, y, t, \dot{x}, \dot{y})\, dt$, stationary.

3. If the range of integration of the integral $I = \int_a^b F(x, y, y_1)\, dx$ is sufficient small, and if the expression $E = \frac{\partial^2 F}{\partial y_1^2} - \lambda\frac{\partial^2 \phi}{\partial y_1^2}$ has the same sign throughout this range, then the integral I has a maximum if E is negative and minimum if E is positive.

Integral with Variable End Points

1. If the end points A and B of the range of integration of the integral $I = \int_a^b F(x, y, y')\, dx$ can be displaced along prescribed curves, then I is stationary when the following conditions (necessary) satisfied :

(i) The ordinate of the external y, satisfies Eulerian equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$,

(ii) At $x = a$, $F + (g'_1 - y')\frac{\partial F}{\partial y'} = 0$, where a is the abscissa of the end point A, which can be displaced along the curve $y = g_1(x)$.

(iii) At $x = b$, $F + (g'_2 - y')\frac{\partial F}{\partial y'} = 0$, where b is the abscissa of the end point B, which can be displaced along the curve $y = g_2(x)$.

Strong Maximum and Minimum

1. **Necessary Condition**—The integral I has strong maximum if :

(i) The equation of $\lceil e$, the arc of integration, satisfy $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$.

(ii) The arc AB of $\lceil e$ contains no point conjugate to either A or B.

(iii) At all points of this arc $E = \frac{\partial^2 F}{\partial y'^2} < 0$.

(iv) At all points of this arc and for all finite values of p, $E(x, y, y', p) \leq 0$.

For strong minimum, the inequality are reversed.

2. **Sufficient Conditions**—The integral I has strong maximum if :

(i) The equation of $\lceil e$, the arc of integration, satisfies $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$.

(ii) The arc AB of $\lceil e$, contains no point conjugate to either A or B.

(iii) At all points of the arc AB, of $\lceil e$, and for all finite values of p, $\frac{\partial^2 F(x, y, p)}{(\partial p^2)} < 0$.

For strong minimum the inequality sign is reversed.

3. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$, have strong maximum iff (i) the variable x and y, which are functions of t, either satisfy two equations,

$$\left.\begin{aligned} \frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) &= 0 \\ \frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) &= 0 \end{aligned}\right\} \quad \ldots(1)$$

or the single equivalent equation $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial x \partial \dot{x}} - S(x, y, \dot{x}, \dot{y})(\ddot{x}\dot{y} - \ddot{y}\dot{x}) = 0$. Here $\lceil e$ is the curve, which passes through the fixed ends A and B satisfies the parametric equations (1).

(ii) The arc AB of $\lceil e$ contains no points conjugate either to A or to B.

(iii) At all points of this arc $S(x, y, \dot{x}, \dot{y}) > 0$, where

$$S(x, y, \dot{x}, \dot{y}) = \frac{1}{\dot{y}^2}\frac{\partial^2 G}{\partial \dot{x}^2} = \frac{-1}{\dot{x}\dot{y}}\frac{\partial^2 G}{\partial \dot{x}\partial \dot{y}} = \frac{1}{\dot{x}^2}\frac{\partial^2 G}{\partial \dot{y}^2}$$

(iv) At all points of this arc $E(x, y, \dot{x}, \dot{y}, p, q) < 0$ for every pair of finite values of p and q other, than $p = \dot{x}$ and $q = \dot{y}$ simultaneously.

4. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where A, B are fixed and where $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$ iff

(i) The variables x and y which are functions of t, either satisfy

$$\left.\begin{aligned} \frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) &= 0 \\ \frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) &= 0 \end{aligned}\right\} \quad \ldots(1)$$

or $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial y \partial \dot{x}} - S(x, y, \dot{x}, \dot{y})(\ddot{x}(\dot{y}) - \ddot{y}(\dot{x})) = 0$

Here $\lceil e$ is the curve which passes through A and B and satisfies equation (1).

(ii) The arc AB of $\lceil e$ contains no points conjugate either to A or to B.

(iii) $S(x, y, p, q) < 0$ at every point of AB and for every pair of p and q.

- For strong minimum the sign of the in equality be reversed.

(2) Integral Equations

Integral Equations

Integral Equations—An unknown function appears under the integral sign in these equations.

Fredholm Integral Equations—Let $k(x, y)$ be a given real function defined for $0 \le x \le 1$, $0 \le y \le 1$ and $f(x)$ a real valued function defined for $0 \le x \le 1$ and λ an arbitrary complex number. Then

(i) Linear Fredholm integral equation of second kind (for a function $\phi(x)$) is $\phi(x) - \lambda \int_0^1 k(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le 1)$.

(ii) Linear Fredholm integral equation of first kind (for a function $\phi(x)$) is $\int_0^1 k(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le 1)$.

Note—For a sake of convenience the basic domain is assumed to be $0 \le x \le 1$, the basic domain can be $a \le x \le b$ or any bounded set on the x-axis.

Algebraic System of Equations—The problem of solving Fredholm integral equations of first and second kind is considered as a generalization of the problem of solving a set of n-linear algebraic equations in n-unknowns :

$$\sum_{S=1}^{n} a_{rs}x_s = b_r \; (r = 1, 2, \ldots, n).$$

Volterra Equations—Let $k(x, y)$ be a given real function defined for $0 \le x \le 1$, $0 \le y \le 1$, $f(x)$ a real valued function for $0 \le x \le 1$ and λ an arbitrary complex number, with $k(x, y) \equiv 0$ if $y > x$, then

(i) Voltera equation of first kind (for a function of $\phi(x)$ is $\int_0^x k(x, y)\, \phi(y)\, dy = f(x)$

(ii) Voltera integral equation of second kind (for a function $\phi(x)$ is $\phi(x) - \lambda \int_0^x k(x, y)\, \phi(y)\, dy = f(x)$.

Volterra Equations

L_2–Kernel—A kernel $k(x, y)$ which is quadratically integrable in the square $(0 \le x \le h, 0 \le y \le h)$. *i.e.*, $k(x, y)$ with norm $\|k\|$ such that $\|k\|^2 = \int_0^h \int_0^h k^2\, k^2(x, y)\, dxdy \le N^2$ exist, where N^2, a constant.

L_2–Function—A function (f), whose norm is given by $\|f\|^2 = \int_0^b f^2(x)\, dx$.

L^*_2–Function—A function f, which is L_2–function and bounded.

Solutions of Volterra Integral Equation of Second Kind

1. Volterra integral equation of second kind can be solved by Picard's process of successive approximations : $k_{n+1}(x, y) = \int_0^x k(x, z)\, k_n(z, y)\, dz$, $(h = 1, 2, \ldots)$.

2. The Volterra integral equation of the second kind $\phi(x) - \lambda \int_0^x k(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le h)$, where the kernel $k(x, y)$ and the function $f(x)$, belongs to the class L_2 has one and essentially only one solution in the same class L_2. The solution is given by $\phi(x) = f(x) - \lambda \int_0^x H(x, y, \lambda)\, f(y)\, dy$, where $H(x, y, \lambda)$ is the resolvent kernel given by the series of iterated kernels $H(x, y, \lambda) = -\sum_{n=0}^{\infty} \lambda^n k_{n+1}(x, y)$. This series converges almost every where. The resolvent kernel satisfies the integral equation $k(x, y) + H(x, y, \lambda) = \lambda \int_x^y k(x, z)\, H(x, y, \lambda) dz = \lambda \int_y^x H(x, z, \lambda)\, k(z, y)\, dz$.

3. Any solution of homogeneous Volterra integral equations of second kind $\phi(x) - \lambda \int_0^x k(x, y)\, \phi(x)\, dy = 0$ in L_2–space is necessarily a zero function.

4. By uniqueness theorem, the equations $k(x, y) + H(x, y, \lambda) = \lambda \int_y^x k(x, z)\, H(z, y, \lambda)\, dz = \lambda \int_y^x H(x, z, \lambda)\, k(z, y)\, dz$ are characteristic for the resolvent kernel $H(x, y, \lambda)$ in the space L_2. For only L_2–function satisfying above equation, it is the resolvent kernel corresponding to the kernel $k(x, y)$.

Solutions of Volterra Equation of First Kind

If in a Volterra equation of first kind $\int_0^x k(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le h)$. ...(1)

The diagonal $k(x, x)$ vanish now here in the basic interval $(0, h)$ and if the derivatives $\frac{df(x)}{dx} \equiv f'(x)$, $\frac{\partial k}{\partial x} \equiv k'_x(x, y)$ and $\frac{\partial k}{\partial x} \equiv k'_x(x, y)$ exist and continuous, the equation can be reduced to one of the two way :

1. Differentiate (1) with respect to x, one obtain

$$\phi(x) - \int_0^x \frac{k'_x(x, y)}{k(x, x)}\, \phi(y)\, dy = \frac{f'(x)}{k(x, x)}$$

2. Using $\int_0^x \phi(x)\, dy = \Phi(x)$, one get $\Phi(x) = \frac{f(x)}{k(x, x)} - \int_0^x \frac{k'_y(x, y)}{k(x, x)}\, \Phi(y)\, dy = \frac{f(x)}{k(x, x)} - \int_0^x H^*(x, y, 1) \frac{f(y)}{k(y, y)}\, dy$, where $H^*(x, y, \lambda)$ is the resolvent kernel corresponding to $\frac{k'_y(x, y)}{k(x, x)}$ and $f(x)$ is differentiable.

Fredholm Equation of Third Kind

The equations $k(x, x)\, \phi(x) + \int_0^x k'_x(x, y)\, \phi(y)\, dy = f'(x)$ and $f(x) = [k(x, y)\, \Phi(y)]_{y=0}^{y=x} - \int_0^x k'_y(x, y)\, \Phi(y)\, dy$, where $k(x, x)$ vanishes at $x = 0$.

Linear Differential Equations and Volterra Integral Equations

Linear differential equation

$$a_0(x) \frac{d^n u}{dx^n} + a_1(x) \frac{d^{n-1} u}{dx^{n-1}} + \ldots + a_n(x)\, u = F(x) \qquad \ldots(1)$$

with initial conditions,

$$u(0) = c_0,\ u'(0) = c_1,\ \ldots,\ u^{n-1}(0) = c_{n-1} \qquad \ldots(2)$$

can be reduced to Volterra equation of second kind.

$$a_0(x)\, \phi(x) + \int_0^x k(x, y)\, \phi(y)\, dy = f(x) \qquad \ldots(3)$$

$$\text{where } k(x, y) = \sum_{n=1}^{n} a_n(x) \frac{(x-y)^{n-1}}{(n-1)!} \qquad \ldots(4)$$

$$f(x) = F(x) - c_{n-1}\, a_1(x) - (c_{n-1}\, x + c_{n-2})\, a_2(x) - \ldots - \left(c_{n-1}\frac{x^{n-1}}{(n-1)!} + \ldots + c_1 x + c_0\right) a_n(x) \quad \ldots(5)$$

1. If the leading coefficient in (1) $a_0(x) = 1$ then the equation (1) reduced to, $\phi(x) + \int_0^x k(x, y)\, \phi(x)\, dy = f(x)$, where k and F are given by equations (4) and (5) respectively.

2. If $a_0(x) = 0$, the equation (1) reduces to Volterra equation of first kind.

Equations of Faltung Type (Closed Cycle/Convolution Type)

The equation $\phi(x) - \lambda \int_0^x k(x, y)\, \phi(y)\, dy = f(x)$ is called equation of Faltung type, which can be written as $y(x) = f(x) + k(x) * y(x)$ and solve by Laplace transformation.

Able Equation—The equation $\int_0^x \frac{\phi(y)}{(x-y)^{\alpha}})\, dy = f(x)$, $(0 < \alpha < 1)$ and its solution is given by

$$\phi(x) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dx} \int_0^x \frac{f(y)}{(x-y)^{1-\alpha}}\, dy$$

$$= \frac{\sin(\alpha\pi)}{\pi} \left[\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(y)}{(x-y)^{1-\alpha}}\, dy\right]$$

Fredholm Equations

Fredholm Linear Operator—$F_x[\phi(y)] \equiv \phi(x) - \lambda \int_0^1 K(x, y)\, \phi(y)\, dy$.

Associated Fredholm Operator—$F_x^*[\phi(y)] \equiv \phi(x) - \lambda \int_0^1 k(y, x)\, \phi(y)\, dy$.

Green's Formula—$\int F_x[\phi(y)]\, \psi(x)\, dx = \int F_x^*[\psi(y)]\, \phi(x)\, dx$.

Pincherie-Goursat Kernel—$k(x, y) = \sum_{k=1}^{n} X_k(x)\, Y_k(x)$, where $\{X_i(x)\}$ and $\{Y_i(x)\}$ are two sets of linearly independent (L_2) functions in the basic interval (0, 1).

1. To each quadratically integrable kernel $k(x, y)$, there corresponds a resolvent kernel H (x, y', λ) which is an analytic function of λ regular at least inside the circle $|\lambda| < \|k\|^{-1}$ and represented there by the power series

$-H(x, y, \lambda) = k(x, y) + \lambda\, k_2(x, y) + \lambda^2\, k_3(x, y) + \ldots$

Let the domain of existence of the resolvent kernel in complex λ-plane be $\hbar$: Then if $f(x)$ belong to the class L_2, the unique, quadratically integrable solution of Fredholm equation of second kind.

$\phi(x) - \lambda \int k(x; y)\, \phi(y)\, dy = f(x)$, valid in $\hbar$ is given by the formula $\phi(x) = f(x) - \lambda \int H(x, y; \lambda)\, f(y)\, dy$.

2. Fredholm integral equation of second kind, $\phi(x) - \lambda \int k(x, y)\, \phi(y)\, dy = f(x)$ has one and only one solution of the class L_2 given by formula $\phi(x) = f(x) - \lambda \int H(x, y; \lambda)\, f(y)\, dy$. Here $H(x, y; \lambda)$ the resolvent kernel is an analytic function of λ ...

If $|\lambda| \le \|k\|^{-1}$, it is give by the Neumann series $-H(x, y, \lambda) = k(x, y) + \lambda k_2(x, y) + \lambda^2 k_3(x, y) + \ldots$, where $k_2, k_3, \ldots$ are iterated kernels. The only exceptions are the singular points of $H(x, y; \lambda)$ which coincide with the zeros (eigen values) of an analytic function $D(\lambda)$ of λ. If kernel is the Pincheri-Goursat kernel then $D(\lambda)$ is a polynomial.

3. If the homogeneous Fredholm integral equation has the only trivial solution, then the corresponding non-homogeneous equation always has one and only one solution. On the contrary if the homogeneous equation has non-trivial solutions, then the non-homogeneous integral equation has either no solution or infinity of solutions, depending on the given function $f(x)$.

4. A Volterra integral equation has no eigen values.

Symmetric Kernels and Orthogonal Systems of Functions

Orthogonal System of Functions—Given a system of functions $\{\phi_n\} = \phi_1(x), \phi_2(x), \ldots$ where $\phi_i, \phi_j\ (i \ne j)$ satisfies orthogonal condition $(\phi_i, \phi_j) = \int_a^b \phi_i(x)\, \phi_j(x)\, dx = 0\ (i \ne j)$ in basic interval (a, b).

Orthogonal System—An orthogonal system of functions $\{\phi_n\}$ satisfying normalizing condition $(\phi_i, \phi_j) = \begin{cases} 0, & i \ne j \\ 1, & i = j \end{cases}$.

Complete Orthonormal System—An orthogonal system is called complete in L_2 if Parseval's equation $\sum_{h=1}^{\infty} a^2{}_h = \int_a^b f^2(x)\, dx$ holds for any

function $f(x)$ of L_2, where a_n is the Fourier coefficient of the function $f(x)$ with respect to the system $[\phi_h]$,

$$a_n = \int_a^b f(x)\, \phi_n(x)\, dx;\ n = 1, 2, \ldots$$

$k \in L^*_2$—if function $A(x)$ related to the kernel k is bounded, *i.e.* $\int k^2(x, y)\, dy = A^2(x) < N^2$.

Theorems

1. For a symmetric kernel $k(x, y) = k(y, x)$, the associated eigen function ψ_h coincide with the proper eigen functions $\{\phi_h\}$.

2. Any pair $\phi_h(x)$ and $\phi_k(x)$ of eigen functions of a symmetric kernel, corresponding to two different eigen values λ_h and λ_k satisfies the orthogonality condition

$$(\phi_h, \phi_k) \equiv \int_a^b \phi_h(x)\, \phi_k(x)\, dx = 0,\ (h \neq k)$$

3. **Process of Orthogonalization**—Given any (finite or denumerable) system of linearly independent L_2–functions $\psi_1(x), \psi_2(x), \ldots$ it is always possible to find constants hrs such that the functions

$$\phi_1(x) = \psi_1(x)$$
$$\phi_2(x) = h_{21}\psi_1(x) + \psi_2(x)$$
$$\phi_3(x) = h_{31}\psi_1(x) + h_{32}\psi_2(x) + h_{33}\psi_3(x)$$
$$\vdots$$
$$\vdots$$
$$\phi_n(x) = h_{n1}\psi_1(x) + h_{n2}\psi_2(x) + \ldots + h_{nn-1}\ \psi_{n-1} + \psi_n(x)$$

are orthogonal in the basic interval (a, b).

4. For a given system $\{\phi_h\}$, a given function $f(x)$ and a given number n of terms of the linear combination $c_1\phi_1(x) + c_2\phi_2(x) + \ldots$, the non-negative integral.

$$I_n = \int_a^b f^2(x)\, dx - 2\sum_{h=1}^{n} a_h c_h + \sum_{n=1}^{n} c^2_h$$
$$= \int_a^b f^2(x)\, dx - \sum_{h=1}^{n} a^2_h + \sum_{h=1}^{n} (a_h - c_h)^2$$

attains its minimal value iff the coefficients c_h coincide with the corresponding Fourier coefficients

$$a_h = \int_a^b f(x)\, \phi_h(x)\, dx,\ (h = 1, 2, \ldots n)$$

of the function $f(x)$.

And this minimal value is given by

$$I_n^* = \int_a^b f^2(x)\, dx - \sum_{h=1}^{n} a^2_h$$
$$= \left[\int_a^b f^2(x)\, dx - \sum_{n=1}^{\infty} a^2_h\right] + \sum_{h=n+1}^{\infty} a^2_h$$

5. A necessary and sufficient condition for an L_2–function $f(x)$ to be approximated in the mean by a linear combination of $f(x)$ of a given orthonormal system is that Parseval's equation $\sum_{n=1}^{\infty} a^2_h = \int_a^b f^2(x)\, dx$ holds for given $f(x)$.

6. **Weyl Lemma**—A necessary and sufficient condition for the convergence in the mean over the interval (a, b) of a sequence. Of L_2–function $\{f_n\}$ to $f(x)$ is that for any $\varepsilon > 0$, $\exists$ an integer N such that for $m, n \geq N$, we have

$$\int_a^b [f_m(x) - f_n(x)]^2 dx < \varepsilon$$

7. **Rietz–Fischer Theorem**—Given sequence of L_2–functions $\{f_n\}$ over the interval (a, b) and for every $\varepsilon > 0$ there exist an integer N such that for $m, n \geq N$ we have $\int_a^b [f_m(x) - f_n(x)]^2\ dx \leq \varepsilon$, then there exist a function $f \in L_2$–function such that $\{f_n\}$ convergence to $f(x)$.

8. Approximation of general L_2–kernel by means of Pincherle–Goursat kernel–Any L_2–kernel $k(x, y)$ *i.e.*, any kernel for which both functions.

$$A(x) = \left[\int_0^1 k^2(x, y)\, dy\right]^{1/2}$$

and
$$B(y) = \left[\int_0^1 k^2(x, y)\, dx\right]^{1/2}$$

exist almost everywhere in the basic interval (0, 1) and belong to the class L_2, can be decomposed into the sum of suitable Pincherle–Goursat kernel

$$S(x, y) = \sum_{k=1}^{n} X_k(x)\, Y_k(x)$$

and L_2–kernel $T(x, y)$ for $\varepsilon > 0$,

$$\|T\|^2 = \int_0^1 \int_0^1 T^2(x, y)\, dxdy < \varepsilon^2$$

9. Every symmetric, non-zero L_2–kernel has at least one eigen value.

10. Any L_2–function $w(x)$ is orthogonal to all the eigen functions $\phi_h(x)$ of symmetric kernel $k(x, y)$ iff $\int k(x, y)\, w(y)\, dy = 0$.

11. **Hilbert-Schmidt Theorem**—If $f(x)$ can be written in the form

$$f(x) = \int k(x, y)\, g(y)\, dy \qquad \ldots(1)$$

where $k(x, y)$ is a symmetric L_2–kernel and $g(y)$ is an L_2–function, then $f(x)$ can also be represented by its Fourier series with respect to the orthonormal system h_n of eigen functions of $k(x, y)$ *i.e.*,

$$f(x) = \sum_{h=1}^{\infty} a_h \phi_h(x) \qquad \ldots(2)$$

where $a_n = \int f(x)\, \phi_h(x)\, dx$, $(h = 1, 2, \ldots)$ (2) moreover if $k \in L^*_2$ *i.e.*, $\int k^2(x, y)\, dy \le N^2$ (N = constant) then series (2) converges absolutely and uniformly for every $f(x)$ of type (1).

1. If the symmetric kernel $k(x, y)$ belong to class L_2, then all the corresponding iterated kernels $k_m(x, y)$, $(m \ge 2)$ can be represented by absolute and uniformly convergent series.

$$k_m(x, y) = \sum_{h=1}^{\infty} \lambda_n^{-m} \phi_h(x)\, \phi_h(y), \ (m = 2, 3, \ldots)$$

If $k \in L^*_2$ then each series converges uniformly.

2. The singular points of the resolvent kernel H corresponding to a symmetric L_2 kernel $K(x, y)$ are simple poles.

3. If the function $f(x)$ can be differentiated twice in the basic interval (0, 1) and if its second derivative $f''(x)$ belongs to the class L_2 and if $f(0) = f(1) = 0$, then $f(x)$ has the absolutely and uniformly convergent expansion

$$f(x) = \sum_{h=1}^{\infty} \alpha_h \sin(h\pi x), \ (0 \le x \le 1),$$

where $\alpha_h = 2\int_0^1 f(x) \sin(h\pi x)\, dx$, $(h = 1, 2, 3, \ldots)$

OBJECTIVE TYPE QUESTIONS

1. An arc of integration such that the integral $I = \int_a^b F\left(x, y, \frac{dy}{dx}\right) dx$, can be determined, where y is a function of x and $F = F(x, y, y')$ process a continuous partial derivatives is reffered as—
   (A) Admissible arc (B) Open arc
   (C) Closed arc (D) None of these

2. Let $y = S(x)$ be the equation of the extremal through the fixed point A and B for which the integral $I = \int_a^b F(x, y, y')\, dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, and $\frac{\partial^2 F}{\partial S'^2}$ is negative. Therefore for weak variation—
   (A) Integral I is a minimum
   (B) Integral I is a maximum
   (C) Integral I is zero
   (D) None of these

3. Let integral $I = \int_0^1 F(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, t)\, dt$ where the q_i, $i = 1, 2, \ldots, n$ satisfy the n second order partial differential equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}m}\right) = 0$ $(m = 1, 2, \ldots, n)$. If at every point of a sufficiently small range (t_0 to t_1) (i) $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ $(r = 1, 2, \ldots, n)$ all have constant sign, (ii) $\left(\frac{\partial^2 F}{\partial \dot{q}^2_r}\right)\left(\frac{\partial^2 F}{\partial \dot{q}^2_s}\right) > \left(\frac{\partial^2 F}{\partial \dot{q}_r \partial \dot{q}_s}\right)$, $\left(\begin{matrix} r = 1, 2, \ldots, n \\ s = 1, 2, \ldots, n \end{matrix}\ r \ne s\right)$. Then if the sign of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ are all negative, then—
   (A) I is minimum (B) I is maximum
   (C) I is constant (D) None is true

4. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one is $\dot{x}$ and $\dot{y}$, have strong minimum, then—
   (A) $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial x \partial \dot{x}} - S(x, y, \dot{x}, \dot{y})(\ddot{x}\dot{y} - \ddot{y}\dot{x}) = 0$
   (B) $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial x \partial \dot{x}} = 0$
   (C) $\frac{\partial^2 G}{\partial x \partial \dot{y}} + \frac{\partial^2 G}{\partial x \partial \dot{x}} + S(x, y, \dot{x}, \dot{y})(\ddot{x}\dot{y} - \ddot{y}\dot{x}) = 0$
   (D) None of these

5. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$ have strong maximum, then—
   (A) $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial x \partial \dot{x}} - S(x, y, \dot{x}, \dot{y})(\ddot{x}\dot{y} - \ddot{y}\dot{x}) = 0$
   (B) $\frac{\partial^2 G}{\partial x \partial \dot{y}} - \frac{\partial^2 G}{\partial x \partial \dot{x}} = 0$
   (C) $\frac{\partial^2 G}{\partial x \partial \dot{y}} + \frac{\partial^2 G}{\partial x \partial \dot{x}} + S(x, y, \dot{x}, \dot{y})(\ddot{x}\dot{y} - \ddot{y}\dot{x}) = 0$
   (D) None of these

6. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$ have strong minimum, then—
   (A) $\left.\begin{array}{l} \frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) = 0 \\ \frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) = 0 \end{array}\right\}$
   (B) $\frac{\partial G}{\partial x} \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) = \frac{\partial G}{\partial y} \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) = 0$
   (C) $\frac{\partial G}{\partial x}\left(\frac{\partial G}{\partial \dot{x}}\right) = \frac{\partial G}{\partial y}\left(\frac{\partial G}{\partial \dot{y}}\right)$
   (D) None of these

7. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$ have strong maximum, then—
   (A) $\left.\begin{array}{l} \frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) = 0 \\ \frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) = 0 \end{array}\right\}$
   (B) $\frac{\partial G}{\partial x} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{x}}\right) = \frac{\partial G}{\partial y} - \frac{d}{dt}\left(\frac{\partial G}{\partial \dot{y}}\right) = 0$
   (C) $\frac{\partial G}{\partial x}\left(\frac{\partial G}{\partial \dot{x}}\right) = \frac{\partial G}{\partial y}\left(\frac{\partial G}{\partial \dot{y}}\right)$
   (D) None of these

8. The integral I has strong maximum if—
   (A) The arc AB of the arc of integration $\overline{|e}$, contains no point conjugate to either A or B
   (B) The arc AB of the arc of integration $\overline{|e}$, contains point conjugate to either A or B
   (C) The arc AB of the arc of integration $\overline{|e}$, contains point conjugate to both A or B
   (D) None of these

9. The integral $I = \int_A^B G(x, y, \dot{x}, \dot{y})\, dt$, where the end points A and B are fixed and $G(x, y, \dot{x}, \dot{y})$ is homogeneous and of degree one in $\dot{x}$ and $\dot{y}$ have strong maximum, then—
   (A) The variables x and y, which are functions of t
   (B) The variables x and y, which are not functions of t
   (C) The variables x and y, which are not functions
   (D) None of these

10. The integral I has strong maximum if—
   (A) The arc AB of the arc of integration $\overline{|e}$, contains no point conjugate to either A or B
   (B) The arc AB of the arc of integration $\overline{|e}$, contains point conjugate to either A or B
   (C) The arc AB of the arc of integration $\overline{|e}$, contains point conjugate to both A or B
   (D) None of these

11. The integral I has .... if—
   (1) The equation of $\overline{|e}$, the arc of integration, satisfy $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$,
   (2) The arc AB of $\overline{|e}$ contains no point conjugate to either A or B,
   (3) At all points of this arc $E = \frac{\partial^2 F}{\partial y'^2} < 0$. And at all points of this arc and for all finite values of p, $E(x, y, y', p) \le 0$
   (A) Strong maximum
   (B) Strong minimum
   (C) Constant
   (D) None of these

12. The integral I has strong maximum if—
   (A) At all points of the arc AB, of the arc of integration $\overline{|e}$, and for all finite values of p, $\frac{\partial^2 F(x, y, p)}{\partial p^2} < 0$

(B) At all points of the arc AB, of the arc of integration $\lceil e$, and for all finite values of p, $\frac{\partial^2 F\,(x, y, p)}{\partial p^2} > 0$

(C) At all points of the arc AB, of the arc of integration $\lceil e$, and for all finite values of p, $\frac{\partial^2 F\,(x, y, p)}{\partial p^2} = 0$

(D) None of these

13. The integral I has strong maximum if—

(A) The equation of $\lceil e$, the arc of integration, satisfies $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

(B) The equation of $\lceil e$, the arc of integration, satisfies $\frac{\partial F}{\partial y} - \left(\frac{\partial F}{\partial y'}\right) = 0$

(C) The equation of $\lceil e$, the arc of integration, satisfies $\frac{\partial F}{\partial y} = 0$

(D) The equation of $\lceil e$, the arc of integration, satisfies $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

14. The integral I has .... if—

(1) The equation of $\lceil e$, the arc of integration, satisfy $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$,

(2) The arc of AB of $\lceil e$, contains no point conjugate to either A or B,

(3) At all points of this arc $E = \frac{\partial^2 F}{\partial y'^2} < 0$ and at all point of this arc and for all finite values of p, $E\,(x, y, y', p) \le 0$

(A) Strong maximum

(B) Strong minimum

(C) Constant

(D) None of these

15. Let $y = S\,(x)$ be the equation of the extremal through the fixed point A and B for which the integral $I = \int_a^b F\,(x, y, y')\,dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, and $\frac{\partial^2 F}{\partial S'^2}$ is positive. Therefore for weak variation—

(A) Integral I is a maximum

(B) Integral I is zero

(C) Integral I is minimum

(D) None of these

16. Let integral $I = \int_{t_0}^{t_1} F\,(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n, ; t)\,dt$ where the q_i, $i = 1, 2, \ldots, n$ satisfy the n second order partial differential equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_m}\right) = 0$ ($m = 1, 2, \ldots, n$). If at every point of a sufficiently small range t_0 to t_1 (i) $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ ($r = 1, 2, \ldots, n$) all have constant sign, (ii) $\left(\frac{\partial^2 F}{\partial \dot{q}^2_r}\right)\left(\frac{\partial^2 F}{\partial \dot{q}^2_s}\right) > \left(\frac{\partial^2 F}{\partial \dot{q}_r \partial \dot{q}_s}\right)$, $\left(\begin{matrix} r = 1, 2, \ldots, n \\ s = 1, 2, \ldots, n \end{matrix}\; r \neq s\right)$. Then if the sign of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ are all negative, then—

(A) I is minimum (B) I is maximum

(C) I is constant (D) None is true

17. The integral I has strong minimum if—

(A) The equation of $\lceil e$, the arc of integration, satisfies $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

(B) The equation of $\lceil e$, the arc of integration, satisfies $\frac{\partial F}{\partial y} - \left(\frac{\partial F}{\partial y'}\right) = 0$

(C) The equation of $\lceil e$, the arc of integration, satisfies $\frac{\partial F}{\partial y} = 0$

(D) The equation of $\lceil e$, the arc of integration, satisfies $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$

18. Let $y = S\,(x)$ be the equation of the extremal through the fixed points A and B for which the integral $I = \int_a^b F\,(x, y, y')\,dx$ is stationary. Let A and A' be two adjacent conjugate points

(kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, then for weak variation I is a minimum—

(A) When $\frac{\partial^2 F}{\partial S'^2}$ is less than zero

(B) When $\frac{\partial^2 F}{\partial S'^2}$ is greater than zero

(C) When $\frac{\partial^2 F}{\partial S'^2}$ is equal to zero

(D) None of these

19. The integral I has strong minimum if—

(A) At all points of the arc AB, of the arc of integration $\lceil e$, and for all finite values of p, $\frac{\partial^2 F(x, y, p)}{\partial p^2} < 0$

(B) At all points of the arc AB, of the arc of integration $\lceil e$, and for all finite values of p, $\frac{\partial^2 F(x, y, p)}{\partial p^2} > 0$

(C) At all points of the arc AB, of the arc of integration $\lceil e$, and for all finite values of p, $\frac{\partial^2 F(x, y, p)}{\partial p^2} = 0$

(D) None of these

20. Let $I = \int_{t_0}^{t_1} F(q_1, q_2, \ldots, q_m, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)$ dt where the q_i, $i = 1, 2, \ldots, n$ satisfy the n second order partial differential equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_m}\right) = 0, (m = 1, 2, \ldots, n)$. If at every point of a sufficiently small range t_0 to t_1 (i) $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ $(r = 1, 2, \ldots, n)$ all have constant sign, (ii) $\left(\frac{\partial^2 F}{\partial \dot{q}^2_r}\right)\left(\frac{\partial^2 F}{\partial \dot{q}^2_s}\right) > \left(\frac{\partial^2 F}{\partial \dot{q}_r \partial \dot{q}_s}\right), \left(\begin{matrix} r = 1, 2, \ldots, n \\ s = 1, 2, \ldots, n \end{matrix}\; r \neq s\right)$.
Then I is a minimum—

(A) If the signs of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ are all negative

(B) If the signs of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ are all positive

(C) If the signs of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ varies from negative to positive

(D) None of these

21. Let $y = S(x)$ be the equation of the extremal through the fixed points A and B for which the integral $I = \int_a^b F(x, y, y')\, dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, then for weak variation I is a minimum—

(A) When $\frac{\partial^2 F}{\partial S'^2}$ is negative

(B) When $\frac{\partial^2 F}{\partial S'^2}$ is positive

(C) When $\frac{\partial^2 F}{\partial S'^2}$ is neither negative nor positive

(D) None of these

22. The integral I has strong minimum if—

(A) The arc AB of the arc of integration $\lceil e$, contains no point conjugate to either A or B

(B) The arc AB of the arc of integration $\lceil e$, contains point conjugate to either A or B

(C) The arc AB of the arc of integration $\lceil e$, contains point conjugate to both A or B

(D) None of these

23. The integral I has strong maximal if (1) the equation of $\lceil e$, the arc of integration, satisfy $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$, (2) The arc AB of $\lceil e$. contains no point conjugate to either A or B, (3) At all points of this arc $E = \frac{\partial^2 F}{\partial y'^2} < 0$ and at all points of this arc and for all finite values of p, E (x, y, y', p) is—

(A) Less than/equals to zero

(B) Less than zero

(C) Greater than/equals to zero

(D) Greater than zero

24. Let F be given functonal form, let $p_r = \dfrac{\partial z}{\partial x_r}$, where z is a function of $x_1, x_2, \ldots, x_n$ and let the integral $I = \iiint \ldots \int F(z, x_1, x_2, \ldots, x_n, p_1, p_2, \ldots, p_n)\, dx_1 dx_2 \ldots dx_n$ be taken through an n-dimensional region bounded by given fixed boundaries of dimension $n - 1$. Then the integral I is stationary when z is a solutions of—
   (A) The second-order partial differential equation
   $$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial p_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial p_2}\right) - \cdots - \frac{\partial}{\partial x_n}\left(\frac{\partial F}{\partial p_n}\right) = 0$$
   (B) The second-order partial differential equation
   $$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x_1}\left(\frac{\partial F}{\partial p_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial F}{\partial p_2}\right) - \cdots - \frac{\partial}{\partial x_n}\left(\frac{\partial F}{\partial p_n}\right) = c$$
   (C) The first order partial differential equation
   $$\frac{\partial F}{\partial z} - \left(\frac{\partial F}{\partial p_1}\right) - \left(\frac{\partial F}{\partial p_2}\right) - \cdots - \left(\frac{\partial F}{\partial p_n}\right) = 0$$
   (D) None of these

25. Let the functional form of F be given, then the integral $I = \int_a^b F(x, y, y_1, y_2, \ldots, y_n)\, dx$ is stationary when y satisfies the equation—
   (A) $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y_1}\right) + \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial F}{\partial y_2}\right) - \cdots + (-1)^n \dfrac{d^n}{dx^n}\left(\dfrac{\partial F}{\partial y_n}\right) = 0$
   (B) $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y_1}\right) + \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial F}{\partial y_2}\right) + \cdots + \dfrac{d^n}{dx^n}\left(\dfrac{\partial F}{\partial y_n}\right) = 0$
   (C) $(-1)^n \dfrac{d^n}{dx^n}\left(\dfrac{\partial F}{\partial y_n}\right) = 0$
   (D) None of these

26. Let $y = S(x)$ be the equation of the extremal through the fixed points A and B for which the integral $I = \int_a^b F(x, y, y')\, dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies between A and A' and (ii) $\dfrac{\partial^2 F}{\partial S'^2}$ has constant sign for all point of the arc AB, then for weak variation I is a minimum—
   (A) When $\dfrac{\partial^2 F}{\partial S'^2}$ is less than zero
   (B) When $\dfrac{\partial^2 F}{\partial S'^2}$ is greater than zero
   (C) When $\dfrac{\partial^2 F}{\partial S'^2}$ is equal to zero
   (D) None of these

27. The integral I has strong minimum if (1) the equation of $\lceil e$, the arc of integration, satisfy $\dfrac{\partial F}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0$, (2) The arc AB of $\lceil e$. contains no point conjugate to either A or B, (3) At all points of this arc $E = \dfrac{\partial^2 F}{\partial y'^2} < 0$ and at all points of this arc and for all finite values of p, $E(x, y, y', p)$ is—
   (A) Less than/equals to zero
   (B) Less than zero
   (C) Greater than/equals to zero
   (D) Greater than zero

28. Let $I = \displaystyle\int_{t_0}^{t_1} F(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)\, dt$ where the q_i, $i = 1, 2, \ldots, n$ satisfy the n second order partial differential equations $\dfrac{\partial F}{\partial q_m} - \dfrac{d}{dt}\left(\dfrac{\partial F}{\partial \dot{q}_m}\right) = 0$ $(m = 1, 2, \ldots, n)$. If at every point of a sufficiently small range t_0 to t_1 (i) $\dfrac{\partial^2 F}{\partial \dot{q}^2_r}$ $(r = 1, 2, \ldots, n)$ all have constant sign, (ii) $\left(\dfrac{\partial^2 F}{\partial \dot{q}^2_r}\right)\left(\dfrac{\partial^2 F}{\partial \dot{q}^2_s}\right) > \left(\dfrac{\partial^2 F}{\partial \dot{q}_r \partial \dot{q}_s}\right)$, $\left(\begin{matrix} r = 1, 2, \ldots, n \\ s = 1, 2, \ldots, n \end{matrix}\; r \neq s\right)$.
   Then I is a minimum—
   (A) If the signs of $\dfrac{\partial^2 F}{\partial \dot{q}^2_r}$ are all negative
   (B) If the signs of $\dfrac{\partial^2 F}{\partial \dot{q}^2_r}$ are all positive

(C) If the signs of $\frac{\partial^2 F}{\partial \dot{q}^2_r}$ varies from negative to positive

(D) None of these

29. Given $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ then second variation is defined as—

(A) $\varepsilon \int_a^b \left(t \frac{\partial F}{\partial S} - \frac{dt}{dx} \frac{\partial F}{\partial S'} \right) dx$

(B) $\int_a^b F(x, y, y')\, dx$

(C) $\frac{\varepsilon}{2!} \int_a^b (t^2 F_{00} + 2tt' F_{01} + t'^2 F_{11}) dx$

(D) None of these

30. In Legender's test the integral $I = \int_a^b F(x, y, y')\, dx$, is minimum of value of $\frac{\partial^2 F}{\partial y'^2}$ is—

(A) Positive

(B) Negative

(C) Neither positive nor negative

(D) None of these

31. Let the values of t_0 and t_1 and the functional form of F be given. Then the integral $\int_{t_0}^{t_1} F(q_1, q_2, \ldots, q_r, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n ; t)\, dt$ where the q_i, $i = 1, 2, \ldots, n$ are arbitrary functions of t, is stationary for weak variations when the q_i, $i = 1, 2, \ldots, n$ satisfy—

(A) n equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{q}^2_m} \right) = 0$

$(m = 1, 2, \ldots, n)$

(B) $n + 1$ equations $\frac{\partial F}{\partial q_m} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{q}_m} \right) = 0$

$(m = 1, 2, \ldots, n + 1)$

(C) $n - 1$ equation $\frac{\partial F}{\partial q_m} - \frac{d}{dt} \left(\frac{\partial F}{\partial \dot{q}_m} \right) = 0$

$(m = 1, 2, \ldots, n - 1)$

(D) None of these

32. Given $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ then second variation is defined as—

(A) $\frac{\varepsilon}{2!} \int_a^b \left(t^2 \frac{\partial^2 F}{\partial S^2} - 2tt' \frac{\partial^2 F}{2 S \partial S'} + t'^2 \frac{\partial^2 F}{\partial S'^2} \right) dx$

(B) $\frac{\varepsilon}{2!} \int_a^b (t^2 F_{00} + 2tt' F_{01} + t'^2 F_{11}) dx$

(C) (A) and (B) both

(D) None of these

33. Given $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ then first variation is defined as—

(A) $\varepsilon \int_a^b \left(t \frac{\partial F}{\partial S} - \frac{dt}{dx} \frac{\partial F}{\partial S'} \right) dx$

(B) $\int_a^b F(x, y, y')\, dx$

(C) $\frac{\varepsilon}{2!} \int_a^b (t^2 F_{00} + 2tt' F_{01} + t'^2 F_{11}) dx$

(D) None of these

34. If $u_1(x)$ and $u_2(x)$ are independent solutions of the accessory equation, then … as x increases—

(A) The ratio $u_1(x)/u_2(x)$ steadily increases or steadily decreases

(B) The $u_1(x)/u_2(x)$ steadily increases only

(C) The ratio $u_1(x)/u_2(x)$ steadily decreases only

(D) None of these

35. If $u(x)$ is a solution of the accessory equation then—

(A) It cannot have double zero always

(B) It can have double zero always

(C) It may have double zero

(D) None of these

36. If $y = S(x, c_1, c_2)$ is the general solution of Euler's characteristic equation, then the equation for abscissa points conjugate to A is—

(A) $\frac{\partial y}{\partial c_1} = \left(\frac{\partial y}{\partial c_1} \right)_{x = a}$

(B) $\frac{\partial y}{\partial c_1} / \frac{\partial y}{\partial c_2} = \left(\frac{\partial y}{\partial c_1} / \frac{\partial y}{\partial c_2} \right)_{x = a}$

(C) $\frac{\partial y}{\partial c_2} = \left(\frac{\partial y}{\partial c_1} \right)_{x = a}$

(D) None of these

37. Let $y = S(x)$ be the equation of the extremal through the fixed point A and B for which the integral $I = \int_a^b F(x, y, y')\, dx$ is stationary. Let A and A' be two adjacent conjugate points (kinetic foci) on the curve. If (i) B lies

between A and A' and (ii) $\frac{\partial^2 F}{\partial S'^2}$ has constant sign for all points of the arc AB, then for weak variation I is a maximum—

(A) When $\frac{\partial^2 F}{\partial S'^2}$ is negative

(B) When $\frac{\partial^2 F}{\partial S'^2}$ is positive

(C) When $\frac{\partial^2 F}{\partial S'^2}$ is neither negative nor positive

(D) None of these

38. If (i) is the abscissa of the point A, (ii) $u_1(x)$ and $u_2(x)$ are independent solutions of the Jacobi's equation, then the equation for the abscissa of the points conjugate to A is—

(A) $\frac{1}{u_2(x)} = \frac{1}{u_2(a)}$ (B) $\frac{u_1(x)}{u_2(x)} = \frac{u_1(a)}{u_2(a)}$

(C) $u_1(x) = u_2(x)$ (D) None of these

39. If $t(a) = t(b) = 0$ and u is a solution of $\left\{ F_{00} - \frac{d}{dx}(F_{01}) \right\} u - \frac{d}{dx}\left\{ F_{11}\frac{du}{dx} \right\} = 0$, then—

(A) $I_2 = \int_a^b F_{11}\left(t' - t\frac{u'}{u}\right)^2 dx$

(B) $I_2 = \frac{\varepsilon}{2!}\int_a^b F_{11}\left(t' - t\frac{u'}{u}\right)^2 dx$

(C) $I_2 = \frac{\varepsilon}{2!}\int_a^b F_{11}\left(\frac{u'}{u}\right)^2 dx$

(D) $I_2 = \frac{\varepsilon}{2!}$

40. The following result

$$\int_a^b (t^2 F_{00} + 2tt' F_{01} + t'^2 F_{11})dx =$$

$$\int_a^b \left\{ t^2 F_{00} - t^2\frac{d}{dx}(F_{01}) - t\frac{d}{dx}(t' F_{11}) \right\} dx$$

holds if—

(A) $t(a) = t(b) = 0$

(B) $t(a) \neq t(b)$

(C) $t(a) = t(b)$

(D) None of these

41. The roots of $u(x) = 0$ are the abscissa of the points on the curve $y = S(x)$ conjugate to A if—

(A) $u(x)$ is a solution of the Jacobi's accessory equation

(B) a is the abscissa of the point A and $u(a) = 0$

(C) Both (A) and (B)

(D) None of these

42. If Euler's characteristic equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$ varishes identically, then the indefinite integral $\int F(x, y, y')\, dx$ can be evaluated as a function of—

(A) x and y (B) x only

(C) y only y (D) None of these

43. The accessory equation $\left\{ F_{00} - \frac{d}{dx}(F_{01}) \right\} u - \frac{d}{dx}\left\{ F_{11}\frac{du}{dx} \right\} = 0$, where $\frac{\partial^2 F}{\partial S^2} = F_{00}$, $\frac{\partial^2 F}{\partial S \partial S'} = F_{01}$ and $\frac{\partial^2 F}{\partial S'^2} = F_{11}$ is referred as—

(A) Jacobi's equation

(B) Euler equation

(C) Volterra equation

(D) None of these

44. If Euler's characteristic equation $\left\{ \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0 \right\}$ vanishes identically, then the indefinite integral $\int F(x, y, y')\, dx$—

(A) Is a function of the end positions only

(B) Is not a function of the end positions

(C) Dependent on path

(D) None of these

45. If Euler's characteristic equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = 0$ vanishes identically, then the indefinite integral $\int F(x, y, y')\, dx$—

(A) Has a value which is independent of the path of integration

(B) Has a value which is dependent of the path of integration

(C) Both (A) and (B) true

(D) None of these

46. The integral $\frac{\varepsilon}{2!}\int_a^b \left(t^2\frac{\partial^2 F}{\partial S^2} - 2tt'\frac{\partial^2 F}{\partial S \partial S'} + t'^2\frac{\partial^2 F}{\partial S'^2} \right) dx$, where $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ is referred as—

(A) First variation (B) Second variation

(C) Third variation (D) Fourth variation

47. In Legendre's test the integral $I = \int_a^b F(x, y, y')\, dx$, is maximum if value of $\frac{\partial^2 F}{\partial y'^2}$ is—
(A) Positive
(B) Negative
(C) Neither positive nor negative
(D) None of these

48. For Legendre's test the sign of $\left(\frac{\partial^2 F}{\partial y'^2}\right)$ is—
(A) Constant throughout this range
(B) Increasing throughout the range
(C) Decreasing throughout the range
(D) None of these

49. For Legendre's test—
(A) Euler's characteristic equation is to be satisfied
(B) Euler's characteristic equation is not to be satisfied
(C) Jacobi's accessory equation is satisfied
(D) Jacobi's accessory equation is not to be satisfied

50. The integral $\int_a^b F(x, y')\, dx$, whose ends fixed, is stationary for weak variations if y satisfies the differential equation—
(A) $\frac{\partial F}{\partial y'} = c$, where c is an arbitrary constant
(B) $\frac{\partial F}{\partial y'} > c$, where c is an arbitrary constant
(C) $\frac{\partial F}{\partial y'} < c$, where c is an arbitrary constant
(D) None of these

51. The integral $\int_a^b F(y, y')\, dx$, whose ends are fixed, is stationary for weak variation if y satisfies the differential equation—
(A) $F - y'\frac{\partial F}{\partial y'} = c$, where c is an arbitrary constant
(B) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$
(C) $F + y'\frac{\partial F}{\partial y'} = c$, where c is and arbitrary constant
(D) None of these

52. The integral $I = \int_a^b F(x, y, y')\, dx$, whose end points are fixed, is stationary for weak variations, if y satisfies—
(A) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$
(B) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = F$
(C) $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \geq 0$
(D) None of these

53. Given the function $F = F(x, y, y')$, the differential equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$, is referred as—
(A) Euler's characteristic equation
(B) Lagrange's equation
(C) Hamilton equation
(D) None of these

54. The integral $\frac{\varepsilon}{2!}\int_a^b \left(t^2\frac{\partial^2 F}{\partial S^2} - 2tt'\frac{\partial^2 F}{\partial S \partial S'} + t'^2\frac{\partial^2 F}{\partial S'^2}\right) dx$, where $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ is referred as—
(A) First variation (B) Second variation
(C) Third variation (D) Fourth variation

55. The solution to Euler's characteristic equation is referred as—
(A) Extremals (B) Zeros
(C) Nulls (D) None of these

56. The integral $\varepsilon \int_a^b \left(t\frac{\partial F}{\partial S} - \frac{dt}{dx}\frac{\partial F}{\partial S'}\right) dx$, where $y = S(x)$, $t = t(x)$ and $F = F(x, y, y')$ is referred as—
(A) First variation (B) Second variation
(C) Third variation (D) Fourth variation

57. Let $y = S(x)$, and $y = S(x) + \in t(x)$ be two admissible curves of integral, $I = \int_a^b F\left(x, y, \frac{\partial y}{dx}\right) dx$ where $\in$ is an arbitrary constant independent of x and y, $t(x)$ is an arbitrary function of x, independent of $\in$, with this restriction on $t(x)$, the ordinate y is referred as—
(A) Strong variation (B) Weak variation
(C) Admissible arc (D) None of these

58. Let $y = S(x)$ and $y = S(x) + \in t(x)$ be two admissible curves of integral and $t(x)$ is an arbitrary function of x, independent of $\in$ then variation y is referred as—
    (A) Strong variation (B) Weak variation
    (C) Admissible arc (D) None of these

59. In integral equations—
    (A) An unknown function appears under the integral sign
    (B) A known function appears under the integral sign
    (C) No function appears under the integral sign
    (D) None of these

60. Let $K(x, y)$ be a given real function defined for $0 \le x \le 1$, $0 \le y \le 1$ and $f(x)$ a real valued function defined for $0 \le x \le 1$ and λ an arbitrary complex number. Then $\phi(x) - \lambda \int_0^1 K(x, y)\,\phi(y)\,dy = f(x)$, $(0 \le x \le 1)$ is the—
    (A) Linear Fredholm integral equation of second kind (for a function $\phi(x)$)
    (B) Linear Fredholm integral equation of first kind (for a function $\phi(x)$)
    (C) Volterra integral equation of first kind (for a function $\phi(x)$)
    (D) Volterra integral equation of second kind (for a function $\phi(x)$)

61. Let $K(x, y)$ be a given real function defined for $0 \le x \le 1$, $0 \le y \le 1$ and $f(x)$ a real valued function defined for $0 \le x \le 1$ and λ an arbitrary complex number. Then $\int_0^1 K(x, y)\,\phi(y)\,dy = f(x)$, $(0 \le x \le 1)$ is the—
    (A) Linear Fredholm integral equation of second kind (for a function $\phi(x)$)
    (B) Linear Fredholm integral equation of first kind (for a function $\phi(x)$)
    (C) Volterra integral equation of first kind (for a function $\phi(x)$)
    (D) Volterra integral equation of second kind (for a function $\phi(x)$)

62. Let $K(x, y)$ be a given real function defined for $0 \le x \le 1$, $0 \le y \le 1$, $f(x)$ a real valued function for $0 \le x \le 1$ and λ an arbitrary complex number, with $K(x, y) \equiv 0$ if $y > x$, the $\int_0^x K(x, y)\,\phi(y)\,dy = f(x)$ is the—
    (A) Linear Fredholm integral equation of second kind (for a function $\phi(x)$)
    (B) Linear Fredholm integral equation of first kind (for a function $\phi(x)$)
    (C) Volterra integral equation of first kind (for a function $\phi(x)$)
    (D) Volterra integral equation of second kind (for a function $\phi(x)$)

63. Let $K(x, y)$ be a given real function defined for $0 \le x \le 1$, $0 \le y \le 1$, $f(x)$ a real valued function for $0 \le x \le 1$ and λ an arbitrary complex number, with $K(x, y) \equiv 0$ if $y > x$, then $\phi(x) - \lambda \int_0^x K(x, y)\,\phi(y)\,dy = f(x)$ is the—
    (A) Linear Fredholm integral equation of second kind (for a function $\phi(x)$)
    (B) Linear Fredholm integral equation of first kind (for a function $\phi(x)$)
    (C) Volterra integral equation of first kind (for a function $\phi(x)$)
    (D) Volterra integral equation of second kind (for a function $\phi(x)$)

64. A kernel $K(x, y)$ which is quadratically integrable in the square $(0 \le x \le h, 0 \le y \le h)$ is referred as—
    (A) L_1–kernel (B) L_2–kernel
    (C) L_3–kernel (D) L_4–kernel

65. The equation $K(x, y)\,\phi(x) + \int_0^x K'_x(x, y)\,\phi(y)\,dy = f'(x)$ and $f(x) = [K(x, y)\,\Phi(y)]_{y=0}^{y=x} - \int_0^x K'_y(x, y)\,\Phi(y)\,dy$, where $K(x, y)$ vanishes at $x = 0$, are referred as—
    (A) Fredholm equation of first kind
    (B) Fredholm equation of second kind
    (C) Fredholm equation of third kind
    (D) Fredholm equation of fourth kind

66. A function f, whose norm is given by $\|f\|^2 = \int_0^h f^2(x)\,dx$ is—
    (A) L–function (B) L_1–function
    (C) L_2–function (D) L_3–function

67. A L_2–function is—
    (A) A function f, which is L_2–function and bounded

(B) A function f, which is L_2–function only
(C) A function f, which is bounded only
(D) None of these

68. The Volterra integral equation of the second kind $\phi(x) - \lambda \int_0^x K(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le h)$, where the kernel $K(x, y)$ and the function (x), belongs to the class L_2 has—
(A) One and essentially only the solution in the same class L_2
(B) Two and essentially only two solutions in the same class L_2
(C) Three and essentially only three solutions in the same class L_2
(D) None of these

69. Any solution of homogeneous Volterra integral equations of second kind $\phi(x) - \lambda \int_0^x K(x, y)\, \phi(x)\, dy = 0$ in L_2–space is—
(A) Necessarily a zero function
(B) Necessarily a non zero function
(C) Absolute function
(D) None of these

70. A kernel $K(x, y)$ with norm $\|K\|$ such that $\|K\|^2 = \int_0^h \int_0^h K^2(x, y)\, dxdy \le N^2$ exist, where N^2, a constant is—
(A) L_1–kernel (B) L_2–kernel
(C) L_3–kernel (D) L_4–kernel

71. The equation $\int_0^x \frac{\phi(y)}{(x-y)^{\alpha}}\, dy = f(x)$, $(0 \le \alpha < 1)$ is referred as—
(A) Fredholm equation
(B) Able equation
(C) Maxwell Equation
(D) Picard's equation

72. The solution to Able equation is given by—
(A) $\phi(x) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dx} \int_0^x \frac{f(y)}{(x-y)^{1-\alpha}}\, dy$
(B) $\phi(x) = \lambda \int_0^1 K(x, y)\, \phi(y)\, dy + f(\alpha)$, $(0 \le x \le 1)$
(C) $\phi(x) = \lambda \int_0^1 K(x, y)\, \phi(y)\, dy$
(D) None of these

73. The solution of Able equation is given by—
(A) $\phi(x) = \frac{\sin \alpha\pi}{\pi}$
(B) $\phi(x) = \frac{\sin(\alpha\pi)}{\pi} \left[\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(y)}{(x-y)^{1-\alpha}}\, dy\right]$
(C) $\phi(x) = x$
(D) None of these

74. The following represents Linear Fredholm integral equation of second kind (for a function $\phi(x)$)—
(A) $\phi(x) - \lambda \int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le 1)$
(B) $\int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$
(C) $\int_0^x K(x, y)\, \phi(y)\, dy = f(x)$
(D) $\phi(x) - \lambda \int_0^x K(x, y)\, \phi(y)\, dy = f(x)$

75. The following represents Linear Fredholm integral equation of first kind (for a function $\phi(x)$)—
(A) $\phi(x) - \lambda \int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le 1)$
(B) $\int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$
(C) $\int_0^x K(x, y)\, \phi(y)\, dy = f(x)$
(D) $\phi(x) - \lambda \int_0^x K(x, y)\, \phi(y)\, dy = f(x)$

76. The following represents Volterra integral equation of second kind (for a function $\phi(x)$)—
(A) $\phi(x) - \lambda \int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le 1)$
(B) $\int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$
(C) $\int_0^x K(x, y)\, \phi(y)\, dy = f(x)$
(D) $\phi(x) - \lambda \int_0^x K(x, y)\, \phi(y)\, dy = f(x)$

77. A Volterra integral equation has—
(A) One eigen values
(B) Two eigen values
(C) Three eigen values
(D) No eigen values

78. The following represents Linear Fredholm integral equation of second kind (for a function $\phi(x)$)—
(A) $\phi(x) - \lambda \int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$, $(0 \le x \le 1)$
(B) $\int_0^1 K(x, y)\, \phi(y)\, dy = f(x)$
(C) $\int_0^x K(x, y)\, \phi(y)\, dy = f(x)$
(D) $\phi(x) - \lambda \int_0^x K(x, y)\, \phi(y)\, dy = f(x)$

79. The Fredholm Linear operator is—
(A) $F_x[\phi(y)] \square\ \Phi(x) - \lambda \overset{1}{\underset{0}{\square}} K_Q(x, y)\, \phi(y)\, dy$
(B) $F^*_x[\phi(y)] \equiv \phi(x) - \lambda \int_0^1 K(y, x)\, \phi(y)\, dy$
(C) $\int F_x[\phi(y)]\, \psi(x)\, dx = \int F^*h[\psi(y)]\, \phi(x)\, dx$
(D) None of these

80. The associated Fredholm operator is—
(A) $F_x[\phi(y)] \square\ \Phi(x) - \lambda \overset{1}{\underset{0}{\square}} K_Q(x, y)\, \phi(y)\, dy$
(B) $F^*_x[\phi(y)] \equiv \phi(x) - \lambda \int_0^1 K(y, x)\, \phi(y)\, dy$
(C) $\int F_x[\phi(y)]\, \psi(x)\, dx = \int F^*_x[\psi(y)]\, \phi(x)\, dx$
(D) None of these

81. The Green's formula is—
(A) $F_x[\phi(y)] \square\ \Phi(x) - \lambda \overset{1}{\underset{0}{\square}} K_Q(x, y)\, \phi(y)\, dy$
(B) $F^*_x[\phi(y)] \equiv \phi(x) - \lambda \int_0^1 K(y, x)\, \phi(y)\, dy$
(C) $\int F_x[\phi(y)]\, \psi(x)\, dx = \int F^*_x[\psi(y)]\, \phi(x)\, dx$
(D) None of these

82. The Pincherie-Goursat kernel is given by—
(A) $K(x, y) = \sum_{k=1}^{n} Y_k(x)$, where $\{Y_i(x)\}$ is a set of linearly independent L_2–functions in the basic interval (0, 1)
(B) $K(x, y) = \sum_{k=1}^{n} X_k(x)$, where $\{X_i(x)\}$ is a set of linearly independent L_2–functions in the basic interval (0, 1)
(C) $K(x, y) = = \sum_{k=1}^{n} X_k(x)\, Y_k(x)$, where $\{X_i(x)\}$ and $\{Y_i(x)\}$ are two sets of linearly independent L_2–functions in the basic interval (0, 1)
(D) None of these

83. An orthogonal system of functions is a system of functions $\{\phi_n\} = \phi_1(x), \phi_2(x), \ldots$, where $\phi_i, \phi_j\ (i \ne j)$ satisfies orthogonal condition—
(A) $(\phi_i, \phi_j) \equiv \int_a^b \phi_i(x)\, \phi_j(x)\, dx = 0\ (i \ne j)$
(B) $(\phi_i, \phi_j) \equiv \int_a^b \phi_i(x)\, \phi_j(x)\, dx = 0$
(C) $(\phi_i, \phi_j) \equiv \int_a^b \phi_i(x)\, \phi_j(x)\, dx = 0\ (i = j)$
(D) None of these

84. An orthonormal system is—
(A) An orthogonal system of functions $\{\phi_n\}$ satisfying normalizing condition $(\phi_i, \phi_j) = \begin{cases} 0, & i = j \\ 1, & i \ne j \end{cases}$
(B) An orthogonal system of functions $\{\phi_n\}$ satisfying non normalizing condition
(C) An orthogonal system of functions $\{\phi_n\}$ satisfying normalizing condition $(\phi_i, \phi_j) = \begin{cases} 0, & i \ne j \\ 1, & i = j \end{cases}$
(D) None of these

85. An orthonormal system for which Parseval's equation $\sum_{h=1}^{\infty} a^2{}_h = \int_a^b f^2(x)\, dx$ holds for any function $f(x)$ of L_2, where a_n is the Fourier coefficient of the function $f(x)$ with respect to the system $\{\phi_h\}$, and $a_n = \int_a^b f(x)\, \phi_n(x)\, dx$, $n = 1, 2, \ldots$ is referrers as—
(A) Complete orthonormal system
(B) Complete normal system
(C) Symmetric system
(D) None of these

86. One states $K \varepsilon L^*{}_2$—
(A) If function $A(x)$ related to the kernel K is bounded

(B) If function A (x) related to the kernel K is not bounded
(C) If function A (x) related to the kernel K is linear function
(D) None of these

87. If function A (x) related to the kernel K is bounded then—
(A) $\int K^2(x, y)\, dy = A^2(x) > N^2$
(B) $\int K^2(x, y)\, dy = A^2(x) < N^2$
(C) $\int K^2(x, y)\, dy = A^2(x) = N^2$
(D) None of these

88. A necessary and sufficient condition for an L_2–function $f(x)$ to be approximated in the mean by a linear combination of $f(x)$ of a given orthonormal system is—
(A) That Parseval's equation $\sum_{h=1}^{\infty} a^2{}_h = \int_a^b f^2(x)\, dx$ does not holds for given $f(x)$
(B) That Parseval's equation $\sum_{h=1}^{\infty} a^2{}_h = \int_a^b f^2(x)\, dx$ holds for given $f(x)$
(C) Weyl lemma is satisfied
(D) None of these

89. The kernel is symmetric if—
(A) $K(x, y) = K(y, x)$
(B) $K(x, y) > K(y, x)$
(C) $K(x, y) < K(y, x)$
(D) None of these

90. For any pair $\phi_h(x)$ and $\phi_k(x)$ of eigen functions of a symmetric kernel corresponding to two different eigen values λ_h and λ_k satisfies orthogonality condition—
(A) $(\phi_h, \phi_k) \equiv \int_a^b \phi_h(x)\, \phi_k(x)\, dx = 0, (h \neq k)$
(B) $(\phi_h, \phi_k) \equiv \int_a^b \phi_h(x)\, \phi_k(x)\, dx = 1, (h = k)$
(C) $(\phi_h, \phi_k) \equiv \int_a^b \phi_h(x)\, \phi_k(x)\, dx = 0, (h = k)$
(D) None of these

91. A necessary and sufficient condition for the convergence in the mean over the interval (a, b) of a sequence of L_2–function $\{f_n\}$ to $f(x)$ is that for any $\varepsilon > 0$, $\exists$ an integer N such that for $m, n \geq N$, one have $\int_a^b [f_m(x) - f_n(x)]^2\, dx < \varepsilon$—
(A) Weyl lemma
(B) Parseval's lemma
(C) Reistz–Fischer theorem
(D) None of these

92. Given sequence of L_2–functions $\{f_n\}$ over the interval (a, b) and for every $\varepsilon > 0$ there exist an integer N such that for $m, n \geq N$ we have $\int_a^b [f_m(x) - f_n(x)]^2\, dx \leq \varepsilon$, then there exist a function $f \in L_2$–function such that $\{f_n\}$ converges to $f(x)$—
(A) Weyl lemma
(B) Parseval's lemma
(C) Riestc–Fischer theorem
(D) None of these

93. If the function $f(x)$ can be differentiated twice in the basic interval $(0, 1)$ and if its …, $f''(x)$ belongs to the class L_2 and if $f(0) = f(1) = 0$, then $f(x)$ has the absolutely and uniformly convergent expansion $f(x) = \sum_{h=1}^{\infty} \alpha_h \sin(h\pi x)$, $(0 \leq x \leq 1)$, where $\alpha_h = 2\int_0^1 f(x) \sin(h\pi x)\, dx$, $(h = 1, 2, 3, \ldots)$—
(A) First derivative (B) Second derivative
(C) Third derivative (D) Fourth derivative

94. If the function $f(x)$ can be differentiated … in the basic interval $(0, 1)$ and if its second derivative $f''(x)$ belongs to the class L_2 and if $f(0) = f(1) = 0$, then $f(x)$ has the absolutely and uniformly convergent expansion $f(x) = \sum_{n=1}^{\infty} \alpha_h \sin(h\pi x)$, $(0 \leq x \leq 1)$, where $\alpha_h = 2 \int_0^1 f(x) \sin(h\pi x)\, dx$, $(h = 1, 2, 3, \ldots)$
(A) Once (B) Twice
(C) Thrice (D) None of these

95. Every symmetric, non-zero L_2–kernel has at least—
(A) One eigen value
(B) Two eigen value
(C) Three eigen value
(D) Four eigen value

96. Any L_2–function $w(x)$ is orthogonal to all the eigen functions $\phi_h(x)$ of symmetric kernel K (x, y) if—
(A) $\int K(x, y)\, w(y)\, dy > 0$

(B) $\int K(x, y)\, w(y)\, dy \neq 0$

(C) $\int K(x, y)\, w(y)\, dy = 0$

(D) None of these

97. If the symmetric kernel K (x, y) belongs to class L_2, then all the corresponding iterated kernels $K_m(x, y)$, $(m \geq 2)$ can be represented by … series—

(A) Absolute convergent

(B) Absolute and uniformly convergent

(C) Uniformly convergent

(D) None of these

98. The singular points of the resolvent kernel H corresponding to a symmetric L_2 kernel K (x, y) are—

(A) Simple poles

(B) Not simple pole

(C) Double poles

(D) None of these

99. If the function $f(x)$ can be differentiated twice in the basic interval (0, 1) and if its second derivative $f''(x)$ belongs to the class L_2 and …, then $f(x)$ has the absolutely and uniformly convergent expansion $f(x) = \sum_{h=1}^{\infty} \alpha_h \sin(h\pi x)$, $(0 \leq x \leq 1)$, where $\alpha_h = 2\int_0^1 f(x) \sin(h\pi x)\, dx$, $(h = 1, 2, 3, \ldots)$—

(A) If $f(0) > f(1)$

(B) If $f(0) \leq f(1)$

(C) If $f(0) = f(1) = 0$

(D) None of these

100. If $\int K(x, y)\, w(y)\, dy = 0$—

(A) Any L_2–function $w(x)$ orthogonal to all the eigen functions $\phi_h(x)$ of symmetric kernel K (x, y)

(B) Any L_2–function $w(x)$ is symmetric to all the eigen functions $\phi_h(x)$ of symmetric kernel K (x, y)

(C) Any L_2–function $w(x)$ if not orthogonal to all the eigen functions $\phi_h(x)$ of symmetric kernel K (x, y)

(D) None of these

Answers

1. (A)	2. (B)	3. (B)	4. (A)	5. (A)
6. (A)	7. (A)	8. (A)	9. (A)	10. (A)
11. (B)	12. (A)	13. (D)	14. (A)	15. (C)
16. (A)	17. (D)	18. (B)	19. (B)	20. (B)
21. (B)	22. (A)	23. (A)	24. (A)	25. (A)
26. (A)	27. (B)	28. (A)	29. (C)	30. (A)
31. (A)	32. (C)	33. (A)	34. (A)	35. (A)
36. (B)	37. (A)	38. (B)	39. (B)	40. (A)
41. (C)	42. (A)	43. (A)	44. (A)	45. (A)
46. (B)	47. (B)	48. (A)	49. (A)	50. (A)
51. (A)	52. (A)	53. (A)	54. (A)	55. (A)
56. (A)	57. (B)	58. (B)	59. (A)	60. (A)
61. (B)	62. (C)	63. (D)	64. (B)	65. (C)
66. (C)	67. (A)	68. (A)	69. (A)	70. (B)
71. (B)	72. (A)	73. (B)	74. (A)	75. (B)
76. (D)	77. (D)	78. (D)	79. (A)	80. (B)
81. (C)	82. (C)	83. (A)	84. (A)	85. (A)
86. (A)	87. (B)	88. (B)	89. (A)	90. (A)
91. (A)	92. (C)	93. (B)	94. (B)	95. (A)
96. (C)	97. (B)	98. (A)	99. (C)	100. (A)

LEVEL OF DIFFICULTY–1
OBJECTIVE TYPE QUESTIONS

1. Let M be a $m \times n$ $(m < n)$ matrix with rank m. Then—
   (A) For every b in R^m, $M_x = b$ has unique solution
   (B) For every b in R^m, $M_x = b$ has a solution but it is not unique
   (C) There exists $b \in R^m$ for which $M_x = b$ has no solution
   (D) None of these

2. Let A be a $m \times n$ matrix with row rank $= r =$ column rank. The dimension of the space of solutions of the system of linear equations $AX = 0$ is—
   (A) r
   (B) $n - r$
   (C) $m - r$
   (D) $\min(m, n) - r$

3. If $f(x)$ is real valued function defined on $[0, \infty]$ such that $f(0) = 0$ and $f''(x) > 0$ for all x, then the function $h(x) = \dfrac{f(x)}{x}$ is—
   (A) Increasing in $[0, \infty]$
   (B) Decreasing in $[0, 1]$
   (C) Increasing in $[0, 1]$ and decreasing in $[1, \infty]$
   (D) Decreasing in $[0, 1]$ and increasing in $[1, \infty]$

4. Let P be a matrix of order $m \times n$ and Q be a matrix of order $n \times p$, $n \neq p$. If rank (P) $= n$ and rank (Q) $= p$, then rank (PQ) is—
   (A) n (B) p
   (C) np (D) $n + p$

5. Let w be the space spanned by $f = \sin x$ and $g = \cos x$. Then for any real value of θ, $f_1 = \sin(x + \theta)$ and $g_1 = \cos(x + \theta)$—
   (A) Are vectors in w
   (B) Are linearly independent
   (C) Do not form a basic for w
   (D) Form a basic for w

6. The eigen values of 3×3 real matrix P are $1, -2, 3$. Then—
   (A) $P^{-1} = \frac{1}{6}(5I + 2P - P^2)$
   (B) $P^{-1} = \frac{1}{6}(5I - 2P + P^2)$
   (C) $P^{-1} = \frac{1}{6}(5I + 2P + P^2)$
   (D) $P^{-1} = \frac{1}{6}(5I - 2P - P^2)$

7. The dimension of the vector space of all 3×3 real symmetric matrices is—
   (A) 3 (B) 9
   (C) 6 (D) 4

8. Let T be an arbitrary linear transformation from R^n to R^n which is not one-one. Then—
   (A) Rank T > 0 (B) Rank T $= n$
   (C) Rank T $< n$ (D) Rank T $= n - 1$

9. Let S and T be two subspaces of R^{24} such that dim (S) = 19 and dim (T) = 17. Then, the—
   (A) Smallest possible value of dim $(S \cap T)$ is 2
   (B) Largest possible value of dim $(S \cap T)$ is 18
   (C) Smallest possible value of dim $(S + T)$ is 19
   (D) Largest possible value of dim $(S + T)$ is 22

10. The set of all $x \in R$ for which the vectors $(1, x, 0)$, $(0, x^2, 1)$ and $(0, 1, x)$ are linearly independent in R^3 is—
    (A) $\{x \in R : x = 0\}$ (B) $\{x \in R : x \neq 0\}$
    (C) $\{x \in R : x \neq 1\}$ (D) $\{x \in R : x \neq -1\}$

11. The dimension of the subspace $\{(x_1, x_2, x_3, x_4, x_5) : 3x_1 - x_2 + x_3 = 0\}$ of R^5 is—
    (A) 1 (B) 2
    (C) 3 (D) 4

12. Consider R^2 with the usual topology. Let $S = \{(x, y) \in R^2 : x \text{ is an integer}\}$. Then S is—
(A) Open but not closed
(B) Both open and closed
(C) Neither open nor closed
(D) Closed but not open

13. Let $M = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$. Then the rank of M is equal to—
(A) 3 (B) 4
(C) 2 (D) 1

14. A matrix M has eigen values 1 and 4 with corresponding eigen vectors $(1, -1)^T$ and $(2, 1)^T$, respectively. Then M is—
(A) $\begin{bmatrix} -4 & -8 \\ 5 & 9 \end{bmatrix}$ (B) $\begin{bmatrix} 9 & -8 \\ 5 & -4 \end{bmatrix}$
(C) $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ (D) $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$

15. $\frac{d}{dx}\int_{2\sin x}^{2\sin x} e^{t^2} dt$ at $x = \pi$ is—
(A) 1 (B) –1
(C) 2 (D) –2

16. Let P and Q be square matrices such that PQ = 1, the identity matrix. Then zero is an eigen value of—
(A) P but not Q (B) Q but not P
(C) Both P and Q (D) Neither P nor Q

17. Consider the basic $S = \{v_1, v_2, v_3\}$ for R^3 where $v_1 = (1, 1, 1)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 0, 0)$ and let $T : R^3 \to R^3$ be a linear transformation such that $Tv_1 = (1, 0)$, $Tv_2 = (2, -1)$, $Tv_3 = (4, 3)$. Then T (2, –3, 5) is—
(A) (–1, 5) (B) (3, 4)
(C) (0, 0) (D) (9, 23)

18. Let $T : C^n \to C^n$ be a linear operator having n distinct eigen values. Then—
(A) T is invertible
(B) T is invertible as well as diagonalizable
(C) T is not diagonalizable
(D) T is diagonalizable

19. Let A be a non-zero upper triangular matrix all of whose eigen values are 0. Then I + A is—
(A) Invertible (B) Singular
(C) Idempotent (D) Nilopotent

20. Let T be a linear transformation from $R^3 \to R^2$ defined by $T(x, y, z) = (x + y, y - z)$. Then the matrix of T with respect to ordered bases {(1, 1, 1), (1, –1, 0), (0, 1, 0) and {(1, 1), (1, 0)} is—
(A) $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$
(C) $\begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}$

21. Let v_1 (1, 2, 0, 3, 0), $v_2 = (1, 2, -1, -1, 0)$, $v_3 = (0, 0, 1, 4, 0)$, $v_4 = (2, 4, 1, 10, 1)$ and $v_5 = (0, 0, 0, 0, 1)$. The dimension of the linear span of $\{v_1, v_2, v_3, v_4, v_5\}$ is—
(A) 2 (B) 3
(C) 4 (D) 5

22. Consider the vector space R^3 and the maps f, $g : R^3 \to R^3$ defined by $f(x, y, z) = (x, |y|, z)$ and $g(x, y, z) = (x + 1, y - 1, z)$. Then—
(A) Both f and g are linear
(B) Neither f nor g linear
(C) g is linear but not f
(D) f is linear but not g

23. Let the linear transformations S and $T : R^3 \to R^3$ be defined by $S(x, y, z) = (2x, 4x - y, 2x + 3y - z)$, $T(x, y, z) = (x \cos\theta - y \sin\theta, \sin\theta + y\cos\theta, z)$ where $0 < \theta < \frac{\pi}{2}$. Then—
(A) S is one to one but not T
(B) T is one to one but not S
(C) Both S and T are one to one
(D) Neither S nor T is one to one

24. Suppose X = {α, β, δ). Let
$J_1 = \{\phi, X, \{\alpha\}, \{\alpha, \beta\}\}$ and
$J_2 = \{\phi, X, \{\alpha), \{\beta, \delta\}\}$. Then—
(A) $J_1 \cap J_2$ and $J_1 \cup J_2$ are topologies
(B) Neither $J_1 \cap J_2$ nor $J_1 \cup J_2$ is a topology
(C) $J_1 \cup J_2$ is a topology but $J_1 \cap J_2$ is not a topology
(D) $J_1 \cap J_2$ is a topology but $J_1 \cup J_2$ is not a topology

25. Let F be a finite field. If $f : F \to F$, given by $f(x) = x^3$ is a ring homomorphism, then—
   (A) $F = Z/3Z$
   (B) $F = Z/2Z$ or characteristic of $F = 3$
   (C) $F = Z/2Z$ or $Z/3Z$
   (D) Characteristic of F is 3

26. Let PID, ED, UFD denote the set of all principal ideal domains, Euclidean domains, unique factorization domains, respectively. Then—
   (A) UFD $\subset$ ED $\subset$ PID
   (B) PID $\subset$ ED $\subset$ UFD
   (C) ED $\subset$ PID $\subset$ UFD
   (D) PID $\subset$ UFD $\subset$ ED

27. Let $f(x, y) = \sqrt{|xy|}$. Then—
   (A) f_x and f_y do not exist at (0, 0)
   (B) $f_x(0, 0) = 1$
   (C) $f_y(0, 0) = 0$
   (D) f is differentiable at (0, 0)

28. An analytic function $f(z)$ is such that Re $\{f'(z)\} = 2y$ and $f(1 + i) = z$. Then the imaginary part of $f(z)$ is—
   (A) $-2xy$ (B) $x^2 - y^2$
   (C) $2xy$ (D) $y^2 - x^2$

29. For $0 < \theta < \pi$, the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$—
   (A) Has no real eigenvalue
   (B) Is orthogonal
   (C) Is symmetric
   (D) Is skew symmetric

30. Let U be a 3×3 complex Hermitian matrix which is unitary. Then the distinct eigen values of U are—
   (A) $\pm i$ (B) $1 \pm i$
   (C) ± 1 (D) $\frac{1}{2}(1 \pm i)$

31. The eigen values of a skew-symmetric matrix are—
   (A) Negative
   (B) Real
   (C) Of absolute value 1
   (D) Purely imaginary or zero

32. Let the characteristics equation of a matrix M be $\lambda^2 - \lambda - 1 = 0$, then—
   (A) M^{-1} does not exist
   (B) M^{-1} exists but cannot be determined from the data
   (C) $M^{-1} = M + 1$
   (D) $M^{-1} = M - 1$

33. The set $V = \{(x, y) \in R^2 : xy \geq 0\}$ is—
   (A) A vector subspace of R^2
   (B) Not a vector subspace of R^2 since every element does not have an inverse in V
   (C) Not a vector subspace of R^2 since it is not closed under scalar multiplication
   (D) Not a vector subspace of R^2 since it is not closed under vector addition

34. Let $M = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$. Then—
   (A) M is diagonalizable but not M^2
   (B) M^2 is diagonalizable but not M
   (C) Both M and M^2 are diagonalizable
   (D) Neither M nor M^2 is diagonalizable

35. Let E be a non-measurable subset of [0, 1]. If $f : [0, 1] \to R$ is defined by
$$f(x) = \begin{cases} -1/2, & x \in E \\ 0, & \text{otherwise} \end{cases}$$ Then—
   (A) f is measurable but not $|f|$
   (B) $|f|$ is measurable but not f
   (C) Both f and $|f|$ are measurable
   (D) Neither f nor $|f|$ is measurable

36. For a positive integer n, let $f_n : R \to R$ be defined by $f_n(x) = \begin{cases} \frac{1}{4n+5}, & \text{if } 0 \leq x \leq n \\ 0, & \text{otherwise} \end{cases}$.
Then $\{f_n(x)\}$ converges to zero—
   (A) Uniformly but not in L^1 norm
   (B) Uniformly and also in L^1 norm
   (C) Pointwise but not uniformly
   (D) In L^1 norm but not pointwise

37. If S is a finite commulative ring with 1, then—
   (A) Each prime is ideal is a maximum ideal
   (B) S may have a prime ideal which is not maximal
   (C) S has no nontrivial maximal ideals
   (D) S is a field

38. Let $f : C \to C$ be given by

$$f(z) = \begin{cases} (\bar{z})^2/z & \text{, when } z \neq 0 \\ 0 & \text{, when } z = 0 \end{cases} \text{, then } f\text{—}$$

(A) Is not continuous at $z = 0$
(B) Is differentiable but not analytic at $z = 0$
(C) Is analytic at $z = 0$
(D) Satisfies the Cauchy-Riemann equations at $z = 0$

39. In a metric space (x, d)—
(A) Every finite set E has a limit point in E
(B) Every closed subset of a compact set is compact
(C) Every closed and bounded set is compact
(D) Every subset of a compact set in closed

40. The value of the integral $\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-4)(z-2)} dz$. Where c is the circle $|z| = 3$ traced anti-clockwise, is—
(A) $-2i\pi$ (B) $i\pi$
(C) $-i\pi$ (D) $2i\pi$

41. For the function $f(z) = \frac{1 - e^{-z}}{z}$, the point $z = 0$ is—
(A) An essential singularity
(B) A pole of order zero
(C) A pole of order one
(D) A removable singularity

42. The function $\sin z$ is analytic in—
(A) $C \cup \{\infty\}$
(B) C except on the negative real axis
(C) $C - \{0\}$
(D) C

43. The function $f(z) = z^2$ maps the first quadrant onto—
(A) Itself
(B) Upper half plane
(C) Third quadrant
(D) Right half plane

44. Consider a function $f(z) = u + iv$ defined on $|z - i| < 1$ where u, v are real valued functions of x, y. Then $f(z)$ is analytic for u equals to—
(A) $x^2 + y^2$ (B) $ln(x^2 + y^2)$
(C) e^{xy} (D) $e^{x^2 - y^2}$

45. Let f: $R^4 \to R$ be a linear functional defined by $f(x_1, x_2, x_3, x_4) = -x_2$. If $\{\cdot, \cdot\}$ denotes the standard inner product on R^4, then the unique vector $v \in R^4$ such that $f(w) = \{v, w\}$ for all $w \in R^4$, is—
(A) (0, –1, 0, 0) (B) (–1, 0, –1, 1)
(C) (0, 1, 0, 0) (D) (1, 0, 1, –1)

46. Let M be a skew symmetric, orthogonal real matrix. The only possible eigen values are—
(A) –1, 1 (B) $-i, i$
(C) 0 (D) $1, i$

47. Let $L^2([0, 1])$ denote the space of all square integrable functions on [0, 1]. Define f_1, f_2 : $[0, 1] \to R$ by—

$$f_1(t) = \begin{cases} t^{-1/3}, & 0 < t \leq 1 \\ 0, & t = 0 \end{cases}$$

$$f_2(t) = \begin{cases} t^{-2/3}, & 0 < t \leq 1 \\ 0, & t = 0 \end{cases}$$

then—
(A) f_1 belongs to $L^2([0, 1])$ but not f_2
(B) f_2 belongs to $L^2([0, 1])$ but not f_1
(C) Both f_1 and f_2 belongs to $L^2([0, 1])$
(D) Neither f_1 nor f_2 belongs to $L^2([0, 1])$

48. Let P_1 and P_2 be two projection operators on a vector space. Then—
(A) $P_1 + P_2$ is a projection if $P_1P_2 = P_2P_1 = 0$
(B) $P_1 - P_2$ is a projection if $P_1P_2 = P_2P_1 = 0$
(C) $P_1 + P_2$ is a projection
(D) $P_1 - P_2$ is a projection

49. Let f be an entice function. If f satisfies the following two equations $f(z + 1) = f(z)$, $f(z + i) = f(z)$ for every z in c, then—
(A) $f'(z) = f(z)$
(B) $f(z) \in R\forall_z$
(C) $f \equiv$ constant
(D) f is a non-constant polynomial

50. The bilinear transformation $w = 2z/(z - 2)$ maps $\{z : |z - 1| < 1\}$ onto—
(A) $\{w : \text{Re } w < 0\}$
(B) $\{w : \text{Im } w > 0\}$
(C) $\{w : \text{Re } w > 0\}$
(D) $\{w : |w + 2| < 1\}$

51. Let (x, d) be a complete matric space and $f : X \to X$ satisfies $d(x), f(y) \leq \alpha(x, y)$ for some $\alpha, 0 \leq \alpha < 1$ for all $x, y, \in X$. Then—
(A) f is bounded function on X
(B) f need not be continuous on X
(C) $\{f(X_n)\}_{n=1}^{\infty}$ may not be a Cauchy sequence even through $\{f(X_n)\}_{n=1}^{\infty}$ is a cauchy sequence in X
(D) $f(p) = p$ for some $p \in X$

52. For the function $f(z) = \frac{z - \sin z}{z^3}$, the point $z = 0$ is—
(A) A pole of order 3
(B) A pole of order 2
(C) An essential singularity
(D) A removable singularity

53. The transformation $w = e^{i\theta}\left(\frac{z-p}{\bar{p}z-1}\right)$, where p is a constant, maps $|z| < 1$ onto—
(A) $|w| < 1$ if $|p| < 1$
(B) $|w| > 1$ if $|p| > 1$
(C) $|w| = 1$ if $|p| = 1$
(D) $|w| = 3$ if $p = 0$

54. The series $\sum_{n=1}^{\infty} \frac{z^n}{n.\sqrt{n+1}}$ $|z| \leq 1$ is—
(A) Uniformly but not absolutely convergent
(B) Uniformly but absolutely convergent
(C) Absolutely convergent but not uniformly convergent
(D) Convergent by not uniformly convergent

55. The radius of convergence of the power series of the function $f(z) = \frac{1}{1-z}$ about $z = \frac{1}{4}$ is—
(A) 1 (B) 1/4
(C) 3/4 (D) 0

56. At $z = 0$, then function $f(z) = z^2\bar{z}$—
(A) Does not satisfy Cauchy-Reimann equations
(B) Satisfy Cauchy-Reimann equations but is not differentiable
(C) Is differentiable
(D) Is analytic

57. If D is the open unit disk in C and $f : C \to D$ is analytic with $f(10) = 1/2$, then $f(10 + i)$ is—
(A) $\frac{1+i}{2}$ (B) $\frac{1-i}{2}$
(C) $\frac{1}{2}$ (D) $\frac{i}{2}$

58. The principle value of $\log(i^{1/4})$ is—
(A) $i\pi$ (B) $\frac{i\pi}{2}$
(C) $\frac{i\pi}{4}$ (D) $\frac{i\pi}{8}$

59. For the ordinary differential equation
$$(x-1)\frac{d^2y}{dx^2} + (\cot \pi x)\frac{dy}{dx} + (\operatorname{cosec}^2 \pi x)\, y = 0,$$
which of the following statements is true ?
(A) 0 is regular and 1 is irregular
(B) 0 is irregular and 1 is regular
(C) Both 0 and 1 are regular
(D) Both 0 and 1 are irregular

60. Consider the system of linear equations
$x + y + z = 3,\ x - y - z = 4,\ x - 5y + kz = 6$
Then the value of k for which this system has an infinite number of solutions is—
(A) $k = -5$ (B) $k = 0$
(C) $k = 1$ (D) $k = 3$

61. The residue of $\frac{\sin z}{z^8}$ at $z = 0$ is—
(A) 0 (B) $-\frac{1}{7!}$
(C) $\frac{1}{7!}$ (D) None of these

62. The statement "the dual space of a non empty normed linear space is non empty" follows from—
(A) Uniform boundedness principle
(B) Hahn-Banach theorem
(C) Riesz representation theorem
(D) Closed graph theorem

63. Let $f_n(x) = \begin{cases} 1/x, & \text{if } n < x < n+1 \\ 0, & \text{otherwise} \end{cases}$

If $f(x) \lim_{x \to \infty} f_n(x)$, then $\int_{(0,\infty)} f(x)\, dx = \lim_{x \to \infty} \int_{(0,\infty)} f_n(x)\, dx$ follows by—
(A) Bounded convergence theorem

(B) Monotone convergence theorem

(C) Dominated convergence theorem

(D) None of the above

64. Let E be the set of all numbers x in [0, 1] such that the decimal expansion of x does not contain the digit 7. Then the Lebesgue measure of E is—

(A) 0 (B) 0·7

(C) 0·9 (D) 1

65. The value of the integral $\oint \frac{dz}{z^2 - 1}$, $c : |z| = 4$ is equal to—

(A) πi (B) 0

(C) $-\pi i$ (D) $2\pi i$

66. If $f(z) = z^3$, then it—

(A) Has an essential singularity at $z = \infty$

(B) Has a pole of order 3 at $z = \infty$

(C) Has a pole of order 3 at $z = 0$

(D) Is analytic at $z = \infty$

67. Let T be any circle enclosing the origin and oriented counter-clockwise. Then the value of the integral $\int_\tau \frac{\cos z}{z^2} dz$ is—

(A) $2\pi i$ (B) 0

(C) $-2\pi i$ (D) Undefined

68. The bilinear transformation w, which maps the points 0, 1, ∞ in the z-plane onto the points $-i$, ∞, 1 in the w plane is—

(A) $\frac{z-1}{z+i}$ (B) $\frac{z-i}{z+1}$

(C) $\frac{z+i}{z-1}$ (D) $\frac{z+1}{z-i}$

69. The real part of the principle value of 4^{4-i} is—

(A) 256 cos (ln 4)

(B) 64 cos (ln 4)

(C) 16 cos (ln 4)

(D) 4 cos (ln 4)

70. Consider the functions $f(z) = x^2 + iy^2$ and $g(z) = x^2 + y^2 + ixy$, at $z = 0$—

(A) f is analytic but not g

(B) g is analytic but not f

(C) Both f and g are analytic

(D) Neither f nor g is analytic

71. For the n-th Legendre polynomial $c_n \frac{d^n y}{dx^n}$ $(x^2 - 1)^n$, the value of c_n is—

(A) $\frac{1}{(n!\, 2^n)}$ (B) $\frac{n!}{(2^n)}$

(C) $(n!)\, 2^n$ (D) $\frac{2^n}{(n!)}$

72. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ x & y & z \end{bmatrix}$ and let $V = \{(x, y, z) \in R^3 : \det(A) = 0\}$. Then the dimension of V equals—

(A) 0 (B) 1

(C) 2 (D) 3

73. Let Γ denoted the boundary of the square whose sides lies along $x = \pm 1$ and $y = \pm 1$, where Γ is described in the positive sense. Then the value of $\int_\Gamma \frac{z^2}{2z+3} dz$ is—

(A) $\frac{\pi i}{4}$ (B) $2\pi i$

(C) 0 (D) $-2\pi i$

74. Let $T : D \subset X \to Y$ be a linear transformation, where X and Y are normed linear spaces. Then T is closed if and only if—

(A) D is closed

(B) T (D) is closed

(C) Graph of T is closed

(D) T (D) is bounded

75. Let $f_n(x) = \frac{\sin x}{\sqrt{n}}$, $n = 1, 2, \ldots$ and $x \in [-1, 1]$. Then as $n \to \infty$—

(A) $\{f_n(x)\}_{n=1}^{\infty}$ does not converge uniformly in [–1, 1]

(B) $\lim_{n \to \infty} \int_{-1}^{1} f_n(n)\, dx \neq 0$

(C) $\{f_n(x)\}$ does not converge uniformly in [–1, 1]

(D) $f_n(x)$ $n = 1, 2$ is not uniformly continuous in [–1, 1]

76. Let $f_n : [0, 1] \to R$ be defined by

$$f_n(x) = 2n, \text{ if } \frac{2}{2n} \le x \le \frac{1}{n}$$
$$= 0, \text{ otherwise}$$

Then the values of $\int_0^1 \lim_{n \to \infty} f_n du$ and $\lim_{n \to \infty} \int_0^1 f_n\, du$ (where u is the Lebesque measure on R) are respectively—

(A) 0, 0 (B) 0, 1

(C) 1, 0 (D) 1, 1

77. Let E be the set of all rationals p such that $2 < p^2 < 3$. Then E is—

(A) Compact in Q

(B) Closed and bounded in Q

(C) Not compact in Q

(D) Closed and unbounded in Q

78. The uniformly continuous function is—

(A) Measurable

(B) Not measurable

(C) Measurable and simple

(D) Integrable and simple

79. Suppose S_1, S_2 and S_3 are measurable subset of [0, 1], each of measure 3/4, such that the measure of $S_1 \cup S_2 \cup S_3$ is 1. Then, the measure of $S_1 \cap S_2 \cap S_3$ lies in—

(A) $\left[1, \frac{1}{16}\right]$ (B) $\left[\frac{1}{16}, \frac{1}{8}\right]$

(C) $\left[\frac{1}{8}, \frac{1}{4}\right]$ (D) $\left[\frac{1}{4}, 1\right]$

80. The continuous function $f : R \to R$ defined by $f(x) = (x^2 + 1)^{2003}$ is—

(A) Onto but not one-one

(B) One-one but not onto

(C) Both one-one and onto

(D) Neither one-one nor onto

81. If $\sin z = \sum_{n=0}^{\infty} a_n \left(z - \frac{\pi}{4}\right)^n$, then a_6 equals—

(A) 0 (B) $\frac{1}{720}$

(C) $\frac{1}{(720\sqrt{2})}$ (D) $\frac{-1}{(720\sqrt{2})}$

82. The coefficient of $\frac{1}{z}$ in the expansion of $\log \left(\frac{z}{z-1}\right)$, valid in $|z| > 1$, is—

(A) – 1 (B) 1

(C) $-\frac{1}{2}$ (D) $\frac{1}{2}$

83. Let G be a cyclic group of order 8, then its group of automorphisms has order—

(A) 2 (B) 4

(C) 6 (D) 8

84. Let $S = \{0\} \cup \left\{\frac{1}{4n+7} : n = 1, 2, \ldots\right\}$. Then the number of analytic functions which vanish only on S is—

(A) Infinite (B) 0

(C) 1 (D) 2

85. Let X be the space of polynomials in one variable.

For $p \in X$ $p(t) = a_0 + a_1 t + \ldots + a_n t^n$, let

$$\|p\| = \text{sub}\{|p(t) : 0 \le t \le 1\},$$

$$\|p\|_1 = |a_0| + |a_1| + \ldots + |a_n|$$

Then—

(A) X is complete with $\|\cdot\|$

(B) X is complete with $\|\cdot\|_1$

(C) X cannot be normal so as to make it complete

(D) None of these

86. The Hermit interpolating polynomial for the function $f(x) = x^6$ based on –1, 0 and 1 is—

(A) $x^4 - 2x^2$ (B) $2x^4 - x^2$

(C) $x^4 + 2x^2$ (D) $2x^4 + x$

87. The function $w(z) = -\left(\frac{1}{z} + bz\right)$, $-1 < b < 1$, maps $|z|$ onto—

(A) A half plane

(B) Exterior of the circle

(C) Exterior of an ellipse

(D) Interior of an ellipse

88. The value of the surface integral $\iint_S F.\, n dS$, where S is the surface of the sphere $x^2 + y^2 + z^2 = 4$, n is the unit outward normal and $F = x\hat{i} + y\hat{j} + z\hat{k}$, is—

(A) 32π (B) 16π

(C) 8π (D) 64π

89. Let A be the set of points in the interval (0, 1) representing the numbers whose expansion as infinite decimal do not contain the digit 7. Then the measure of A is—

(A) 1 (B) 0

(C) $\frac{1}{2}$ (D) ∞

90. Which of the following pair of functions is not a linearly independent pair of solutions of $y'' + 9y = 0$?

(A) $\sin 3x$, $\sin 3x - \cos 3x$

(B) $\sin 3x + \cos 3x$, $3\sin x - 4\sin^3 x$

(C) $\sin 3x$, $\sin 3x \cos 3x$

(D) $\sin 3x + \cos 3x$, $4\cos^3 x - 3\cos x$

91. Let $f : [0, 1] \to \mathrm{R}$ be a bounded Riemann integrable function and let $g : \mathrm{R} \to \mathrm{R}$ be continuous. Then $g o f$ is—

(A) Riemann integrable

(B) Continuous

(C) Lebesque integrable, but not Reimann integrable

(D) Nor necessarily measurable

92. Diameter of a set S in a metric space with metric d defined by diam (S) = $1{\cdot}u{\cdot}b{\cdot}$ $\{d(x, y) / x, y \text{ in S}\}$

Thus diameter of cylinder C = $\{(x, y, z)$ in R^3 $|x^2 + y^2 = 1, -1 < z < 1\}$ in R^3 with standard metric, is—

(A) 2 (B) $2\sqrt{2}$

(C) $\sqrt{2}$ (D) $\pi + 2$

93. The equation $x^6 - x - 1 = 0$ has—

(A) No positive real roots

(B) Exactly one positive real root

(C) Exactly two positive real roots

(D) All positive real roots

94. Under the usual topology in R^3, if O = $\{(x, y, z) \in \mathrm{R}^3 : x^2 + y^2 < 1\}$ and F = $\{(x, y, z) \in \mathrm{R}^3 : z = 0\}$, then $\mathrm{O} \cap \mathrm{F}$ is—

(A) Both open and closed

(B) Neither open nor closed

(C) Open but not closed

(D) Closed but not open

95. Let M_3 (R) be the ring of 3×3 real matrices. If $\mathrm{I}, \mathrm{J} \subseteq \mathrm{M}_3(\mathrm{R})$ are defined as

$$\mathrm{I} = \left\{ \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| a, b, c \in \mathrm{R} \right\},$$

$$\mathrm{J} = \left\{ \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix} \middle| a, b, c \in \mathrm{R} \right\}, \text{ then}$$

(A) I is a right ideal and J a left ideal

(B) I and J are both left ideals

(C) I and J are both right ideals

(D) I is a left ideal and J a right ideal

96. It is given that $\sum_{n=0}^{\infty} a_n z^n$ converges at $z = 3 + i^4$. Then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ is—

(A) ≤ 5 (B) ≥ 5

(C) < 5 (D) > 5

97. Let X and Y be normal linear spaces. Every linear map $\mathrm{T} : \mathrm{X} \to \mathrm{Y}$ is continuous if—

(A) dim (X) $< \infty$ (B) dim (Y) $< \infty$

(C) dim (Y) = 1 (D) X = l^2

98. The system of equations

$$\begin{aligned} 3x + 2y &= 4{\cdot}5 \\ 2x + 3y - z &= 5{\cdot}0 \\ -y + 2z &= -0{\cdot}5 \end{aligned}$$

is to be solved by successive over relaxation method. The optimal relaxation factor, w_{opt}, rounded upto two decimal places is given by—

(A) 1·23 (B) 0·78

(C) 1·56 (D) 0·63

99. The function $f(z) = \begin{cases} (\bar{z})^2/z^2, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}$ —

(A) Satisfies the Cauchy-Riemann equations at $z = 0$

(B) Is not continuous at $z = 0$

(C) Is differentiable at $z = 0$

(D) Is analytic at $z = 1$

100. The value of line integral $\int_{\mathrm{L}} yzdx + xzdy + xydz$ where L is the line segment from (1, 1, 0) to (2, 3, 2), is—

(A) 0 (B) 9

(C) 7 (D) 12

101. The Fourier expansion in the interval [–4, 4] of the function

$$\begin{aligned} f(x) &= -x, -4 \leq x \leq 0, \\ &= x, 0 \leq x \leq 4 \end{aligned}$$

has—

(A) No cosine term

(B) No sine term

(C) Both cosine and sine term

(D) None of these

102. Determine the type of following differential equation

$$\frac{d^2y}{dx^2} + \sin(x + y) = \sin x—$$

(A) Linear, homogeneous
(B) Non-linear, homogeneous
(C) Linear, non-homogeneous
(D) Non-linear, non-homogeneous

103. Let V be the volume of a region bounded by a smooth closed surfaces. Let r denote the position vector and $\hat{n}$ denote the outward unit normal to S. Then the integral $\iint_s r \cdot \hat{n}\, ds$ equals—

(A) V (B) $\frac{V}{3}$
(C) 3V (D) 0

104. Let X = (0, 1) U (2, 3) be on open set in R. Let f be a continuous function on X such that the derivative $f'(x) = 0$ for all x. Then the range of f has—

(A) Uncountable number of points
(B) Countably infinite number of points
(C) At most 2 points
(D) At most 1 point

105. Let $f, g : (0, 1) \times (0, 1) \to R$ be two continuous functions defined by $f(x, y) = \frac{1}{1 + x(1-y)}$ and $g(x, y) = \frac{1}{1 + x(y-1)}$. Then on $(0, 1) \times (0, 1)$—

(A) f and g are both uniformly continuous
(B) f is uniformly continuous but g is not
(C) g is uniformly continuous but f is not
(D) Neither f nor g is uniformly continuous

106. Suppose E is a non measurable subset of [0, 1]. Let $P = E^\circ \cup \left\{\frac{1}{n} : n \in N\right\}$ and $Q = \bar{E} \cup \left\{\frac{1}{n} : n \in N\right\}$ where E° is the interior of E and $\bar{E}$ is the closure of E. Then—

(A) P is measurable but not Q
(B) Q is measurable but not P
(C) Both P and Q are measurable
(D) Neither P nor Q is measurable

107. Consider the Hitbert space

$l^2 =$

$$\left\{ (x_1, x_2, \ldots) \mid x_i \in R, i = 1, 2, \ldots \text{ and } \sum_{i=1}^{\infty} x_i^2 < \infty \right\}$$

under the inner product $<(x_1, x_2, \ldots), (y_1, y_2, \ldots)> = \sum_{i=1}^{\infty} x_i y_i$.

Let $S = \left\{(x_1, x_2, \ldots) \in l^2 \mid \sum_{n=1}^{\infty} \frac{x_n}{n} = 0\right\}$.

Then the number of interior points of S is—

(A) 0
(B) Non zero but finite
(C) Countably infinite
(D) Uncountably infinite

108. The value of α for which G = {α, 1, 3, 9, 19, 27} is a cyclic group under multiplication modulo 56 is—

(A) 5 (B) 15
(C) 25 (D) 35

109. All the eigenvalues of matrix $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ lie in the disc—

(A) $|\lambda + 1| \le 1$
(B) $|\lambda - 1| \le 1$
(C) $|\lambda + 1| \le 0$
(D) $|\lambda - 1| \le 2$

110. The Fourier series of the 2π periodic function $f(x) = x + x^2$, $-\pi < x \le \pi$ at $x = \pi$ converges to—

(A) π (B) 2π
(C) π^2 (D) $\pi + \pi^2$

111. The value of $\int_{|z|=2} \frac{e^2}{(z+1)^4}\, dz$ is—

(A) $2\pi i\, e^{-1}$ (B) $\frac{8\pi i}{3} e^{-2}$
(C) $\frac{2\pi i}{3} e^{-2}$ (D) 0

112. Let $f(x, y) = (x - 2)^2 (y + 3)$. Then—

(A) (2, –3) is not a stationary point of f
(B) f has a local maximum at (2, –3)
(C) f has a local minimum at (2, –3)
(D) f has neither a local maximum nor a local minimum at (2, –3)

113. The particular solution of the equation $y'\sin x = y\, ln\, y$ satisfying the initial condition $y\left(\frac{\pi}{2}\right) = e$, is—

(A) $e^{\tan(x/2)}$ (B) $e^{\cot(x/2)}$

(C) $ln \tan\left(\frac{x}{2}\right)$ (D) $ln \cot\left(\frac{x}{2}\right)$

114. Which of the following is not an integrating factor of $x\,dy - y\,dx = 0$?

(A) $\frac{1}{x^2}$ (B) $\frac{1}{x^2+y^2}$

(C) $\frac{1}{xy}$ (D) $\frac{x}{y}$

115. Let G be a cyclic group of order σ. Then the number of elements $g \in$ G such that $G = < g >$ is—

(A) 5 (B) 3

(C) 4 (D) 2

116. The orthogonal trajectory to the family of circles $x^2 + y^2 = 2cx$ (c arbitrary) is described by the differential equation—

(A) $(x^2 + y^2)\, y' = 2xy$

(B) $(x^2 - y^2)\, y' = 2xy$

(C) $(y^2 - x^2)\, y' = xy$

(D) $(y^2 - x^2)\, y' = 2xy$

117. Let S be the surface bounding the region $x^2 + y^2 \le 1$, $x \ge 0$, $y \ge 0$ $|z| \le 1$, and $\hat{n}$ be the unit outer normal to S. Then $\iint_S [(\sin^2 x)\,\hat{i} + 2y\,\hat{j} - z(1 + \sin 2x)\,\hat{k}] \cdot \hat{n}\, ds$ equals—

(A) 1 (B) $\frac{\pi}{2}$

(C) π (D) 2π

118. The value of $\int_0^{\pi}\int_x^{\pi}\int_0^{2} \frac{\sin y}{y}\, dzdydx$ is—

(A) –2 (B) 2

(C) –4 (D) 4

119. Let C ([0, 1]) be the space of all real valued continuous functions on [0, 1] with the norm $\|f\|_0 = \{|f(x)| : x \in [0, 1]\}$. Sup consider the subspace P_n ([0, 1]) of all polynomials of degree less than or equal to n and subspace P ([0, 1]) of all polynomials on [0, 1]. Then—

(A) P_n ([0, 1] is closed in C ([0, 1]) but not P ([0, 1])

(B) P ([0, 1]) is closed in C ([0, 1]) but not P_n ([0, 1])

(C) Both P ([0, 1]) and P_n ([0, 1]) are closed in C ([0, 1])

(D) Neither P ([0, 1]) nor P_n ([0, 1]) is closed in C ([0, 1])

120. Consider Z_{24} as the additive group modulo 24. Then the number of elements of order 8 in the group Z_{24} is—

(A) 1 (B) 2

(C) 3 (D) 4

121. $\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k}{n}\right) x^k (1 - x)^{n-k}$ is equal to—

(A) x^2 (B) 1

(C) x (D) x^n

122. Let S be non empty Lebesque measurable subset of R such that every subset of S is measurable. Then the measure of S in equal to the measure of any—

(A) Subset of S

(B) Countable subset of S

(C) Bounded subset of S

(D) Closed subset of S

123. Let (X, τ) be a topology space. Where X = $\{a, b, c, d\}$ and $\tau = \phi$ X, $\{a\}$, $\{a, b\}$, $\{a, c\}$, $\{a, b, c\}$. The limit points of the set A = {A, c, d} are—

(A) a and b (B) b and c

(C) c and d (D) d and a

124. The differential equation $(3a^2x^2 + by\cos x)\, dx + (2\sin x - 4ay^3)\, dy = 0$ is exact for—

(A) $a = 3, b = 2$ (B) $a = 2, b = 3$

(C) $a = 3, b = 4$ (D) $a = 2, b = 5$

125. The differential equation $\frac{dy}{dx} = k(a - y)(b - y)$, when solved with the condition $y(0) = 0$, yields the result—

(A) $\frac{b(a-y)}{a(b-y)} = e^{(a-b)kx}$

(B) $\frac{b(a-x)}{a(b-x)} = e^{(b-a)ky}$

(C) $\frac{a(b-y)}{b(a-y)} = e^{(a-b)kx}$

(D) $xy = ke$

126. Let G be a group of order 49. Then—
    (A) G is Abelian
    (B) G is cyclic
    (C) G is non-Abelian
    (D) Centre of G has order 7

127. The number of elements of order 5 in the symmetric group S_5 is—
    (A) 5 (B) 20
    (C) 24 (D) 12

128. Let $y_1(x)$ and $y_2(x)$ be solutions of $y''x^2 + y' + (\sin x)\,y = 0$, which satisfy the boundary conditions $y_1(0) = 0$, $y_1'(1) = 1$ and $y_2(0) = 1$, $y_2'(1) = 0$ respectively. Then—
    (A) y_1 and y_2 do not have common zeros
    (B) y_1 and y_2 have common zeros
    (C) Either y_1 or y_2 has a zero of order 2
    (D) Both y_1 and y_2 have zeroes of order 2

129. Let $f : [0, \infty] \to R$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{\sqrt{x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Consider the two improper integral $I_1 = \int_0^1 f(x)\,dx$ and $I_2 = \int_1^\infty f(x)\,dx$, then—
    (A) Both I_1 and I_2 exist
    (B) I_1 exists but I_2 does not
    (C) I_1 does not exist but I_2 does
    (D) Neither I_1 nor I_2 exists

130. Let $S\backslash = \left\{\frac{1}{n} : n \in N\right\} \cup \{0\}$ and $T = \left\{n + \frac{1}{n} : n \in N\right\}$ be the subset of the metric space R with the usual metric. Then—
    (A) S is complete but not T
    (B) T is complete but not S
    (C) Both T and S are complete
    (D) Neither T nor S is complete

131. In the region $x > 0, y > 0$, the partial differential equation $(x^2 - y^2)\dfrac{\partial^2 u}{\partial x^2} + 2(x^2 + y^2)\dfrac{\partial^2 u}{\partial x \partial y} + (x^2 - y^2)\dfrac{\partial^2 u}{\partial y^2} = 0$—
    (A) Changes type (B) Is elliptic
    (C) Is parabolic (D) Is hyperbolic

132. Define $f : R^2 \to R$ y $f(x, y) = \begin{cases} 1, & \text{if } xy = 0 \\ 2, & \text{otherwise} \end{cases}$

If $S = \{(x, y) : f$ is continuous at the point $(x, y)\}$, then—
    (A) S is open (B) S is connected
    (C) $S \neq \phi$ (D) S is closed

133. The equation $e^x - 4x^2 = 0$ has a root between 4 and 5. Fixed point iteration function $\frac{1}{2}e^{x/2}$—
    (A) Diverges
    (B) Converges
    (C) Oscillates
    (D) Converges monotonically

134. If $y' - x \neq 0$, a solution of the differential equation $y'(y' + y) = x(x + y)$ is given by $y =$
    (A) $1 - x - e^{-x}$ (B) $1 - x + e^x$
    (C) $1 + x + e^{-x}$ (D) $1 + x + e^x$

135. Let A and B be $n \times n$ matrices with the same minimal polynomial. Then—
    (A) A is the similar to B
    (B) A is diagonalizable if B is diagonalizable
    (C) A – B is singular
    (D) A and B commute

136. The eigen values of the boundary value problem $x'' + \lambda x = 0$; $x(0) = 0$, $x(\pi) + x'(\pi) = 0$, satisfy—
    (A) $\lambda + \tan \lambda\pi = 0$
    (B) $\sqrt{\lambda} + \tan \lambda\pi = 0$
    (C) $\sqrt{\lambda} + \tan \sqrt{\lambda}\,\pi = 0$
    (D) $\lambda + \tan \sqrt{\lambda}\pi = 0$

137. The Sturm-Liouville problem : $y'' + \lambda^2 y = 0$, $y'(0) = 0$, $y'(\pi) = 0$ has its eigen vectors given by $y =$
    (A) $\sin\left(n + \frac{1}{2}\right)x$
    (B) $\sin nx$
    (C) $\cos\left(n + \frac{1}{2}\right)x$
    (D) $\cos nx$; where $n = 0, 1, 2, \ldots$

138. The polynomial $f(x) = x^5 + 5$ is—

(A) Irreducible over C

(B) Irreducible over R

(C) Ineducible over Q

(D) Not irreducible over Q

Where Q denotes the field of rational number

139. The order of element $(\bar{2}, \bar{2})$ in $Z_4 \times Z_6$ is—

(A) 2 (B) 6

(C) 4 (D) 12

140. For the Sturm Liouville problems : $(1 + x^2) y'' + 2xy' + \lambda x^2 y = 0$ with $y'(1) = 0$ and $y'(10) = 0$ the eigen-values, λ, satisfy—

(A) $\lambda \geq 0$ (B) $\lambda < 0$

(C) $\lambda \neq 0$ (D) $\lambda \leq 0$

141. The orthogonal trajectory to the family of straight lines $y = k(x - 1)$, $k \in$ R, are given by—

(A) $(x - 1)^2 + (y - 1)^2 = c^2$

(B) $x^2 + y^2 = c^2$

(C) $x^2 + (y - 1)^2 = c^2$

(D) $(x - 1)^2 + y^2 = c^2$

142. In a sufficiently small neighbourhood around $x = 2$, the differential equation $\frac{dy}{dx} = \frac{y}{\sqrt{x}}$, $y(2) = 4$ has—

(A) No solution

(B) A unique solution

(C) Exactly two solutions

(D) Infinitely many solutions

143. Consider the partial differential equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ satisfying the initial condition $u(x, 0) = \alpha + \beta x$. If $u(x, t) = 1$ along the characteristic $x = t + 1$, then—

(A) $\alpha = 1, \beta = 1$ (B) $\alpha = 2, \beta = 0$

(C) $\alpha = 0, \beta = 0$ (D) $\alpha = 0, \beta = 1$

144. Consider the linear programming problem $M_{a \times z} = c_1 x_1 + c_2 x_2$, $c_1, c_2 > 0$

Subject to $x_1 + x_2 \leq 3$

$2x_1 + 3x_2 \leq 4$

$x_1, x_2 \geq 0$

then—

(A) The primal has an optimal solution but the dual does not have an optimal solution

(B) Both the primal and the dual have optimal solutions

(C) The dual has an optimal solution but the primal does not have an optimal solution

(D) Neither the primal nor the dual have optimal solutions

145. The formula $A_0 f\left(-\frac{1}{2}\right) + A_2 f(0) + A_2 f\left(\frac{1}{2}\right)$ which approximates the integral $\int_{-1}^{1} f(x)\, dx$ is exact for polynomials of degree less than or equal to z if—

(A) $A_0 = A_2 = \frac{4}{3}, A_1 = \frac{2}{3}$

(B) $A_0 = A_1 = A_2 = 1$

(C) $A_0 = A_2 = \frac{4}{3}, A_1 = -\frac{2}{3}$

(D) None of the above

146. For the differential equation $4x^3 y'' + 6x^2 y' + y = 0$, the point at infinity is—

(A) An ordinary point

(B) A regular singular point

(C) An irregular singular point

(D) A critical point

147. Let $A \in C^{m \times n}$ and A', A^* denote respectively the transpose and conjugate and conjugate transpose of A. Then—

(A) Rank (AA^*A) = rank (A)

(B) Rank (A) = rank (A^2)

(C) rank (A) = rank (A'A)

(D) rank (A^2) = rank (A) = rank (A^3) – rank (A^2)

148. If $2x(1 - y) = k$ and $g(x, y) = L$ are orthogonal families of curves where K and L are constants, then $g(x, y)$ is—

(A) $x + 2y - y^2$ (B) $2y(1 - x)$

(C) $x^2 + 2x - y^2$ (D) $x^2 + 2y + y^2$

149. Let G be the additive group of integers I and G' be the multiplicative group of the fourth

roots of unity. Let $f : G \to G'$ be a homomorphism mapping given by $f(n) = i^n$; where $i = \sqrt{-1}$. Then the kernel of f is—

(A) Empty set
(B) $\{4m : m \in I\}$
(C) $\{(2m)^2 + 1 : m \in I\}$
(D) $\{2m + 1 : m \in I\}$

150. Let U be a 3×3 complex Hermitian matrix which is unitary. Then the distinct eigen values of U are—

(A) $\pm i$ (B) $1 \pm i$
(C) ± 1 (D) $\frac{1}{2}(1 \pm i)$

151. Which of the following Banach spaces is not separable ?

(A) $L^1 [0, 1]$ (B) $L^\infty [0, 1]$
(C) $L^2 [0, 1]$ (D) $C [0, 1]$

152. The number of groups of order n (upto isomorphism) is—

(A) Finite for all values of n
(B) Finite only for finitely many values of n
(C) Finite for infinitely many values of n
(D) Infinite for some values of n

153. If $y = \phi(x)$ is a particular solution of $y'' + (\sin x)\, y' + 2y = e^x$ and $y = \psi(x)$ is a particular solution of $y'' + (\sin x)\, y' + 2y = \cos 2x$, then a particular solution of $y'' + (\sin x)\, y' + 2y = e^x + 2\sin^2 x$, is given by—

(A) $\phi(x) - \psi(x) + \frac{1}{2}$
(B) $\psi(x) - \phi(x) + \frac{1}{2}$
(C) $\phi(x) - \psi(x) + 1$
(D) $\psi(x) - \phi(x) + 1$

154. Let D_8 denote the group of symmetries of square (dihedral group). The minimal number of generators for D_8 is—

(A) 1 (B) 2
(C) 4 (D) 8

155. Consider the usual topology on R. Let
S = {U $\subseteq$ R : U is either bounded open or empty or R)
and T = {U $\subseteq$ R : U is either unbounded open or empty or R)
Then, on R—

(A) S is a topology but not T
(B) T is a topology but not S
(C) Both S and T are topologies
(D) Neither S nor T is a topology

156. Let $f(x) = x^{10} + x - 1, x \in R$ and let $x_k = k$, $k = 0, 1, 2, \ldots, 10$. Then the value of the divided difference $f[x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}]$ is—

(A) –1 (B) 0
(C) 1 (D) 10

157. If $x = \xi$ is a double root of the equation $f(x) = 0$ and if $f'''(x)$ is continuous in a neighbourhood of ξ, then the iteration scheme for determining ξ :

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$$

has the order—

(A) 2 (B) 1
(C) Less than 2 (D) Less than 1

158. Using transformation $u = \frac{w}{y}$ in the partial differential equation $xu_x = u + yu_y$, the transformed equation has a solution of the form $w =$

(A) $f\frac{x}{y}$ (B) $f(x + y)$
(C) $f(x - y)$ (D) $f(xy)$

159. Consider 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $a + d = I = ad - bc$, then A^3 equals—

(A) 0 (B) –I
(C) 3I (D) None of these

160. Let S_3 be the group of all permutations on three symbols, with the identity element e. Then the number of elements in S_3 that satisfy the equation $x^2 = e$ is—

(A) 1 (B) 2
(C) 3 (D) 4

161. If A is the subspace of l^∞ consisting of all sequences of zeros and ones and d is the induced metric on A, then the rare sets in (A, d) are—

(A) Empty set
(B) All singleton subset of A

(C) Power set of A
(D) Set A itself

162. If Δ and ∇ are the forward and the backward difference operators respectively, then $\Delta - \nabla$ is equal to—
(A) $-\Delta\nabla$ (B) $\Delta\nabla$
(C) $\Delta + \nabla$ (D) $\frac{\Delta}{\nabla}$

163. For a subset A of a metric space, which of the following implies the other three ?
(A) A is closed
(B) A is bounded
(C) Closure of B is compact for every $B \subseteq A$
(D) A is compact

164. The set of all real 2×2 invertible matrices as on R^2 by matrix multiplication. The number of orbits for this action is—
(A) 1 (B) 2
(C) 4 (D) Infinite

165. Let $P_n(x)$ be the Legendre polynomial of degree $n \le 0$. If $1 + x^{10} = \sum_{n=0}^{10} c_n P_n(x)$, then c_5 equals—
(A) 0 (B) $\frac{2}{11}$
(C) 1 (D) $\frac{11}{2}$

166. For the differential equation $x^2(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$
(A) $x = 1$ is an ordinary point
(B) $x = 1$ is a regular singular point
(C) $x = 0$ is an irregular singular point
(D) $x = 0$ is an ordinary point

167. Let x, y and z be events which are mutually independent, with probabilities a, b, c respectively. Let the random variable N denote the number of X, Y or Z which occur. Then, the probability that N = 2 is—
(A) $ab + bc + ca - abc$
(B) $ab + bc + ca - 3abc$
(C) $2(a + b + c) - abc$
(D) $ab + bc + ca$

168. Let X and Y be jointly distributed random variables having the joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{x}, & \text{if } x^2 + y^2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Then P (Y > max (X – Y)—
(A) $\frac{1}{2}$ (B) $\frac{1}{3}$
(C) $\frac{1}{4}$ (D) $\frac{1}{6}$

169. Let $f(x, y) = \frac{xy}{x^2 + y^2}$, $(x, y) \ne (0, 0)$: $f(0, 0) = 0$ then—
(A) f is continuous at (0, 0) and the partial derivatives f_x, f_y exist at every point of R^2
(B) f if discontinuous at (0, 0) and f_x, f exist at point of R^2
(C) f is discontinuous at (0, 0) and f_x, f exist only at (0, 0)
(D) None of the above

170. The complete integral of the partial differential equation $xp^3q^2 + yp^2q^3 + (p^3 + q^3) - zp^2q^2 = 0$, is z—
(A) $ax + by + (ab^{-2} + ba^{-2})$
(B) $ax - by + (ab^{-2} - ba^{-2})$
(C) $-ax + by + (ba^{-2} - ab^{-2})$
(D) $ax + by - (ab^{-2} + ba^{-2})$

171. Let $C_{2\pi}$ denote the set of all 2π periodic continuous functions. Let S_f denote the Fourier series of f. That $S_f \notin C_{2\pi}$ for some $f \notin C_{2\pi}$, follows from—
(A) Principle of uniform boundedness
(B) Halm Banach theorem
(C) Open mapping theorem
(D) Closed graph theorem

172. Let G be a group of order 15. Then the number of sylow subgroups of G of order 3 is—
(A) 0 (B) 1
(C) 3 (D) 5

173. The norms of the linear functional f defined on C [–1, 1] by $f(x) = \int_{-1}^{0} x(t)dt - \int_{0}^{1} x(t)dt$ is—
(A) Zero (B) One
(C) Two (D) Three

where C [–1, 1] denotes a Banach space of all real valued functions $x(t)$ on [–1, 1] with norm given by $\|x\| = \max |x(t)|\ t \in [-1, 1]$

174. One root of equation $e^x - 3x^2 = 0$ lies in the interval (3, 4). The least number of iterations of the bisection method so that $|\text{error}| < 10^{-3}$ are—

(A) 10 (B) 8
(C) 6 (D) 4

175. Let n be a non negative integer. The eigen values of the Sturm-Liouville problem

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

with boundary conditions $y(0) = y(2\pi)$, $\frac{dy}{dx}(0) = \frac{dy}{dx}(2\pi)$ are—

(A) n (B) $n^2\pi^2$
(C) $n\pi$ (D) n^2

176. Let l_2 be the set of real sequence $\{x_n\}$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. For x in l_2 define $|x|^2 = \sum_{n=1}^{\infty} |x_n|^2$. Consider the set $S = \{x \in I_2$ such that $|x| < 1\}$. Then—

(A) Interior of S is compact
(B) S is compact
(C) Closure of S is compact
(D) Closure of S is not compact

177. Let I be the set of irrational real numbers and let G = IU {0}. Then under the usual addition of real numbers, G is—

(A) A group, since R and Q are groups under addition
(B) A group, since additive identity is in G
(C) Not a group, since addition on G is not a binary operation
(D) Not a group, since not all elements in G have an inverse

178. Let D_8 denote the group of symmetries of square (dihedral group). The minimal number of generators for D_8 is—

(A) 1 (B) 2
(C) 4 (D) 8

179. Assume that 45 per cent of population favour a certain candidate in an election., If a random sample of size 200 is chosen, then the standard deviation of the number of members of the sample that favours the candidate is—

(A) 6·12 (B) 5·26
(C) 8·18 (D) 7·04

180. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed chi-square random variable, each having 4 degree of freedom. Define $S_n = \sum_{i=1}^{n} X_1^2, n = 1, 2, \ldots$

If $\frac{S_n}{n} \xrightarrow{p} u$, as $n \longrightarrow \infty$, then u—

(A) 8 (B) 16
(C) 24 (D) 32

181. Let $f : [a, b] \rightarrow (0, \infty)$ be continuous. Let $G_f = \{(x, y) : y = f(x)\}$ be the graph of f. Then—

(A) G_f is measurable with measure zero in R^2
(B) G_f is measurable only if f is differentiable in (a, b)
(C) G_f is measurable and the measure of G_f lies between $(b - a) f(a)$ and $(b - a) f(b)$
(D) G_f need not be measurable

182. Introducing the new co-ordinate $Q = q \sin \omega t$ in the Lagrangian $L = \frac{m}{2}(\dot{q}^2 \sin^2 \omega t + \dot{q}\omega \sin 2\omega t + q^2\omega^2)$

the Hamiltonian H equals (p is conjugate momentum)—

(A) $\frac{m}{2}\left(\frac{p^2}{m^2} - Q^2\omega^2\right)$

(B) $\frac{m}{2}\left(\frac{p^2}{m^2} + Q^2\omega^2\right)$

(C) $\frac{m}{2}\left(\frac{p^2}{m^2} - Q^2\omega^2 + 2\omega^2 Q \operatorname{cosec} \omega t\right)$

(D) $\frac{m}{2}\left(\frac{p^2}{m^2} + Q^2\omega^2 - 2\omega^2 Q \operatorname{cosec} \omega t\right)$

183. The sequence $\{x_n\}$ of $m \times m$ matrices defined by the iterations

$X_{n+1} = 2X_n - X_n, AX_n, n = 0, 1, 2, \ldots$

When $X_0 = 1$, the identity matrix converges to A^{-1}, if and only if each eigen value λ of A satisfies—

(A) $|\lambda| < 1$ (B) $|\lambda - 1| < 1$
(C) $|\lambda + 1| < 1$ (D) None of the above

184. For $j = 1, 2, \ldots$, let $e_j = (a_1, a_2, a_3, \ldots)$ where $a_i = 1$ and $a_i = 0$ for $i \neq j$ then $E = \{e_1, e_2, e_3, \ldots\}$ is a—
(A) Hamel basis of l^1
(B) Hamel basis of l^∞
(C) Schauder basis of l^1
(D) Schauder basis of l^∞

185. The smallest value x ($|x| < 1$) correct to two decimal places satisfying the equation
$x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \ldots =$ 0·4431135 is—
(A) 0·58 (B) 0·47
(C) 0·44 (D) 0·88

186. If (r, θ, ϕ) is a harmonic function in a domain D, where (r, θ, ϕ) are spherical polar co-ordinates, then so is—
(A) $\frac{1}{r} f(r, \theta, \phi)$
(B) $\frac{1}{r^2} f\left(\frac{1}{r}, \theta, \phi\right)$
(C) $\frac{1}{r^2} f\left(\frac{1}{r^2}, \theta, \phi\right)$
(D) $\frac{1}{r} f\left(\frac{1}{r}, \theta, \phi\right)$

187. The Bessel's function $\{J_0(\alpha_k x)\}_{k=1}^{\infty}$ with α_k denoting the k-th zero of $J_0(x)$ form an orthogonal system on [0, 1] with respect to weight function—
(A) 1 (B) x^2
(C) x (D) $\sqrt{x}$

188. On $X = C[0, 1]$ define $T : X \to X$ by $T(f)(x) = \int_0^x f(t)\,dt$, for all f in X. Then—
(A) T is one-one and onto
(B) T is one-one but not two
(C) T is not one-one but onto
(D) T is neither one-one nor onto

189. In the group (Z, +), the subgroup generated by 2 and 7 is—
(A) Z (B) 5Z
(C) 9Z (D) 14Z

190. Let the set $\frac{Z}{nZ}$ denote the ring of integers modulo n under addition and multiplication modulo n. Then $\frac{Z}{9Z}$ is not a subring of $\frac{Z}{12Z}$ because—
(A) Z/9Z is not a subset of Z/12Z
(B) G.C.D. $(9, 12) = 3 \neq 1$
(C) 12 is not a power of 3
(D) 9 does not divide 12

191. Let X and Y be independent poisson random variables with parametre 1 and 2 respectively.
Then, P is $\left(X = 1 \mid \frac{X+Y}{2} = 2\right)$—
(A) 0·426 (B) 0·293
(C) 0·395 (D) 0·512

192. Let $\{E_n : n = 1, 2, \ldots\}$ be a decreasing sequence of lebesque measurable sets on R and let F be a lebesque measurable set on R such that $E_1 \cap F = \phi$. Suppose that F has lebesque measure 2 and the lebesque measure of E_n equals $\frac{2n+2}{3n+1}$, $n = 1, 2, \ldots$. Then the lebesque measure of the set $\left[\bigcap_{n=1}^{\infty} E_n\right] \cup F$ equals—
(A) $\frac{5}{3}$ (B) 2
(C) $\frac{7}{3}$ (D) $\frac{8}{3}$

193. Let $F : R \to R$ be a continuous function. If $f(\phi) \subseteq N$, then—
(A) $f(R) = N$
(B) $f(R) \subseteq N$ but f need not be constant
(C) f is unbounded
(D) f is a constant function

194. As a subset of [0, 1] equipped with the usual topology, the contain set is—
(A) Closed but not compact, no where dense and uncountable
(B) Not closed, dense and countable
(C) Closed, dense and uncountable
(D) Compact, no where dense and uncountable

195. A nontrivial solutions of $x^2y'' + xy' + 4y = 0$, $x > 0$ are—
(A) Bounded and non periodic
(B) Unbounded and non periodic
(C) Bounded and periodic
(D) Unbounded and periodic

196. Let X be a space of all real valued continuously differentiable functions on [0, 1] and Y be the space of all real valued continuous functions on [0, 1], both with supremum norm. Let T : X → Y be defined by
$(T_x)(t) = x'(t) + \int_0^t x(s)\,ds,\ 0 \le t \le 1, x \in X$
Then T is—
(A) Continuous and the graph of T is not closed
(B) Not continuous and the graph T is closed
(C) Continuous and the graph of T is closed
(D) Not continuous and the graph of T is not closed

197. The Jacobi's iteration method for the set of equations
$x_1 + a\,x_2 = 2,\ 2a\,x_1 + x_2 = 7, \left(a \ne \frac{1}{\sqrt{2}}\right)$
converges for—
(A) All values of a (B) $a = 1$
(C) $|a| < \frac{1}{\sqrt{2}}$ (D) $\frac{1}{\sqrt{2}} < a < \sqrt{\frac{3}{2}}$

198. The order of the numerical differentiation formula $f''(x_0) = \frac{1}{12h^2}[-\{f(x_0 - 2h) + f(x_0 + 2h)\} + 16\{f(x_0 - h) + f(x_0 + h) - 30f(x_0)]$ is—
(A) 2 (B) 3
(C) 4 (D) 1

199. Solve the integral equation
$$g(s) = s + \int_0^1 s\mu^2\, g(\mu)\, du$$
is given by—
(A) $g(t) = \frac{3t}{4}$ (B) $g(t) = \frac{4t}{3}$
(C) $g(t) = \frac{2t}{3}$ (D) $g(t) = \frac{3t}{2}$

200. Let M be the length of the initial interval $[a_0, b_0]$ containing a solution of $f(x) = 0$. Let $[x_0, x_1, x_2, \ldots]$ represent the successive points generated by the bisecting method. Then the minimum number of iteration required to quarantee an approximation to the solution with an accuracy of $\in$ is given by—
(A) $-2 - \frac{\log\left(\frac{\in}{M}\right)}{\log 2}$
(B) $-2 + \frac{\log\left(\frac{\in}{M}\right)}{\log 2}$
(C) $-2 + \frac{\log(M\in)}{\log 2}$
(D) $-2 - \frac{\log\left(\frac{\in}{M}\right)}{(\log 2)^2}$

Answers with Explanation

1. (B) $M_x = b$
where $M = m \times n$ matrix
rank $r = m$ (given)
Rank, M = Rank $[M, b] = m$
Hence equations are consistent $(n - r)$ *i.e.*. $n–m$ → linearly independent solution.
Now any linear combination of the $(n - m)$ independent solution will also be a solution. Hence other are infinite number of solutions.

2. (B) Let the equation be $\sum_{j=1}^{n} a_{ij}x_j = 0, i = 1, 2, \ldots, m$.
Since the rank of A which is $m \times n$ matrix, is r, the number of solutions (linearly independent) is $n - r$.
This is because you are looking at the annidilator of the subspace ω generated by $(a_{i1}, a_{i2}, \ldots, a_{in})$,
$i = 1, \ldots, m$ vectors in R^n, which is an r dimensional vector space
$$\dim A(\omega) = \dim R^n - \dim \omega = n - r$$

3. (A) $h(x) = \frac{f(x)}{x}, x > 0$

$$h'(x) = \frac{xf'(x) - f(x)}{x^2}$$

To P. T. $h'(x) > 0$

$$g(x) = xf'(x) - f(x)$$

$$g(0) = 0$$

$\Rightarrow$ g is strictly increasing

$$g(x) > 0 > 0$$

$\therefore$ $h'(x)$

$\Rightarrow$ h is strictly increasing in $(0, \infty)$.

4. (B) By the theorem which states "If A and B be $m \times n$ and $n \times p$ matrices respectively, then rank (AB) $\leq$ min (rank A, rank B)", we have

$$\text{rank (P)} = n$$

$$\text{rank (Q)} = p$$

Since P in a $m \times n$ matrix with rank $n, n \leq m$ and Q is $n \times p$ matrix with rank $p, p \leq n$.

$\therefore$ rank (PQ) = min $(n, p) = p$

5. (D) 6. (A) 7. (B) 8. (D) 9. (A)

10. (A) 11. (A) 12. (B)

13. (C) $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

or $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Rank = 2

14. (D) The matrix is $\begin{bmatrix} a_{11} & a_{11} \\ a_{21} & a_{22} \end{bmatrix}$

Given $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\therefore \quad a_{11} - a_{12} = 1 \qquad \ldots(1)$$

$$a_{21} - a_{22} = -1 \qquad \ldots(2)$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

$$2a_{11} + a_{12} = 8 \qquad \ldots(3)$$

$$2a_{21} + a_{22} = 4 \qquad \ldots(4)$$

Solving equations (1), (2), (3) and (4), we get

$$a_{11} = 3; \; a_{12} = 2$$

$$a_{21} = 1; \; a_{22} = 2$$

$\therefore$ Matrix is $\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}$.

15. (D) $G(x) = \int_{\sin^2 x}^{2\sin x} e^{t^2}\, dt$

$$G'(x) = e^{(2\sin^2 x)} [2\cos x] - e^{\sin^4 x \sin^2 x}$$

$$G'(x) = 2e^0(-1) - e^0(0)$$

$$= -2$$

16. (D) If $\lambda \neq 0$ is an eigen value of P, then eigen value of Q is $1/\lambda$, *i.e.* ∞. It is not possible. Hence neither P nor Q.

17. (D) 18. (B)

19. (D) $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$

Then $A + I = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

It is also non-zero upper triangular matrix.

Since each eigen value of A = 0

$\Rightarrow$ Characteristic equation of A is, $\lambda^n = 0$

$\Rightarrow$ A is nilopotent because a non zero matrix A is said to be nilopotent if for some '+' ve integer r, $A^r = 0$

20. (B) $f_1 = (1, 1, 1), f_2 = (1, -1, 0), f_3\ (0, 1, 0)$

$$g_1 = (1, 1), g_2 = (1, 0)$$

$$(a, b) = xg_1 + yg_2$$

$$= x(1, 1) + y(1, 0)$$

$$= (x + y, x + 0 \cdot y)$$

$$x + y = 0$$

$$x + 0 \cdot y = b$$

$x = b$ and $y = a - b$

$$(a, b) = bg_1\ (a \cdot b)\ g_2$$

$$T(x, y, z) = (x + y, y - z)$$

$$F(f_1) = F(1, 1, 1)$$

$$= (2, 0) = 0 \cdot g_1 + 2g_2$$

$$F(f_2) = F(1, -1, 0)$$

$$= (0, -1) = -g_1 + g_2$$

$$Ff_3) = F(0, 1, 0)$$

$$= (1, 1) = g_1 + 0 \cdot g_2$$

$$[F]_f^g = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

21. (A) 22. (B) 23. (A) 24. (A) 25. (B)

26. (C) Euclidean domain is a PID but the converse is not true.

$\therefore$ ED $\subseteq$ PID

PID and ED are unique factorisation domains UFD need not be PID or ED.

27. (C) $$\frac{\partial f}{\partial y}\Big|_{(0,0)} = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

28. (D) $f(z) = f(x+iy) = u(x,y) + iv(x,y)$

$f(1+i) = u(1,1) + iv(1,1)$

Given $f(1+i) = 2$, so $v(1,1) = 0$

$\therefore$ Either $v = x^2 - y^2$ or $y^2 - x^2$

If $v = x^2y^2$

$v_x = 2x = u_y$

$v_y = -2y = u_x$

Then, $u(x,y) = -2xy$

$u(1,1) = -2$ a contradiction

Let $v = y^2 - x^2$

$v_x = -2x = -u_y$

$v_y = 2y = u_x$

Then, $u(x,y) = 2xy$

$u(1,1) = 2$

$v(x,y) = y^2 - x^2$

29. (B) 30. (C)

31. (D) Suppose A is a Skew-Hermitian matrix. Then iA is Hermitian. Let λ be a characteristic root of A.

Then $AX = XX$

$(iA)X = (i\lambda)X$

From this it follows that $i\lambda$ is a characteristic root of A, which is Hermitian. Hence $i\lambda$ is real, therefore either λ must be zero or pure imaginary.

32. (D) Characteristic equation is

$\lambda^2 - \lambda - 1 = 0$

$\Rightarrow 1 = \lambda^2 - \lambda$

$\Rightarrow \lambda^{-1} = \lambda - 1$

$\Rightarrow M^{-1} = M - 1$

33. (A) 34. (D) 35. (C) 36. (D) 37. (A)

38. (D) $f(z) = \dfrac{\bar{z}^2}{z}$, $f(z) = 0$ at $z = 0$

Continuous at $z = 0$

For $$\underset{z\to 0}{\text{Lt}}\left(\frac{\bar{z}^2}{z}\right) = \underset{r\to 0}{\text{Lt}}\; re^{-i3\theta} = 0 = f(0)$$

$\therefore$ Continuous at $t = 0$

$$\underset{z\to 0}{\text{Lt}}\left(\frac{\bar{z}^2}{z}\right) = \underset{r\to 0}{\text{Lt}}\;\frac{\bar{z}^2}{z^2} = \underset{r\to 0}{\text{Lt}}\; e^{-4i\theta}$$

where $z = re^{i\theta}, \bar{z} = re^{-i\theta}, z^2 = r^2e^{i2\theta}, \bar{z}^2 = r^2e^{-2i\theta}$

$$\therefore \frac{\bar{z}^2}{z^2} = e^{-4i\theta}$$

If $\theta = 0$, $\underset{z\to 0}{\text{Lt}}\, f(z) = 1$ (along the real axis)

If $\theta = \dfrac{\pi}{2}$, $\underset{z\to 0}{\text{Lt}}\, f(z) = e^{-4i\times\frac{\pi}{2}}$

$= e^{-2i\pi} = \cos 2\pi - i\sin 2\pi$

$= -1 - 0 = -1$

Not differentiable at 0. So not analytic.

39. (B)

40. (C) Let $f(z) = \displaystyle\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-4)(z-2)}$

Poles are $(z-4)(z-2) = 0$

$\therefore$ $z = 4$ and $z = 2$

$z = 2$ lie inside the circle $|z| = 3$

$\therefore$ Residue of F(z) at $z = 2$ $\Big\}$

$$= \lim_{z\to 2}(z-2)f(z)$$

$$= \lim_{z\to 2}\frac{\sin \pi z^2 + \cos \pi z^2}{(z-4)(z-2)}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{-2}$$

$$= -\frac{1}{2}$$

$\therefore \displaystyle\int_c F(z)\,dz = 2\pi i$ (some of the residue)

$$= 2\pi i\left(-\frac{1}{2}\right)$$

$$= -\pi i$$

41. (D) 42. (D) 43. (B) 44. (B) 45. (A)

46. (A) Let $$M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

But M is a Skew-Symmetric matrix

i.e. $M' = -M$

$$\Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ -b_1 & -b_2 & -b_3 \\ -c_1 & -c_2 & -c_3 \end{bmatrix}$$

On comparing, we get

$$a_1 = -a_1 \Rightarrow a_1 = 0$$

$$a_2 = -b_1 \text{ and so on}$$

Then $$M = \begin{bmatrix} 0 & -b_1 & -c_1 \\ -a_2 & 0 & -c_2 \\ -a_3 & -b_3 & 0 \end{bmatrix}$$

But M is also orthogonal real matrix

$$MM' = I$$

$$\Rightarrow \begin{bmatrix} b_1^2 + c_1^2 & c_1c_2 & b_1b_3 \\ c_1c_2 & a_2^2 + c_2^2 & a_2a_3 \\ b_1b_3 & a_2a_3 & a_3^2 + b_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On comparing, we get

$$M = \begin{bmatrix} b_1^2 + c_1^2 & 0 & 0 \\ 0 & a_2^2 + c_2^2 & a_2a_3 \\ b_1b_3 & a_1a_3 & a_3^2 + b_3^2 \end{bmatrix}$$

For eigen values,

$$|M - \lambda I| = 0$$

Hence, $\lambda = -1, 1$

47. (A) 48. (C) 49. (C)

50. (A) Given, $w = \dfrac{2z}{z-2}$

It maps the real axis in the interior of $|z - 1| < 1$ onto points on the real axis of the w-plane '–' ve part as $\dfrac{2x}{x-2}$ as $0 < x < 2$ is '–' ve and any point of $|z - 1|\ 1 <$ is of the form,

$1 + re^{i\theta}$ $\{0 \le \theta \le 2\pi), r < 1\}$

$\dfrac{2(1 + re^{i\theta})}{1 + re^{i\theta} - 2}$ can have both '+' ve as well as '–' ve imaginary part.

51. (D)

52. (D) $f(z) = \dfrac{z - \sin z}{z^3}$

$$\lim_{z\to 0} f(z) = \lim_{z \to 0} \frac{z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right)}{z^3}$$

$$= \lim_{z\to 0} \frac{\frac{z^3}{3!} - \frac{z^5}{5!} + \dots}{z^3}$$

$$= \frac{1}{6} - \frac{z^2}{5!} + \dots$$

$$= \frac{1}{6}$$

Hence $z = 0$ is a removable singularity of F (z)

53. (A) 54. (B)

55. (A) $f(z) = \dfrac{1}{1-z} = (1 - z)^{-1}$

$$= 1 + z + z^2 + \dots$$

Radius of convergence,

$$R = \underset{n\to\infty}{\text{Lt}} \left|\frac{a_n}{a_{n+1}}\right|$$

$$= \underset{n\to\infty}{\text{Lt}} \left(\frac{1}{1}\right) = 1$$

56. (D) $f(z) = (z^2 \cdot z)$

$$= (x + iy)^2 (x - iy)$$

$$= (x^2 + y^2)(x + iy)$$

$$= x(x^2 + y^2) + iy(x^2 + y^2)$$

$$= w + iv$$

Since, $u_x = 0 = v_y$, $u_y = -v_x = 0$

Thus, $f(z)$ satisfies C – R equations and also $f(z)$ is a polynomial, so $f(z)$ is analytic.

57. (A)

58. (B) Let $z = \log i^{1/4}$

$\Rightarrow$ $z = \dfrac{1}{4}\log i$

But $i = e^{i\pi/2}$

$$z = \frac{1}{4}\log e^{i\pi/2}$$

Hence principal value is $\dfrac{i\pi}{2}$.

59. (B) 60 (C) 61. (A) 62. (B) 63. (C)

64. (A) From [0, 1] remove $\left[\dfrac{7}{10}, \dfrac{8}{10}\right]$

Then (0·07, 0·08)
(0·17, 0·18)
| |
| |
(0·97, 0·98) and so on

$\therefore$ Required set measure

$$= 1 - \sum_{n=1}^{\infty} \frac{9^n - 1}{10^n}$$

$$= 1 - 1 = 0$$

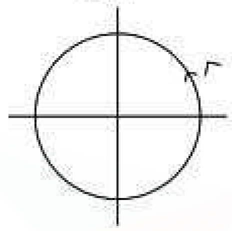
65. (B) 66. (B)

67. (B) $$I = \int_{\tau} \frac{\cos z}{z^2}\, dz$$

Residue at origin

$$R = \underset{z \to 0}{\text{Lt}} \frac{d}{dz} (\cos z)$$

$$= \underset{z \to 0}{\text{Lt}} (-\sin z)$$



$$\therefore \quad I = 2\pi i \cdot R = 0$$

68. (C) Required transformation is

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

or $$(w, 0, 1, \infty) = (z, -i, \infty, 1)$$

or $$\frac{(w-0)(1-\infty)}{(0-1)(\infty - w)} = \frac{(z+i)(\infty - 1)}{(-i-\infty)(1-z)}$$

or $$\frac{w(-\infty)}{(w-\infty)} = \frac{(z+i)(\infty)}{(-\infty)(1-z)}$$

or $$w = \frac{z+i}{z-i}$$

69. (B)

70. (C) $$f(z) = x^2 + iy^2$$

$$g(z) = x^2 + y^2 + ixy$$

$$u = x^2, v = y^2$$

$$u = x^2 + y^2 \quad v = xy$$

For analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

But at $z = 0$, *i.e.*, $x = 0, y = 0$

Hence f is analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Hence, g is also analytic

71. (C) 72. (B) 73. (C) 74. (C)

75. (C) $$f'_n(x) = \sqrt{n} \cos nx$$

$$f'_n(0) = \sqrt{n} \to \infty$$

76. (B) 77. (A) 78. (A) 79. (C) 80. (D)

81. (B)

82. (A) $$\log\left(\frac{z}{z-1}\right) = \log\left(\frac{z-1}{z}\right)$$

$$= \log\left(1 - \frac{1}{z}\right)$$

$$= \frac{-1}{z} - \frac{1}{2z^2} \cdots$$

83. (A) 84. (A) 85. (B)

86. (B) Since $f(x) = x^6$, we take the Hermite interpolating polynomial to be even function.

$$P(x) = a_0 + a_1 x^2 + a_2 x^4$$

$$P(0) = f(0)$$

$$P(0) = f'(0)$$

$$P(1) = f(1)$$

and $P'(1) = f'(1)$ yield the equations

$a_0 = 0,\ a_1 + a_2 = 1,\ 2a_1 + 4a_2 = 6$

Solving, we get

$$a_1 = -1$$

$$a_2 = 2$$

$$P(x) = 2x^4 - x^2$$

87. (B)

88. (A) Given surface integral $\iint_S \vec{F} \cdot \vec{n}\, dS$

Where S in the sphere, $x^2 + y^2 + z^2 = 4$

By Stoke's theorem,

$$\iiint_V \nabla . \vec{F} \cdot dV = \iint_S F \cdot \hat{x}\, dS$$

$$\text{Now } \nabla . \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= 1 + 1 + 1 = 3$$

$$\iiint_V \nabla . \vec{F} \cdot dV = \iiint_V 3 dV$$

Volume of the sphere with radius $r = \frac{4}{3}\pi r^3$

Here $r = 2$

$$\therefore \quad \text{Volume} = \frac{4}{3}\pi (2)^3$$

$$\therefore \quad \iiint_V \nabla \vec{F} \cdot dV = 3\left(\frac{4}{3}\right)\pi (2)^3$$

$$\therefore \quad \iiint_V \nabla . \vec{F} \cdot dV = \iint_S \vec{F} \cdot \hat{x}\, dS$$

$$= 32\pi$$

89. (B) 90. (C) 91. (B) 92. (C) 93. (A)

94. (B) $$O = \{(x, y, z) \in R^3 : x^2 + y^2 < 1\}$$

i.e., it is a sphere of radius less than 1.

$$F = \{(x, y, z) \in R^3 : z = 0\}$$

i.e., it is a plane of equation $z = 0$

Hence, $O \cap F$ is neither open nor closed.

95. (C) 96. (A) 97. (B)

98. (A) $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4{\cdot}5 \\ 5 \\ -0{\cdot}5 \end{bmatrix}$

Where $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$A = D + L + U$$

Where, $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H = -D^{-1}(L + U) = \begin{bmatrix} 0 & -2/3 & 0 \\ -2/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{bmatrix}$$

Characteristic equation is,

$$\lambda^3 - \lambda\frac{11}{18} = 0$$

$\therefore$ $\lambda = 0, \pm\sqrt{\frac{11}{18}}$

$\therefore$ $\rho(H) = \sqrt{\frac{11}{18}}$

and $\omega_{opt} = \frac{2}{\mu^2}\sqrt{1-\mu^2}$

$$= 1{\cdot}23 \text{ (opprox.)}$$

99. (A)

100. (D) $\int_L yzdx + xzdy + xydz$

$$= \int d(xyz)$$

$$= xyz \Big|_{(1,1,0)}^{(2,3,\,2)}$$

$$= 12$$

101. (B) 102. (B)

103. (C) By divergence theorem, we get

$$\iint_S \vec{r} \cdot \hat{n} = \iiint_V \Delta \cdot \vec{r}\, dV$$

$$= \iiint_V 3dV$$

$$= 3V$$

where V is the volume enclosed by S.

104. (D) 105. ()

106. (A) E is a non-measurable subset of [0, 1].

Let $P = E^\circ \cup \left\{\frac{1}{n} : n \in N\right\}$

and $Q = \bar{E} \cup \left\{\frac{1}{n} : n \in N\right\}$

Here E° is the interior of E, *i.e.*, $E^\circ < E$

and $\bar{E}$ is the closure of E, *i.e.*, $\bar{E} \approx E$

Hence, P is measurable but not ϕ.

107. (B) 108. (C)

109. (D) Eigen equation of the given matrix is

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\{(1-\lambda)(-1-\lambda) - 0\}$
$- 2\{2(-1-\lambda) - 0\} + 0 = 0$

or $-(1-\lambda)(1-\lambda)^2 + 4(1+\lambda) = 0$

$$\lambda^3 - \lambda^2 - 5\lambda - 3 = 0$$

$$(\lambda + 1)(\lambda^2 - 2\lambda - 3) = 0$$

$$(\lambda + 1)(\lambda + 1)(\lambda - 3) = 0$$

$$(\lambda + 1)^2(\lambda - 3) = 0$$

$\therefore$ $\lambda = -1, -1, 3$

For $|\lambda - 1| \le 2$

$$-2 \le (\lambda - 1) < 2$$

or $-2 + 1 \le \lambda \le 2 + 1$

or $-1 \le \lambda \le 3$

$\therefore$ $\lambda = -1, 3$ lies in the disc $(\lambda - 1) \le 2$.

110. (C) The Fourier series for $x + x^2$ in $(-\pi, \pi)$ is given by (after computation)

$$a_n = \frac{4(-1)^n}{n^2}, \frac{1}{2}a_0 = \frac{\pi^2}{3}$$

$$b_n = (-1)^{n-1} \cdot \frac{2}{n}$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\left(\frac{\cos n\pi}{n^2} - \frac{\sin n\pi}{z^n}\right)$$

At $x = \pi$,

$$f(\pi) = \frac{\pi^2}{3} + 4\Sigma\frac{1}{n^2}$$

$$= \frac{\pi^2}{3} + \frac{4\pi^2}{6}$$
$$= \pi^2$$

It should converse to π^2.

111. (B) $f^n(z) = \frac{n!}{2\pi i} \int_{|z|=2} \frac{f(z)}{(z-a)^{n+1}} dz$

$$f^3(z) = \frac{3!}{2\pi i} \int_{|z|=2} \frac{e^{2z}}{[z-(-1)]^4} dz$$

$$I = \int_{|z|=2} \frac{e^{2z}}{(z+1)^4} dz$$
$$= \frac{2\pi i}{6} f^3(-1)$$
$$= \frac{2\pi}{6} 8e^{-2}$$
$$= \frac{8\pi i}{3} e^{-2}$$

112. (D) 113. (A) 114. (D) 115. (D)

116. (B) Given,

$$x^2 + y^2 = 2cx$$
$$\Rightarrow \quad c = \frac{x^2+y^2}{2x}$$

Differentiating with respect to x, we get

$$2x + 2y\frac{dy}{dx} = 2c$$
$$\Rightarrow \quad x + y\frac{dy}{dx} = c$$
$$= \frac{x^2+y^2}{2x}$$
$$\frac{dy}{dx} = \frac{\frac{x^2+y^2}{2x} - x}{y}$$
$$= \frac{-x^2+y^2}{2xy}$$

Its orthogonal trajectory is given by $-\frac{dx}{dy}$

$$\therefore \quad \frac{-dx}{dy} = \frac{y^2-x^2}{2xy}$$
$$\Rightarrow \quad \frac{dy}{dx}(y^2-x^2) = -2xy$$
$$\Rightarrow \quad y'(x^2-y^2) = 2xy$$

117. (A)

118. (C) $\int_0^{\pi}\int_x^{\pi}\int_0^{2} \frac{\sin y}{y} dzdydx$

$$= \int_{y=0}^{\pi}\int_{z=x}^{\pi}\int_{x=0}^{2} \frac{\sin y}{y} dzdydx$$
$$= -4 \text{ (on integrating)}$$

119. (C) 120 (D)

121. (B) $1 = 1^n = (1-x+x)^n$

$$= \{(1-x)+x\}^n, \text{ where } 0 < x < 1$$
$$= \sum_{k=0}^{\infty} \binom{n}{k} x^k \binom{k}{n} (1-x)^{n-k}$$

122. (B)

123. (C) By definition, X is a limit point of A of every open set U containing X should have other point of A.

or $\quad A - \{X\} \cap U = \phi$,

A is not limit point since $\{a, b\}$ is one such U.

But $\quad A - \{a\} \cap U = \phi$,

b is not a limit point since $\{b\}$ is one such U.

$\therefore$ c is a limit fit, d is limit point.

124. (A)

$$M = 3a^2x^2 + by\cos x$$
$$N = 2\sin x - 4ay^3$$

For the equation to be exact,

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

or $\quad 2\cos x = b\cos x$

or $\quad b = 2$

125. (A) 126. (A)

127. (C) $(S_5) = 5! = 120$

$\therefore$ Numbers of elements of order,

$$S = \frac{120}{5} = 24$$

128. (B) 129. (A)

130. (C) $S = \left\{\frac{1}{n} : n \in N\right\}$

$$U\{0\} = 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots$$
$$T = \left\{n + \frac{1}{n} : n \in N\right\}$$
$$= 2, \frac{5}{2}, \frac{10}{3}, \ldots$$

Hence by definition of complete, T and S both are complete

131. (A) 132. (B) 133. (A)

134. (A)
$$y'(y'+y) = x(x+y)$$
$$y'^2 + yy' = x^2 + xy$$
$$y'(y'+x+y) - x(y'+x+y) = 0$$
$$(y'-x)(y'+x+y) = 0$$
Since $y' = x$

$\therefore$ $y' + x + y = 0$

or $\dfrac{dy}{dx} + x + y = 0$

or $\dfrac{dy}{dx} + y = -x$

Now I. F. $= e^{\int P dx} = e^x$
$$y \times e^x = \int -xe^x\,dx + c$$
$$= -[x \cdot e^x - \int 1 \cdot e^x\,dx] + c$$
or $ye^x = -xe^x + e^x + c$

or $y = -x + 1 + ce^{-x}$

where c is a constant.

Choosing $c = -1$, we get
$$y = 1 - x - e^{-x}$$

135. (B)

136. (C) Given $x'' + \lambda x = 0$, $x(0) = 0$, $x(\pi) + x'(\pi) = 0$

For $\lambda > 0$, $m^2 + \lambda = 0$;

or $m^2 = -\lambda$

or $m = \sqrt{\lambda}\, i$

$\therefore$ $x(t) = A\cos\sqrt{\lambda}\, t + B\sin\sqrt{\lambda}\, t$
$$x(0) = A = 0$$
$$x'(t) = -A\sqrt{\lambda}\sin\sqrt{\lambda}\, t + B\sqrt{t}\cos\sqrt{\lambda}\, t$$
$$x(\pi) = B\sin\sqrt{\lambda}\,\pi$$
$$x'(\pi) = B\sqrt{\lambda}\cos\sqrt{\lambda}\, x$$
Given, $x(\pi) + x'(\pi) = 0$

or $B\sin\sqrt{\lambda}\pi + B\sqrt{\lambda}\cos\sqrt{\lambda}\,\pi = 0$

or $B\,[\sin\sqrt{\lambda}\,\pi + \sqrt{\lambda}\cos\sqrt{\lambda}\,\pi\} = 0$

Since $B \neq 0$, therefore
$$\sin\sqrt{\lambda}\pi + \sqrt{\lambda}\cos\sqrt{\lambda}\pi = 0$$
or $\tan\sqrt{\lambda}\, x + \sqrt{\lambda} = 0$

137. (B) 138. (A) 139. (B) 140. (A) 141. (C)

142. (B) $\dfrac{dy}{dx} = \dfrac{y}{\sqrt{x}}$

$\Rightarrow$ $\dfrac{1}{y}dy = x^{-1/2}\,dx$

$\Rightarrow$ $y = e^{2\sqrt{x}} \cdot e^c$

At $x = 2$, $y = 4$

$\therefore$ $e^c = 4e^{-2\sqrt{2}}$

$\therefore$ $y = e^{2\sqrt{x}} \cdot 4e^{-2\sqrt{2}}$

Hence, differential equation has a unique solution

143. (A) 144. (A)

145. (C) $\int_{-1}^{1} f(x)\,dx = A_0\left(-\frac{1}{2}\right) + A_1 f(0) + A_2 f\left(\frac{1}{2}\right)$

when $f(x) = 1$

then $2 = A_0 + A_1 + A_2$

when $f(x) = x$

then $0 = -\dfrac{A_0}{2} + 0 + \dfrac{A_2}{2}$

when $f(x) = x^2$

then $\dfrac{2}{3} = A_0\left(-\dfrac{1}{2}\right)^2 + A_1(0)^2 + A_2\left(\dfrac{1}{2}\right)$
$$= \frac{A_0}{4} + 0 + \frac{A_2}{4}$$
Solving, we get
$$A_0 = A_2 = \frac{4}{3}$$
$$A_1 = \frac{-2}{3}$$

146. (B) Given, $4x^3y'' + 6x^2y' + y = 0$

We transform the independent variable x to t by the relation
$$t = \frac{1}{x}$$
or $x = \dfrac{1}{t}$ ...(1)

Now $y' = \dfrac{-1}{x^2}\dfrac{dy}{dt}$

or $y'' = \dfrac{1}{x^4}\dfrac{d^2y}{dt^2} + \dfrac{2}{x^3}\dfrac{dy}{dt}$

and the given differential equation transform to
$$\frac{4td^2y}{dt^2} + 2\frac{dy}{dt} + y = 0 \quad ...(2)$$

The point at ∞ is transformed to the origin

From equation (2), we note that the origin is a regular singular point.

Hence, the point at ∞ is a regular singular point of the given equation.

147. (C)

148. (A) None of the answer is correct.

Given, $2x(1-y) = k$

Differentiating with respect to x, we get

$$2x\left(-\frac{dy}{dx}\right) + (1-y)2 = 0$$

or $$-x\frac{dy}{dx} + (1-y) = 0 \quad \ldots(1)$$

We know $m_1 m_2 = -1$

or $$\left(\frac{dy_1}{dx}\right)\left(\frac{dy_2}{dx}\right) = -1$$

or $$\left(\frac{dy_2}{dx}\right) = \frac{-1}{\left(\frac{dy_1}{dx}\right)}$$

From equation (1),

$$= -x\left[\frac{-1}{\left(\frac{dy}{dx}\right)}\right] + (1-y) = 0$$

or $$\frac{x}{\left(\frac{dy}{dx}\right)} = (y-1)$$

or $$x = (y-1)\frac{dy}{dx}$$

or $$x\,dx = (y-1)\,dy$$

Integrating we have,

$$\frac{x^2}{2} = \frac{y^2}{2} - y + c$$

or $$x^2 - y^2 + 2y = k_1 = L$$

$\therefore$ $$g(x, y) = x^2 - y^2 + 2y = L$$

Hence the given answers A is correct.

149. (B) 150. (A) 151. (C) 152. (A) 153. (B)

154. (A) Given $D^4 - D^2 = 0$

$\Rightarrow$ $D^2(D-1)(D+1) = 0$

$$D = 0, 0, 1, -1$$

$$y = 1 + c_1 x + c_2 c^x + c_3 e^{-x}$$

Hence, the set of linearly independent solutions is $\{1, x, e^{-x}, e^x\}$

155. (A) 156. (C)

157. (A) $$\in_{n+1} = \in_n - \frac{2f(a+\in_n)}{f(a+\in_n)}$$

where $a, \in_n, \in_{n+1}$ have their usual meaning. Further $f(a+\in_n)$ and $f'(a+\in_n)$ is powers of $\in_n$ and using $f(a) = 0, f'(a) = 0$, since $x =$ G is a bouble root near $x = x_n$. We get

$$\in_{n+1} = \in_n - \frac{2\in_n^2\left[\frac{1}{2!}f''(a) + \frac{1}{3!}\in_n f'''(a) + \ldots\right]}{\in_n\left[f''(a) + \frac{\in_n}{2!}f'''(a) + \ldots\right]}$$

$$\cong \in_n - \frac{2\in_n\left[\frac{1}{2!}f''(a) + \frac{\in_n}{3!}f'''(a)\right]}{\in_n\left[f''(a) + \frac{\in_n}{2!}f'''(a)\right]}$$

$$\cong \frac{\in_n^2}{6}\frac{f'''(a)}{\left[f''(a) + \frac{\in_n f'''(a)}{2!}\right]}$$

$$\in_{n+1} \approx \frac{1}{6}\in_n^2\frac{f'''(a)}{f''(a)}$$

Thus

$\Rightarrow$ $\in_{n+1} \;\alpha\; \in_n^2$

and hence the convergence is quadratic.

Hence, $\in_n$ has the order of 2.

158. (D) $$u = \frac{w}{y}$$

$$u_x = \frac{w_x}{y}$$

$$u_y = -\frac{w}{y^2} + \frac{w_y}{y}$$

The equation $xu_n = u + yu_y$ reduces to

$$xw_x - yw_y = 0$$

This is Lagrange's equation whose characteristic equation are

$$\frac{dy}{dx} = \frac{dy}{-y} = \frac{dw}{0}$$

$\therefore$ $$x\,dy + y\,dx = 0$$

$\Rightarrow$ $$xy = c_1$$

$$dw = 0$$

$\Rightarrow$ $$w = c_2$$

$\therefore$ The solution is $w = f(xy)$.

159. (D)

160. (D) S_3 = group of all permutation on three symbols.

$$\left\{\begin{bmatrix}1&2&3\\1&2&3\end{bmatrix}; \begin{bmatrix}1&2&3\\1&3&2\end{bmatrix}; \begin{bmatrix}1&2&3\\2&1&3\end{bmatrix}; \begin{bmatrix}1&2&3\\3&1&2\end{bmatrix}; \begin{bmatrix}1&2&3\\2&3&1\end{bmatrix}; \begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\right\}$$

elements in S_3 that satisfy the equation

$$x^2 = e, \text{ is}$$

$$\left\{\begin{bmatrix}1&2&3\\1&2&3\end{bmatrix}; \begin{bmatrix}1&2&3\\1&3&2\end{bmatrix}; \begin{bmatrix}1&2&3\\1&3&2\end{bmatrix}; \begin{bmatrix}1&2&3\\3&2&1\end{bmatrix}\right\}$$

The cardinality of the set is 4.

161. (A) 162. (B) 163. (D) 164. (B) 165. (C)

166. (B) Given,

$$x^2(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + \frac{1}{x(1-x)}\frac{dy}{dx} + \frac{1}{x^2(1-x)}y = 0$$

$$\therefore \quad P = \frac{1}{x(1-x)}$$

$$Q = \frac{1}{x^2(1-x)}$$

$$\Rightarrow \quad P - xQ = 0$$

Hence, the solution of C. F. is $u = -x$

So, the $x = 1$ is a regular singular point.

167. (B) 168. (D)

169. (A) Given,

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

Test the continuity at (0, 0)

Let $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore f(r\cos\theta, r\sin\theta) = \frac{r^2\sin\theta\cos\theta}{r^2} = \frac{\sin 2\theta}{2}$$

$$\therefore |f(r\cos\theta, r\sin\theta) - f(0,0)| = \left|\frac{\sin 2\theta}{2} - 0\right|$$

$\therefore |f(r\cos\theta, r\sin\theta) - f(0,0)| < E$ for every value of 0.

Therefore function is uniformly continuous in r for all values of θ.

By $$f_{(x)}(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

We find that $f(x)$ is differentiable at every point of R^2.

170. (A) The given differential equation is of Clairaut's types,

$$z = xp + yq + \left(\frac{p}{q^2} + \frac{q}{p^2}\right)$$

Its complete integral is

$$z = ax + by + \left(\frac{a}{b^2} + \frac{b}{a^2}\right)$$

171. (A)

172. (B) To find number of Sylow subgroup of G of order 3

$15 = 5 \times 3$, By a known theorem,

Number of 3 Sylow subgroup $= 3k + 1$

Now $(3k+1)$ must divide 15, $(3k+1)$ being prime to 3 should divide 5. Thus is true only for $k = 0$.

Thus $\quad 3k + 1 = 1$

$\therefore$ Number of 3 Sylow subgroup = 1

173. (C) 174. (D)

175. (D) $\quad y^{(2)} + \lambda y = 0$

Where λ is a parameter.

The general solution is

$$y = c_1 \sin\sqrt{\lambda}\,x + c_2\cos\sqrt{\lambda}x$$

Putting boundary conditions

$$y(0) = y(2\pi)$$

and $\quad y'(0) = y'(2\pi)$

where $\quad y^{(1)} = c_1\sqrt{\lambda}\cos\sqrt{\lambda}\,x - c_2\sqrt{\lambda}\sin\sqrt{\lambda}\,x$

we have $\quad \lambda = n^2$

176. (C) 177. (A) 178. (B) 179. (C) 180. (B)

181. (B)

182. (A) $L = \frac{m}{2}(\dot{q}^2\sin\omega t + \dot{q}\,q\omega\sin 2\omega t + q^2\omega^2)$

$Q = q\sin\omega t$

$\dot{Q} = \dot{q}\sin\omega t + q\omega\cos\omega t$

Subtracting for q and $\dot{q}$ in term of Q and $\dot{Q}$, we get

$$L = \frac{m}{2}[\dot{Q}^2 - \omega^2Q^2]$$

But $$H = \frac{\partial L}{\partial \dot{Q}}\ddot{Q} - L = \frac{m}{2}[\dot{Q}^2 - \omega^2 Q^2]$$

Further $\quad P = \dfrac{\partial L}{\partial \dot{Q}} = m\dot{Q}$

$\therefore \quad H = \dfrac{m}{2}\left[\dfrac{P^2}{m^2} - \omega^2 Q^2\right]$

183. (A) 184. (C) 185. (B) 186. (A)

187. (C) Because if λ_i and λ_j are roots of equation. $J_n(\lambda a) = 0$, then

$$\int_0^a xJ_n(\lambda_{ix})\, J_n(\lambda_{jx})\, dx = \begin{cases} 0, \text{ if } i = j \text{ (diff. roots)} \\ \text{ if } i = j \text{ (equal roots)} \end{cases}$$

for orthogonality of system on [0, 1], the weight function is x.

188. (A) 189. (D) 190. (B) 191. (C) 192. (A)

193. (C) 194. (D) 195. (A) 196. (B) 197. (C)

198. (A)

199. (B) A linear homogeneous equation can always be reduced to c_1 linear differentiation equation with constant coefficients of the form (by substitution).

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_1 y = x \quad \ldots(1)$$

$$f(D) = x$$

We will now show that if $y_1, y_2, \ldots y_n$ are n linearly independent solutions of equation (1), then $c_1y_1 + c_2y_2 + \ldots + c_ny_n$ is also a solution of equation (1),

where $c_1, c_2, \ldots, c_n$ being arbitrary constants.

Since $y_1, y_2, \ldots, y_n$ are solution of (1), we get

$f(D)\, y_1 = 0, f(D)\, y_2 = 0, f(D)\, y_n = 0 \quad \ldots(2)$

$f(D)\,[c_1y_1 + c_2y_2 + \ldots + c_ny_n]$

$= f(D)(c_1y_1) + f(D)(c_2y_2) + \ldots + f(D)(c_ny_n)$

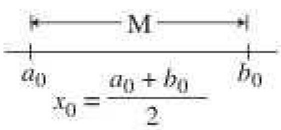
$= c_1 f(D)\, y_1 + c_2 f(D)\, y_2 + \ldots + c_n f(D)\, y_n$

$= c_1 \cdot 0 + c_2 \cdot 0 + \ldots + c_n \cdot 0$ (using (2))

This proves that statements

200. (A) Required : $x_n - x_{n-1} \leq \in$

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}$$



or $\quad \dfrac{x_{n-1} + x_{n-2}}{2} - x_{n-1} \leq \in$

or $\quad \dfrac{x_{n-2} + x_{n-1}}{2} \leq \in$

or $\quad \left(\dfrac{x_{n-1} - x_{n-2}}{2}\right) \leq \in$

or $\quad \dfrac{x_{n-2} - \left(\dfrac{x_{n-2} + x_{n-3}}{2}\right)}{2} \leq \in$

or $\quad \dfrac{x_{n-2} - x_{n-3}}{2^2} \leq \in$

$\vdots \quad \vdots$

or $\quad (-1)^{n-1}\left(\dfrac{x_{n-(n-1)} - x_{n-n}}{2^{n-1}}\right) \leq \in$

or $\quad (-1)^{n-1}\dfrac{(x_1 - x_0)}{2^{n-1}} \leq \in$

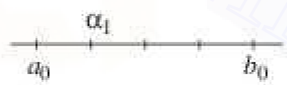
Now $\quad x_1 = \dfrac{x_0 + a_0}{2}$

$x_1 - x_0 = \dfrac{x_0 + a_0}{2} - x_0$

$= \dfrac{a_0 - x_0}{2}$

$= \dfrac{a_0 - \left(\dfrac{a_0 + b_0}{2}\right)}{2}$

$= \dfrac{a_0 - b_0}{2^2}$



$= \dfrac{-M}{2^2}$

or $\quad (-1)^{n-1}\left(\dfrac{-M}{2^2}\right) \leq \in$

or $\quad \dfrac{(-1)^n M}{2^{n+1}} \leq \in$

or $\quad \dfrac{(-1)^n}{2^n} \cdot \dfrac{M}{2} \leq \in$

or $\quad \left(\dfrac{-1}{2}\right)^n \leq \dfrac{2\in}{M}$

For n even

or $\quad \dfrac{1}{2^n} \leq \dfrac{2\in}{M}$

or $$2^n \geq \frac{M}{2\in}$$

or $$n\ln 2 \geq \ln\left(\frac{M}{2\in}\right)$$

or $$n\ln 2 \geq -\ln\left(\frac{2\in}{M}\right)$$

or $$n \geq \frac{-\ln 2 - \ln\left(\frac{\in}{M}\right)}{\ln 2}$$

or $$\frac{x_0 + b_0}{2}$$

$$x_1 - x_0 = \frac{x_0 + b_0}{2} - x_0$$

$$= \frac{b_0 - x_0}{2}$$

$$= \frac{b_0 - \left(\frac{a_0 + b_0}{2}\right)}{2}$$

$$= \frac{b_0 - a_0}{2^2}$$

$$= \frac{M}{2^2}$$

or $$\frac{(-1)^{n-1}}{2^{n-1}}\frac{M}{2^2} \leq \in$$

or $$(-1)^{n-1}\frac{M}{2^{n+1}} \leq \in$$

or $$\frac{(-1)^{n-1}}{2^{n-1}}\frac{M}{2^2} \leq \in$$

or $$\frac{(-1)^{n-1}}{2^{n-1}} \leq \frac{2^2 \in}{M}$$

or $$2^{n+1} \geq \frac{M}{\in}$$

or $$(n+1) \geq \frac{-\ln\left(\frac{\in}{M}\right)}{\ln 2}$$

or $$n \geq -1 - \frac{\ln \in / M}{\ln 2}$$

●●

LEVEL OF DIFFICULTY–2

OBJECTIVE TYPE QUESTIONS

1. The cordinality of the centre of Z_{12} is—
   (A) 1 (B) 2
   (C) 3 (D) 12

2. Let C [0, 1], be the space of all continuous real valued functions on [0, 1]. The identity map $I : (C[0, 1], \|\cdot\|_\infty) \to (C[0, 1], \|\cdot\|_1)$ is—
   (A) Continuous but not open
   (B) Open but not continuous
   (C) Both continuous and open
   (D) Neither continuous nor open

3. For a linear programming primal maximization problem P with dual Q, which of the following statements is correct ?
   (A) The optimal values of P and Q exist and are the same
   (B) Both optimal values exists and the optimal value of P is less than the optimal value of Q
   (C) P will have an optimal solution, if and only if Q also has an optimal solution
   (D) Both P and Q cannot be infeasible

4. The extremum for the variational problem

$$\int_0^{\pi/8} [(y')^2 + 2yy' - 16y^2]\, dx$$

$$y(0) = 0,$$

$$y\left(\frac{\pi}{8}\right) = 1,$$

   occurs for the curve—
   (A) $y = \sin(4x)$ (B) $y = \sqrt{2}\sin(2x)$
   (C) $y = 1 - \cos(4x)$ (D) $y = \dfrac{1-\cos(8x)}{2}$

5. Let $f : [a, b] \to R$ be a monotonic function. Then—
   (A) f is continuous
   (B) f is discontinuous at atmost two points
   (C) f is discontinuous at finitely many points
   (D) f is discontinuous at atmost countable points

6. If A and B are two events and the probability $P(B) \neq 1$, then $\dfrac{P(A) - P(A \cap B)}{1 - P(B)}$ equals—
   (A) $P(A \mid \bar{B})$ (B) $P(A \mid B)$
   (C) $P(\bar{A} \mid B)$ (D) $P(\bar{A} \mid \bar{B})$

7. Every solution of $y'' + ay' + by = 0$, where a and b are constants, approximate to zero as $x \to \infty$ provided—
   (A) $a > 0, b > 0$ (B) $a > 0, b < 0$
   (C) $a < 0, b < 0$ (D) $a < 0, b > 0$

8. The maximum step size h such that the error in linear interpolation for the function $y = \sin x$ in $[0, \pi]$ h less than 5×10^{-5} is—
   (A) 0·02 (B) 0·002
   (C) 0·04 (D) 0·06

9. The interpolating polynomial of highest degree which corresponds the functional values $f(-1) = 9, f(0) = 5, f(2) = 3, f(5) = 15$ is—
   (A) $x^3 + x^2 + 2x + 5$
   (B) $x^2 - 3x + 5$
   (C) $x^4 + 4x^3 + 5x^2 + 5$
   (D) $x + 5$

10. In the motion of a two particle system, if two particles are connected by a rigid weightless rod of constant length, then the number of degrees of the system is—
    (A) 2 (B) 3
    (C) 5 (D) 6

11. Which of the following, concerning the solution of the Neumann problem for Laplace's equation, on a smooth bounded domain, is true ?
    (A) Solution is unique
    (B) Solution is unique upto an additive constant
    (C) Solution is unique upto a multiplicative constant
    (D) No conclusion can be drawn about uniqueness

12. On evaluating $\int_1^2 \int_1^2 \frac{1}{x+y} \, dx \, dy$ numerically by Trapezoidal rule one would get the value—

(A) $\frac{17}{48}$ (B) $\frac{11}{48}$

(C) $\frac{21}{48}$ (D) $\frac{17}{48}$

13. Suppose X = $(1, \infty)$ and T : X $\rightarrow$ X is such that $d(T_x, T_y) < d(x, y)$ for $x \neq y$. Then—

(A) T has at most one fixed point

(B) T has a unique fixed point, by Banach contradiction theorem

(C) T has infinitely many fixed points

(D) For every $x \in (x)$, $\{T^n(x)\}$ converges to a fixed point

14. Consider the Hilbert Space

$l^2 = \{(x_1, x_2, \ldots), x_i \in C$ for all i and $\sum_{i=1}^{\infty} |x_i|^2 < \infty\}$

with the inner product

$<(x_1, x_2, \ldots) (y_1, y_2, \ldots)> = \sum_{i=1}^{\infty} x_i \, \overline{y_i}$

Define T : $l^2 \rightarrow l^2$ by

$T(x_1, x_2, \ldots 1) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right)$. Then T is

(A) Neither self-adjoint nor unitary

(B) Both self-adjoint and unitary

(C) Unitary but not adjoint

(D) Self-adjoint but unitary

15. Let a convex set in 9-dimensional space be given by solution set of the following system of linear inequalities

$$\sum_{J=1}^{3} x_{ij} = 1, \qquad i = 1, 2, 3$$

$$\sum_{J=1}^{3} x_{ij} = 1 \qquad j = 1, 2, 3$$

$$x_{ij} \geq 0 \qquad i, j = 1, 2, 3$$

Then the number of extreme points of this set is—

(A) 3 (B) 4

(C) 9 (D) 6

16. Suppose $y_p(x) = x \cos(2x)$ is a particular solution of $y'' + \alpha y = -4 \sin(2x)$. Then the constant α equals—

(A) 4 (B) -2

(C) 2 (D) 4

17. Let $f : [0, 10] \rightarrow [0, 10]$ be a continuous mapping. Then—

(A) f need not have any fixed point

(B) f has at least 10 fixed points

(C) f has at least 9 fixed points

(D) f has at least one fixed point

18. Let A, B and C be three independent events. If

$\alpha = P_r$ (A occurs)

$\beta = P_r$ (not all three events occur simultaneously)

$\gamma = P_r$ (at least one of the three events does not occur)

$\delta = P_r$ (C occurs, but neither A nor B occur)

then the probability that C occurs equals—

(A) $\delta/(\beta + \delta)$ (B) $\beta/(\beta + \delta)$

(C) $\alpha\gamma/(\beta + \delta)$ (D) $\gamma/(\beta + \delta)$

19. The boundary value problem

$$y'' + ky = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$$

has non trivial solution for—

(A) All negative value of k

(B) All the values of k

(C) $k = 0$

(D) $k = \pm n, n = 1, 2, \ldots,$

20. The second order Runge-Kutta method is applied to the initial value problem $y' = -y$, $y(0) = y_0$, with step size h. Then $y(h)$ is—

(A) $y_0 (h-1)^2$ (B) $\frac{y_0}{2}(h^2 - 2h + 2)$

(C) $\frac{y_0}{6}(h^2 - 2h + 2)$ (D) $y_0\left(1 - h + \frac{h^2}{2} + \frac{h^3}{6}\right)$

21. The equation

$$x^2(y-1) Z_{xx} - x(y^2-1) Z_{xy} + y(y^2-1) Z_{yy} + Z_z = 0$$

is hyperbolic in the entire xy-plane except along.

(A) x-axis

(B) y-axis

(C) A line parallel to y-axis

(D) A line parallel to x-axis

22. The following statement is false—

(A) Any product of compact spaces is compact

(B) Any product of Housdorff spacess is Hausdorff

(C) Any product of connected spaces is connected

(D) Any product of metrizable spaces is metrizable

23. Which of the following satisfies the heat equation (without source terms and with diffusion constant 1) in one space dimension ?

(A) $\sin\left(\frac{x^2}{4t}\right)$ (B) $e^t \sin x$

(C) $x^2 - t$ (D) $\frac{e^{-x^2/4t}}{\sqrt{t}}$

24. Complete integral for the partial differential equation $z = px + qy - \sin(pq)$ is—

(A) $z = ax + by + \sin(ab)$

(B) $z = ax + by - \sin(ab)$

(C) $z = ax + y + \sin(b)$

(D) $z = x + by - \sin(a)$

25. Consider R^2 with $\|\,.\,\|_1$ norm and $M = \{(x, 0) : x \in R\}$. Define $g : M \to R$ by $g(x, y) = x$. Then a Hann-Banach extension f of g is given by—

(A) $f(x, y) = 2x$ (B) $f(x, y) = x + y$

(C) $f(x, y) = x - 2y$ (D) $f(x, y) = x + 2y$

26. An iterative method to find the n^{th} root $(n > N)$ of a positive number a is given by

$$x_{k+1} = \frac{1}{2}\left[x_k + \frac{a}{x_k^{n-1}}\right]$$

A value of n for which this iterative method fails to converges is—

(A) 1 (B) 2

(C) 3 (D) 8

27. Let I be the functional defined by

$$I(y(x)) = \int_0^{\pi/2} \left\{\left(\frac{dy}{dx}\right)^2 - y^2\right\} dx\,;$$

$$y(0) = 0\ y(\pi/2) = 1$$

where the unknown function $y(x)$ possess two derivatives everywhere in $(0, \pi/2)$. Then—

(A) The functional has an extremum which cannot be achieved in the class of continuous functions

(B) The corresponding Euler's equation does not have an unique solution satisfying the given boundary conditions

(C) I is not linear

(D) I is linear

28. If $F(S) = \tan^{-1}(S) + k$ is the Laplace transformation of some function $f(t)$, $t \geq 0$, then $k =$

(A) $-\pi$ (B) $-\frac{\pi}{2}$

(C) 0 (D) 3

29. Let X be a non-empty set and τ_1, τ_2 be two topologies on X. Let

$f : (x, \tau_1) \to (x, \tau_2)$

be a mapping. If f is a homomorphism, then

(A) $\tau_1 = \tau_2$

(B) $\tau_1 = \tau_2$ and f is the identity map

(C) τ_1 is a finer than τ_2 implies x is infinite

(D) either τ_1 is finer than τ_2 or τ_2 is finer than τ_1

30. A degenerated basic feasible solution of the convex region formed by the following closed half spaces in R^2—

$$\begin{aligned} x_1 + 3x_2 &\leq 12 \\ x_1 + x_2 &\leq 6 \\ 2x_1 - x_2 &\leq 6 \\ 0 &\leq x_1 \leq 4, \\ x_2 &\geq 0 \end{aligned}$$

is—

(A) (3, 0) (B) (0, 4)

(C) (4, 2) (D) (3, 3)

31. The interative scheme $x_{n+1} = \frac{x_n}{2}\left(1 + \frac{a}{x_n^2}\right)$ converges to $\sqrt{a}$. The converges is—

(A) Linear (B) Quadratic

(C) Cubic (D) Biquadratic

32. Let $f(x, y) = x - y$ and c be the boundary of the square $S = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$

Then $\int_c f\frac{df}{dn}\,ds$ equals—

(A) 0

(B) 1

(C) 2

(D) None of these

33. The solution of the Cauchy problem

$$\begin{aligned} u_{yy}(x, y) - u_{xx}(x, y) &= 0; \\ u(x, 0) &= 0, \\ u_y(x, 0) &= x \end{aligned}$$

is u (x, y) =

(A) $\frac{x}{y}$ (B) xy

(C) $xy + \frac{x}{y}$ (D) 0

34. The random variable X has a t-distribution with v degrees of freedom. Then the probability distribution of X^2 is—
(A) Chi-square distribution with 1 degrees of freedom
(B) Chi-square distribution with v degrees of freedom
(C) F-distribution with $(1, v)$ degrees of freedom
(D) F-distribution with $(v, 1)$ degrees of freedom

35. Which of the following is elliptic ?
(A) Laplace equation
(B) Wave equation
(C) Heat equation
(D) $u_{xx} + 2u_{xy} - 4u_{yy} = 0$

36. Pick the region in which following differential equation is hyperbolic

$$y\, u_{xx} + 2\, xy\, u_{xy} + x\, u_{yy} = u_x + u_y$$

(A) $xy \neq 1$ (B) $xy \neq 0$
(C) $xy > 1$ (D) $xy > 0$

37. Let X be an inner product space and $S \subset X$. Then it follows that—
(A) $S\perp$ has non empty interior
(B) $S\perp = \{0\}$
(C) S is a closed subspace
(D) $(S\perp)\perp = S$

38. Suppose the function u (x) intrpolates $f(x)$ at $x_0, x_1, x_2, \ldots, x_{n-1}$ and the function $v(x)$ interpolates $f(x)$ at $x_1, x_2, \ldots, x_{n-1}, x_n$. Then a function F$(x)$ which interpolates $f(x)$ at all points $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$ is given by—

(A) $F(x) = \frac{(x_n - x)\, u(x) - (x - x_0)\, v(x)}{(x_n - x_0)}$

(B) $F(x) = \frac{(x_n - x)\, u(x) + (x - x_0)\, v(x)}{(x_n - x_0)}$

(C) $F(x) = \frac{(x_n - x)\, v(x) + (x - x_0)\, u(x)}{(x_n - x_0)}$

(D) $F(x) = \frac{(x_n - x)\, v(x) - (x - x_0)\, u(x)}{(x_n - x_0)}$

39. Solution of the initial value problem

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\, y = F(x),$$
$$0 \leq x \leq 1$$

$$y\,(0) = C_0, \qquad \left(\frac{dy}{dx}\right)_{x=0} = c_1$$

where $a_1(x)$, $a_2(x)$ and F(x) are continuous functions on [0, 1], may be reduced, in general, to a solution of some linear—
(A) Fredholm integral equation of first kind
(B) Volterra's integral equation of first kind
(C) Fredholm integral equation of second kind
(D) Volterra's integral equation of second kind

40. Let $S = \{(0, 1, 1), (1, 0, 1), (-1, 2, 1)\} \subseteq R^3$. Suppose R^3 is endowed with the standard inner product $<, >$. Define $M = \{x \in R^3 : <x, y> = 0 \text{ for all } y \in s\}$. Then the dimension of M equals—
(A) 0 (B) 1
(C) 2 (D) 3

41. The orthogonal trajectories of the family $x^2 - y^2 = c$, are given by—
(A) $x^2 + y^2 = c_2$
(B) $xy = c_2$
(C) $y = c_2$
(D) $x = c_2$

42. If $\theta \in R \mid \{n\pi : n \in z\}$ and P is a 2×2 matrix with complex entries such that

$$P^{-1}\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}.\, P = \begin{bmatrix}e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{bmatrix}$$

then P = ...

(A) $\begin{bmatrix}1 & 1 \\ -i & i\end{bmatrix}$ (B) $\begin{bmatrix}1 & -i \\ i & 1\end{bmatrix}$

(C) $\begin{bmatrix}-i & 0 \\ 1 & -i\end{bmatrix}$ (D) None of these

43. The workdone in moving a particle in the force field $\vec{F} = \hat{i}\, 5x^2 + (xz - y)\hat{j} + 3z\hat{k}$ along the straight line joining (0, 0, 0) to (1, 1, 1) is—
(A) 0 (B) 1
(C) 2 (D) 4

44. The value of the integral

$$\int_{-\pi}^{\pi} \frac{dt}{4 + r^2 - 4r\cos(\theta - t)}, -\pi < 0 \le \pi, 0 < r < 2$$

is—

(A) 0

(B) $2\pi r^2 (\cos\theta - \sin\theta)$

(C) $\frac{4 - r^2}{2\pi}$

(D) $\frac{2\pi}{4 - r^2}$

45. The characteristics curves of the equation

$$x^2 u_{xx} - y^2 u_{yy} = x^2y^2 + x; x > 0, u = (x, y)$$

are—

(A) Rectangular hyperbola

(B) Parabola

(C) Circle

(D) Straight line

46. Let $S_1 = \{(x, y) \in R^2 : x^2 + y^2 \le 1\}$ and $S_2 = \{(x, y) \in R^2 : y \le x^2\}$, then

(A) S_1 and S_2 both are convex sets

(B) S_1 is a convex set but S_2 is not a convex set

(C) S_2 is a convex set but S_1 is not a convex set

(D) Neither S_1 nor S_2 is a convex set

47. Consider the motion of a three-particle system in the 3-D space. Suppose that all pairs of particles are at invariant distance apart, so that the system constitutes a rigid body. For such a system, the number of degrees of freedom is—

(A) 6 (B) 3

(C) 1 (D) 9

48. If the total kinetic energy of a system of particles about the origin is equal to its kinetic energy about the centre of mass, then the centre of mass is—

(A) At rest

(B) Moving along a circle

(C) Moving on a straight line

(D) Moving along an ellipse

49. An iterative scheme is given by $x_{n+1} = \frac{1}{5}\left(16 - \frac{12}{x_n}\right)$, $n \in$ N U {o}. Such a scheme, with suitable x_0, will—

(A) Not converges (B) Converges to 1·6

(C) Converges to 1·8 (D) Converges to 2

50. The integral surface of the partial differential equation $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ satisfying the condition $u(1,y) = y$ is given by—

(A) $u(x_1y) = \frac{y}{x}$ (B) $u(x_1y) = \frac{2y}{x+1}$

(C) $u(x_1y) = \frac{y}{2-x}$ (D) $u(x_1y) = y + x - 1$

51. Let V be the vector space of all real polynomials. Consider the subspace W spanned by $t^2 + t + 2$, $t^2 + 2t + 5$, $5t^2 + 3t + 4$ and $2t^2 + 2t + 4$. Then, the dimension of W is—

(A) 4 (B) 3

(C) 2 (D) 1

52. Let X be an uncountable set and let J = {U ⊆ X : U = ϕ or U^c is finite}. Then the topological space (X, J)—

(A) Is separable

(B) Is H*ausdorff*

(C) Has a countable basis

(D) Has a countable basis at each point

53. For the differential equation $t(t-2)^2 y'' + ty' + y = 0$, $t = 0$ is—

(A) An ordinary point

(B) A branch point

(C) An irregular point

(D) A regular singular point

54. The order of the group generated by the matrices $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, where $i = \sqrt{-1}$, under matrix multiplication is ....

(A) 8 (B) 5

(C) 6 (D) 7

55. The linear programming problem Max. $(-3x_1 + x_2)$ subject to the constraints $2x_1 + x_2 \le 4$, $-x_1 + x_2 \le 1$, $x_1 + x_2 \ge 1$, $-3x_1 + x_2 \ge -3$, where $x_1, x_2 \ge, 0$ has—

(A) Unique of optimal solution

(B) Alternative optimal solution
(C) Constraints leading to an infeasible region
(D) Constraints leading to an unbounded region

56. Let $u(x, t)$ be the solution of—
$u_{tt} = u_{xx}$; $0 < x < 1$, $t > 0$, $u(x, 0) = x(1 - x)$, $u(x, 0) = 0$. Then $u\left(\frac{1}{2}, \frac{1}{4}\right)$ is—
(A) $\frac{3}{16}$ (B) $\frac{1}{4}$
(C) $\frac{3}{4}$ (D) $\frac{1}{16}$

57. The number of generalized co-ordinates for a pair of scissors that can move in a plane is—
(A) 1 (B) 2
(C) 3 (D) 4

58. Let T be the matrix (occurring in a typical transportation problem) given by
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$. Then—
(A) Rank T is 4 and T is unimodular
(B) Rank T = 4 and T is not unimodular
(C) Rank T = 3 and T is unimodular
(D) Rank T = 3 and T is not unimodular

59. If $f(x)$ has an isolated zero of multiplicity 3 at $x = \xi$, and the iteration $x_{n+1} = x_n - \frac{3f(x_n)}{f^1(x_n)}$, $n = 0, 1, 2, \ldots\ldots$ converges to ξ, then the rate of convergence is—
(A) Linear
(B) Faster than linear but slower than quadratic
(C) Quadratic
(D) Cubic

60. The number of generalized co-ordimates required to describe motion of a rigid body with one of its points fixed is—
(A) 9 (B) 6
(C) 3 (D) 1

61. In the (x, t) plane, the characteristics of the initial value problem $U_t + UU_x = 0$, with $U(x, 0) = x$, $0 \le x \le 1$, are—
(A) Parallel straight lines
(B) Straight lines which intersect at (0, –1)
(C) Non-intersecting parabolas
(D) Concentric circle with centre at the origin

62. If $f(x)$ and $g(y)$ are the arbitrary functions, then the general solution of the partial differential equation $U\frac{\partial^2 y}{\partial x \partial y} - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = 0$ is given by—
(A) $U(x, y) = f(x) + g(y)$
(B) $U(x, y) = f(x + y) + g(x - y)$
(C) $U(x, y) = f(x)\, g(y)$
(D) $U(x, y) = xg(y) + yf(x)$

63. Consider the inner product P([0,1]) of all real polynomials on [0, 1] with the inner product $<f, g> = \int_0^1 f(x)dx$ and V = Span $\{t^2\}$. Let $h(t)$ $\in$ be such that—
$\| (2t - 1) - h(t) \| \le \| (2t - 1) - x(t) \|$ for $x(t) \in$ V. Then, $h(t)$ is—
(A) $\frac{5}{6}t^2$ (B) $\frac{5}{3}t^2$
(C) $\frac{5}{12}t^2$ (D) $\frac{5}{24}t^2$

64. Suppose (X, J) is a topological space. Let $\{S_n\}_{n \ge 1}$ be a sequence of subset of X. Then—
(A) $(S_1 \cup S_2)^0 = S_1^0 \cup S_2^0$
(B) $(\cup_n S_n)^0 = \underset{n}{\cup} S_n^0$
(C) $\overline{\cup_n S_n} = \underset{n}{\cup} \overline{S_n}$
(D) $\overline{S_1 \cup S_2} = \overline{S_1} \cup \overline{S_2}$

65. Suppose that y_1 and y_2 form a fundamental set of solutions of a second order ordinary differential equation on the interval $-\infty < t < +\infty$, then—
(A) There is only one zero of y_1 between consecutive zeros of y_2
(B) There are two zeros of y_1 between consecutive zeros of y_2
(C) There are finite number of zeros of y_1 between consecutive zeros of y_2
(D) None of these

66. The complex analytic function, $f(z)$, with the imaginary part $e^x(y \cos y + x \sin y)$ is ....
(A) ze^z (B) ze^{-z}
(C) ze^{-1} (D) None of these

67. If X and Y are independent N(0,1) variables, then the characteristic function of XY is,
   (A) e^{-t^2}
   (B) $(1 + t^2)^{-1/2}$
   (C) $(1 + t)^{1/2}$
   (D) $(1 + t^2)^{-1}$

68. Under the influence of a force field F, a particle of mass '*m*' moves along the ellipse $r = a \cos wt\hat{i} + b \sin wt\hat{j}$. Then ($\gamma$ X F) is—
   (A) 0
   (B) $mw^2(a^2\cos^2 wt + b^2\sin^2 wt)k$
   (C) $mw^2(a^2 + b^2)\cos wt \sin wt\ k$
   (D) $ma^2w^2\ k$

69. The topology τ on the real line R generated by the class J of all closed intervals $[d, d + l]$ with length l is—
   (A) Indiscrete
   (B) Discrete
   (C) Standard topology
   (D) Neither discrete nor Housdorff

70. Consider the primal problem (LP)

   $max\ 4x_1 + 3x_2$

   Subject to

   $x_1 + x_2 \leq 8$

   $2x_1 + x_2 \leq 10$

   $x_1 \geq 0, x_2 \geq 10$

   together with its dual (LD). Then
   (A) (LP) and (LD) both are infeasible
   (B) (LP) and (LD) both are feasible
   (C) (LP) is feasible but (LD) is infeasible
   (D) (LP) is infeasible but (LD) is feasible

71. The best possible error estimate in the Gauss-Hermite formula with 3 points for calculating the integral $\int_{-\infty}^{\infty} x^4 e^{-x^2} dx$ is—
   (A) 0 (B) 0·30
   (C) 0·65 (D) 1·20

72. Let Q = {(*x*, *y*) in R^2 | $x \geq 0, y \geq 0$} be the first quadrant in R^2 with usual topology. Then Q is closed because—
   (A) It is compact
   (B) It does not contain all its limit points
   (C) Its complement is open
   (D) It is connected

73. Suppose $u(x, y)$ satisfies Laplace's equation : $\nabla^2 u = 0$ in R^2 and $u = x$ on the unit circle. Then, at the origin—
   (A) u tends to infinity
   (B) u attains a finite minimum
   (C) u attains a finite maximum
   (D) u is equal to 0

74. A bead slides on smooth rod which is rotating about one end in a vertical plane with uniform angular velocity ω. If g denotes the acceleration due to gravity, then the Lagrange's equation of motion is—
   (A) $\ddot{r} = r\omega^2 - g \sin \omega t$
   (B) $\ddot{r} = r\omega^2 - g \cos \omega t$
   (C) $\ddot{r} = - g \sin \omega t$
   (D) $\ddot{r} = - g \cos \omega t$

75. Let $M = \begin{bmatrix} 1 & a & b \\ 0 & 2 & c \\ 0 & 0 & 1 \end{bmatrix}, a, b, c \in R$.

   Then, M is diagonalizable, if and only if—
   (A) $a = bc$ (B) $b = ac$
   (C) $c = ab$ (D) $a = b = c$

76. Let (X, *d*) is a metric space. Consider the metric ρ on X defined by—

   $\rho(x, y) = \min \left\{ \frac{1}{2}, d(x, y) \right\}\ x, y \in X.$

   Suppose T_1 and T_2 are topologies on X defined by d and ρ, respectively. Then
   (A) T_1 is a proper subset of T_2
   (B) T_2 is a proper subset of T_1
   (C) Neither $T_1 \subseteq T_2$ nor $T_2 \subseteq T_1$
   (D) $T_1 = T_2$

77. The general solution of the system of differential equations $\frac{dX}{dt} = MX + b$, where $X = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$: M, a 2×2 matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and b, a 2×1 constant vector. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is given by—
   (A) $e^{Mt} c + b$ (B) $e^{Mt} c + bt$
   (C) $e^{Mt} c - b$ (D) $e^{Mt} c - bt$

   Where c is any 2×1 constant vector.

78. The range of absolute stability of the multistep method—

$y_{n-1} = y_n + \frac{h}{12}(23f_n - 16 f_{n-1} + 5f_{n-2})$ for the differential equation $y' = f(x, y)$ is …….

(A) $\left(-\frac{6}{11}, 0\right)$ (B) $\left(\frac{6}{11}, 0\right)$

(C) $\left(\frac{11}{6}, 0\right)$ (D) None of these

79. Let $F(x, y) = \begin{Bmatrix} 0 & \text{if} & y < e^{-x} \\ 1 & \text{if} & y \geq e^{-x} \end{Bmatrix}$

Then F is not a probability distribution function because—

(A) It is discontinuous
(B) It is not monotone in x and in y
(C) It is not right continuous in x and in y
(D) It does not satisfy

80. A particle of mass 4 units moves along x-axis attracted towards the origin by a force whose magnitude is $8x$. If it is initially at rest at $x = 10$, then the frequency of the particle is—

(A) $10\sqrt{2}\pi$ (B) π

(C) $\frac{\pi}{10\sqrt{2}}$ (D) None of these

81. Let norms $\| x \|_1 = \sum_{i=1}^{n} |\xi_i|$ and $\|x\|_2 = \left[\sum_{i=1}^{n} |\xi_i|^2\right]^{1/2}$ induce topologies τ_1 and τ_2 on R^n, the n-dimensional Euclidean space, then—

(A) τ_1 is weaker than τ_2
(B) τ_1 is stronger than τ_2
(C) τ_1 is equivalent to τ_2
(D) τ_1 and τ_2 are incomparable

82. The initial value problem corresponding to the integral equation $y(x) = 1 + \int_0^x y(t)\, dt$ is—

(A) $y' - y = 0, y(0) = 1$
(B) $y' + y = 0, y(0) = 0$
(C) $y' - y = 0, y(0) = 0$
(D) $y' + y = 0, y(0) = 1$

83. Let $X = [0, 1] \text{ X } [0, 2]\, x \ldots x\, [0, 10]$ and $f : X \to R$ be a continuous function. Then $f(X)$ is—

(A) $[r_1, r_2] \cup [r_3, r_4]$ for some r_1, r_2, r_3, r_4 in R such that $r_1 \leq r_2 < r_3 \leq r_4$
(B) $[-\infty, r]$ for some r in R
(C) $[r_1, r_2]$ for some r_1, r_2 in R such that $r_1 \leq r_2$
(D) $[-\infty, r_1] \cup [r_2, \infty]$ for some r_1, r_2 in R such that $r_1 \leq r_2$

84. A circular disk of radius a and mass m is supported on a needle at its centre. The disk is set spinning with initial angular velocity ω_0 about an axis making an angle $\pi/6$ with the normal to the disk. If $\overline{\omega}\ (t)$ is the angular velocity of the disk at any time t, then its component along the normal equals—

(A) $\frac{\sqrt{3}\,\omega_0}{2}$ (B) ω_0

(C) $\omega_0 \sin t$ (D) $\left(\frac{\sqrt{3}\,\omega_0}{2}\right)\cos t$

85. The Lagrangian L of a dynamical systems with two degrees of freedom is given by $L = \alpha + \beta\dot{q}_1 + \gamma\dot{q}_2$ where α, β and γ are functions of the generalized co-ordinates q_1, q_2 only. If p_1, p_2 denote the generalized momenta, then Hamiltonian H—

(A) Depends on p_1, p_2 but not on q_1, q_2
(B) Depends upon q_1, q_2 but not p_1, p_2
(C) Depends on p_1, q_1 but not p_2, q_2
(D) Is dependent of p_1, p_2, q_1, q_2

86. Let M be the real 5×5 matrix having all of its entries equal to 1. Then—

(A) M is not diagonalizable
(B) M is idempotent
(C) M is nilpotent
(D) The minimal polynomial and the characteristic polynomial of M are not equal

87. A basis of

$V = \{(x, y, z, w) \in R^4 : x + y - z = 0, y + z + w = 0, 2x + y - 3z - w = 0\}$

is—

(A) $\{(1, 1, -1, 0), (0, 1, 1, 1), (2, 1, -3, 1)\}$
(B) $\{(1, -1, 0; 1)\}$
(C) $\{(1, 0, 1, -1)\}$
(D) $\{(1, -1, 0, 1), (1, 0, 1, -1)\}$

88. The general solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is of the form—
(A) $u = f(x + iy) + g(x - iy)$
(B) $u = f(x + y) + g(x - y)$
(C) $u = cf(x - iy)$
(D) $u = g(x + iy)$

89. The line integral of $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ on the circle $x^2 + y^2 = a^2$, $z = 0$ described in the clockwise sense is ....
(A) πa^2 (B) $-\pi a^2$
(C) $\pi^2 a$ (D) None of these

90. $\lim_{n \to \infty} e^{-n} \sum_{k=n}^{\infty} \frac{n^k}{k!}$
(A) Is 0 (B) Is 1
(C) Is $\frac{1}{2}$ (D) Does not exist

91. The Hamiltonian of a system is given by
$$H(q_1, q_2, p_1, p_2) = k\,p_1^2 + \frac{k}{q_1^2} p_2^2 + \frac{1}{q_1}$$
where q_1, q_2 are the generalized co-ordinates, p_1, p_2 are the generalized momenta and k, l are constants. Then—
(A) $p_1 = kt + l$
(B) $p_2 = kt + l$
(C) p_1 is the independent of time
(D) p_2 is independent of time

92. The probability that exactly one of the events E or F occurs is equal to—
(A) P(E) + P(F) – P(EF)
(B) P(E) + P(F) – 2P(EF)
(C) $P(EF^C) + P(E^CF)$
(D) P(E) + P(F)

93. Let A be an $n \times n$ complex matrix whose characteristic polynomial is given by
$$f(t) = t^n + c_{n-1} t^{n-1} + \dots + c_1 t + c_0$$
Then—
(A) $\det(A) = c_{n-1}$
(B) $\det(A) = c_0$
(C) $\det(A) = (-1)^n c_{n-1}$
(D) $\det(A) = (-1)^n c_0$

94. Let $P(X = n) = \frac{\lambda}{n^2 (n+1)}$, where λ is an appropriate constant. Then E (X) is—
(A) $2\lambda + 1$ (B) λ
(C) ∞ (D) 2λ

95. Let X_1 and X_2 be independent bionomial random variables with $E(X_i) = n_i p$ and $v_{ar}(X_i) = n_i\,p\,(1 - p)$. $o < p < 1$, $i = 1, 2$. Then the distribution of the random variable $z = n_1 + n_2 - X_1 - X_2$ is—
(A) Binomial with mean $(n_1 + n_2)\,p$
(B) Binomial with mean $(n_1 + n_2)(1 - p)$
(C) Poisson with mean $(n_1 + n_2)\,p$
(D) Poisson with mean $(n_1 + n_2)(1 - p)$

96. In R^2 with usual topology, the set
$U = \{(x, -y) \in R^2 : x = 0, 1, -1 \text{ and } y \in N\}$
is—
(A) Neither closed nor bounded
(B) Closed but not bounded
(C) Bounded but not closed
(D) Closed and bounded

97. Let $A_1, A_2, \dots A_n$ be n independent events which the probability of occurrence of the event A_i given by
$$P(A_i) = 1 - \frac{1}{\alpha^i}, \ \alpha > l,\ i = 1, 2, \dots n$$
Then the probability that at least one of the events occurs is—
(A) $1 - \frac{1}{\frac{n(n+1)}{2}}$ (B) $\frac{1}{\frac{n(n+1)}{2}}$
(C) $\frac{1}{\alpha^n}$ (D) $1 - \frac{1}{\alpha^n}$

98. Let $\{v_1, v_2, \dots, v_{16}\}$ be an ordered basis for $V = C^{16}$. If T is linear transformation on V defined by $T(v_i) = v_i + 1$ for $1 \le i \le 15$ and $T(V_{16}) = -(v_1 + v_2 + \dots + v_{16})$ Then—
(A) T is singular with rational eigen values
(B) T is singular but has no rational eigen values
(C) T is regular (invertible) with rational eigen values
(D) T is regular but has no rational eigen values

99. Consider R^3 with the standard inner product. Let $S = \{(1, 1, 1), (2, -1, 2), (1, -2, 1)\}$

For a subset w of R^3, let L(W) denote the linear span of W in R^3. Then an orthonormal set T with L(S) = L(T) is—

(A) $\left\{\frac{1}{\sqrt{3}}(1, 1, 1), \ \frac{1}{\sqrt{6}}(1, -2, 1)\right\}$

(B) $\{(1, 0, 0), (0, 1, 0) (0, 0, 1)\}$

(C) $\left\{\frac{1}{\sqrt{3}}(1, 1, 1), \ \frac{1}{\sqrt{2}}(1, -1, 0)\right\}$

(D) $\left\{\frac{1}{\sqrt{3}}(1, 1, 1), \ \frac{1}{\sqrt{2}}(0, 1, -1)\right\}$

100. Let x be an non-optimal feasible solution of a linear programming maximization problem and y a dual feasible solution. Then—

(A) The primal objective value at x is greater than the dual objective value at y

(B) The primal objective value at x could equal the dual objective value at y

(C) The primal objective value at x is less than the dual objective value at y

(D) The dual could be unbounded

101. Suppose P and Q are independent events of an experiment E. Then it follows that—

(A) P^C and Q^C are dependent

(B) P and Q are mutually exclusive

(C) P and Q^C are dependent

(D) P^C and Q are independent

102. The geen's function for the boundary value problem

$$y'' + f(x) = 0, \quad y(0) = 1, \quad y(\pi) = -1$$

is ....

103. The largest internal in which $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$ converges is—

(A) $(-1, 1]$ (B) $[-1, 1)$

(C) $(-1, 1)$ (D) $[-1, 1]$

104. Let $X = [0, 1] \cup [2, 3]$ with the subspace to pology induced by the usual topology on R_1 and let $f : X \to R$ be defined by

$$f(x) = x \text{ if } 0 \le x \le 1$$
$$= 2 \text{ if } 2 \le x \le 3$$

Then—

(A) f(G) is open in R for every open set G in X

(B) f(H) is closed in R for every closed set H in X

(C) f is a continuous function

(D) f is a discontinuous function

105. If $\{A_n, n \ge 1\}$ is a sequence of events, then

$$\lim_{n \to \infty} P(A_n) = P\left(\lim_{n \to \infty} A_n\right)$$

if—

(A) $\{A_n\}$ is an increasing sequence of events

(B) $\{A_n\}$ is a decreasing sequence of events

(C) $\{A_n\}$ is neither increasing nor decreasing sequence of events

(D) None of these

106. Let A be any $n \times n$ non-singular complex matrix and let $B = (\overline{A})^t A$, where $(\overline{A})^t$ is the conjugate transpose of A. If λ is an eigen value of B, then—

(A) λ is real and $\lambda < 0$

(B) λ is real and $\lambda \le 0$

(C) λ is real and $\lambda \ge 0$

(D) λ is real and $\lambda > 0$

107. Let A be a 3×3 matrix with eigen values 1, –1, 0. Then the determinant of $I + A^{100}$ is—

(A) 6 (B) 4

(C) 9 (D) None of these

108. Let – 2, 5, – 6, 9, – 5, – 9 be the observed values of a random sample of size 6 from a distribution having probability density function,

$$f_\theta(x) = \int_0^{e-1} (x - \theta) d\theta \text{ if } x > 0 \text{ otherwise}$$

Then the maximum likelihood estimate of θ is—

(A) 9 (B) – 9

(C) $-\frac{4}{3}$ (D) $\frac{4}{3}$

109. In R^3 with usual topology, let

$$V = \{(x, y, z) \in R^3 : x^2 + y^2 + z^2 = 1, y \ne 0)$$

and $W = \{(x, y, z) \in R^3 : y = 0\}$
Then V U W is—
(A) Connected and compact
(B) At least one odd n
(C) Compact but not connected
(D) Neither connected nor compact

110. The life time of two brands of bulbs X and Y are exponentially distributed with a mean life time of 100 hours. Bulb X is switched on 15 hours after bulbs Y has been switched on. The probability that the bulb X fails before Y is—
(A) $\frac{15}{100}$ (B) $\frac{1}{2}$
(C) $\frac{85}{100}$ (D) 0

111. The value of $\int_0^{2\pi} \exp(e^{i\theta} - i\theta)\, d\theta$ equals—
(A) $2\pi i$ (B) 2π
(C) π (D) $i\pi$

112. Let A be 3×3 matrix. Suppose that the eigen values of A are $-1, 0, 1$ with respective eigen vectors $(1, -1, 0)^t$, $(1, 1, -2)^t$ and $(1, 1, 1)^t$. Then 6A equals—
(A) $\begin{bmatrix} -1 & 5 & 2 \\ 5 & -1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
(B) $\begin{bmatrix} 1 & 5 & 3 \\ 5 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix}$ (D) $\begin{bmatrix} -3 & 9 & 0 \\ 9 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

113. Let X_n denote the sum of points obtained when n fair dice are rolled together. The expectation and variance of X_n are—
(A) $\frac{7}{2}n$ and $\frac{35}{12}n^2$ respectively
(B) $\frac{7}{2}n$ and $\frac{35}{12}n$ respectively
(C) $\left(\frac{7}{2}\right)^n$ and $\left(\frac{35}{12}\right)^n$ respectively
(D) $\left(\frac{7}{2}\right)^n$ and $\left(\frac{35}{12}\right)^n$ respectively

114. The characteristic of the partial differential equation
$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \cos^2 x\,\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 0$$
when it is of hyperbolic type are ……… and ……….
(A) $y - x + \cos x = c_1$ and $y - x - \cos x = c_2$
(B) $y + x + \cos x = c_1$ and $y - x + \cos x = c_2$
(C) $y + x - \cos x = -c_1$ and $y - x + \cos x = c_2$
(D) None of these

115. $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{3n + k}$ is—
(A) log 4/3 (B) log 3/4
(C) log 3/2 (D) log 5/4

116. Let X = R with confinite topology. Then X is a—
(A) First countable space
(B) T_1-Space
(C) Regular space
(D) Normal space

117. Suppose that the five random variables X_1, ……, X_5 are independent and each has standard normal distribution. A constant C such that the random variable $\frac{C(X_1 + X_2)}{(X_3^2 + X_4^2 + X_5^2)^{1/2}}$ will have a t-distribution, has the value—
(A) $\frac{\sqrt{3}}{2}$ (B) $\sqrt{\frac{3}{2}}$
(C) $\frac{3}{2}$ (D) $\sqrt{\frac{2}{3}}$

118. Let $T : C^n \to C^n$ be a linear operator of rank n-2. Then—
(A) 0 is not an eigenvalue of T
(B) 0 must be an eigenvalue of T
(C) 1 can never be an eigenvalue of T
(D) 1 must be an eigenvalue of T

119. Let A be 2×2 orthogonal matrix of trace and determinant 1. Then the angle between Au and ($u = [1\ \ 0]^t$) is—
(A) 15° (B) 30°
(C) 45° (D) 60°

120. Suppose that the linear programming problem $P = \text{Min } z = C^1 x$ S.t. $A_x \ge b$, $x \ge 0$, where A is an $m \times n$ matrix, C an $n \times 1$ vector and b and 3×1 vector, is being solved by the Dual Sim-plex Algorithm. Then—
(A) The value of primal objective function increases at every iteration

(B) The algorithm will always terminate with an optimal solution for the dual

(C) The algorithm will always terminate with an optimal solution to the primal

(D) It is not always possible to obtain a starting basis for this Algorithm

121. Suppose X is a random variable, C is a constant and $a_n = E(X - C)^n$ is finite for all $n \geq 1$. Then $P(X = C) = 1$ if and only if $a_n = 0$ for—

(A) At least one $n \geq 1$

(B) At least one odd n

(C) At least one even n

(D) At least two values of n

122. A random sample of size n is chosen from a population with probability density function

$$f(x, \theta) = \begin{cases} \frac{1}{2} e^{-(x-\theta)}, & x \geq \theta \\ \frac{1}{2} e^{(x-\theta)}, & x < \theta \end{cases}$$

Then, the maximum likelihood estimator of θ is the—

(A) Mean of the sample

(B) Standard deviation of the sample

(C) Median of the sample

(D) Maximum of the sample

123. The sum of the residues at all poles of

$$f(z) = \frac{\cot \pi z}{(z+a)^2},$$

where a is a constant, $(a \neq 0, \pm 1, \pm 2, \ldots)$ is—

(A) $\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \pi \operatorname{cosec}^2 \pi a$

(B) $-\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} + \pi \operatorname{cosec}^2 \pi a$

(C) $-\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} - \pi \operatorname{cosec}^2 \pi a$

(D) $\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} + \pi \operatorname{cosec}^2 \pi a$

124. Let $T : R^3 \to R^3$ be a linear transformation defined by

$$T[(x, y, z)] = (x + y - z, x + y + z, y - z)$$

Then the matrix of the linear transformation T with respect to the ordered basis B = {(0, 1, 0), (0, 0, 1), (1, 0, 0)} of R^3 is 0—

(A) $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

125. Let b_{yx} and b_{xy} denote the regression coefficient of Y on X and of X on Y respectively. Then $b_{yx} = b_{xy}$ implies that—

(A) $\sigma_y = \sigma_x$ (B) $\rho = 1$

(C) $\rho = 0$ (D) None of the above

126. For the Hamiltonian $H = \frac{1}{2}(q^2 + p^2q^4)$ the equation of motion for q can be written as $f(q, \dot{q}, \ddot{q}) = 0$, where $f(q, \dot{q}, \ddot{q}) = \ldots\ldots\ldots$

(A) $q - \frac{2q^2}{q} - q$ (B) $q + \frac{3q^2}{q} - q$

(C) $P - \frac{2P^2}{P} - P$ (D) None of these

127. Let E_k, $k = 1, 2, \ldots, n$ be pairwise disjoint subset of real line R with Lebesgue measure $m(E_k) = k^3$. If $E = \bigcup_{k=1}^{n} Ek$ and $\phi(x) = \sum_{k=1}^{n} \frac{1}{k} X_{E_k}$, where $X_{(E_k)}$ is the characteristic event function of E_k, then the lebesgue integral $\int_E \phi \, dx$ equals—

(A) 0 (B) Log n

(C) $\frac{n(n+1)}{2}$ (D) $\frac{n(n+1)(2n+1)}{6}$

128. Two independent events E and F such that $P(E \cap F) = \frac{1}{6}$, $P(E^C \cap F^C) = \frac{1}{3}$ and $P(E) > P(F)$. Then P(E) is—

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$

(C) $\frac{1}{3}$ (D) $\frac{1}{4}$

129. Suppose that $X_1, \ldots, X_n$ are random variable such that the variance of each variable is 1 and the correlation between each pair of different variables is $\frac{1}{4}$. Then $V_{ar}(X_1 + X_2 + \ldots + X_n)$ is—

(A) $\frac{n(n+1)}{2}$ (B) $\frac{n(n+2)}{4}$

(C) $\frac{n(n+3)}{4}$ (D) $\frac{n(n+3)}{2}$

130. The fixed points of $f(z) = \frac{2iz+5}{z-2i}$ are—

(A) $1 \pm i$ (B) $1 \pm 2i$

(C) $2i \pm 1$ (D) $i \pm 1$

131. Let $w(z) = \frac{az+b}{cz+d}$ and $f(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ be bilinear (Mobius) transformations. Then the following is also a bilinear transformation—

(A) $f(z)\, w(z)$ (B) $f(w(z))$

(C) $f(z) + w(z)$ (D) $f(z) + \frac{1}{w(z)}$

132. Extremals $y = y(x)$ for the variational problem $v[y(x)] = \int_0^1 (y + y')^2\, dx$ satisfy the differen-tial equation—

(A) $y'' + y = 0$ (B) $y'' - y = 0$

(C) $y'' + y' = 0$ (D) $y' + y = 0$

133. If the random vector $(X_1, X_2)^T$ has a bivariate normal distribution with mean vector $(\mu, \mu)^T$ and the matrix $((E(X_iX_j))_{1 \le i,j \le 2}$ equals $\begin{bmatrix} \alpha_1 & \mu^2 \\ \mu^2 & \alpha_2 \end{bmatrix}$, where $\mu \in R$ and $\alpha_1, \alpha_2 > \mu^2$, then X_1 and X_2 are—

(A) Independent for all α_1 and α_2

(B) Independent if and only if $\alpha_1 = \alpha_2$

(C) Uncorrelated, but not independent for all α_1, α_2

(D) Uncorrelated, if and only if $\alpha_1 = \alpha_2$ and in this case they are not independent

134. Consider the following Linear Programming Problem (LPP)—

Minimize $z = 2x_1 + 3x_2 + x_3$

Subject to $x_1 + 2x_2 + 2x_3 - x_4 + x_5 = 3$

$2x_1 + 3x_2 + 4x_3$

$x_i \ge 0, \quad i = 1, 2, \ldots\ldots, 6$

A nondegenerate basic feasible solution $(x_1, x_2, x_3, x_4, x_5, x_6)$ is—

(A) (1, 0, 1, 0, 0, 0)

(B) (1, 0, 0, 0, 0, 7)

(C) (0, 0, 0, 0, 3, 6)

(D) (3, 0, 0, 0, 0, 0)

135. Let $f(z)$ be an entire function such that for same constant α, $|f(z)| \le \alpha |z|^3$ for $|z| \ge 1$ and $f(z) = f(iz)$ for all $z \in C$. Then—

(A) $f(z) = \alpha z^3$ for all $z \in C$

(B) $f(z)$ is a constant

(C) $f(z)$ is a quadratic polynomial

(D) No such $f(z)$ exists

136. Let $Y(x) = (y_1(x), y_2(x)')$

and let $A = \begin{bmatrix} -3 & 1 \\ k & -1 \end{bmatrix}$.

Further, let S be the set of values of k for which all the solutions of the system of equa-tion $Y'(x) = A\, Y(x)$ tend to zero as $x \to \infty$. Then S is given by—

(A) $\{k : k \le -1\}$ (B) $\{k : k \le 3\}$

(C) $\{k : k < -1\}$ (D) $\{k : k < 3\}$

137. The test of a simple null hypothesis $H_0 : \theta = \theta_0$ of size α is biased if—

(A) The power function of the test assume a value less than α for some $\theta \ne \theta_0$

(B) The power function has a value less than α when $\theta = \theta_0$

(C) The probability of accepting the null hypothesis when it is true is larger than the probability of rejecting the null hypothesis when it is true

(D) The power function has a value greater than α when $\theta = \theta_0$

138. If we identify points $(x, 0)$ with $(2\pi - x, 2\pi)$ in $[0, 2\pi) \times [0, 2\pi]$ equipped with the product topology with the identification space under this equivalence is ……… .

139. The conjugate (also called symmetric) point of $1 + i$ with respect to the circle $|z - 1| = 2$ is—

(A) $1 - i$ (B) $1 + 4i$

(C) $1 + 2i$ (D) $-1 - i$

140. A box contains two coins, one of which is fair and the other two headed. One coin is

chosen at random and tossed twice. If two head appear, then the probability that the chosen coin was two headed is—

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$

(C) $\frac{1}{3}$ (D) $\frac{4}{5}$

141. The objective function of the dual problem for the following primal linear programming problem :

Maximize $f = 2x_1 + x_2$

Subject to $x_1 - 2x_2 \geq 2,$

$x_1 + 2x_2 = 8$

$x_1 - x_2 \leq 11,$

with $x_1 \geq 0$ and x_2 unrestricted in sign, is given by—

(A) Minimize $z = 2y_1 - 8y_2 + 11y_3$

(B) Minimize $z = 2y_1 + 8y_2 + 11y_3$

(C) Minimize $z = 2y_1 - 8y_2 - 11y_3$

(D) Minimize $z = 2y_1 + 8y_2 - 11y_3$

142. The function $f(z) = |z|^2$ is—

(A) Differentiable everywhere

(B) Differentiable only at the origin

(C) Not differentiable anywhere

(D) Differentiable on real x-axis

143. For the function $f(z) = \sin \frac{1}{z}$, $z = 0$ is a—

(A) Removable singularity

(B) Simple pole

(C) Branch point

(D) Essential singularity

144. Let $k(x, t) = \begin{cases} x + t, & 0 \leq t \leq x \\ 0 & \text{otherwise} \end{cases}$

Then the integral equation

$$y(x) = 1 + \lambda \int_0^1 y(t)\, k(x, t)\, dt, \text{ has—}$$

(A) A unique solution for every value of λ

(B) No solution for any value of λ

(C) A unique solution for finitely many values of λ only

(D) Infinitely many solutions for finitely many values of λ

145. The cost matrix for an assignment problem is given by

$$\begin{bmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{bmatrix}$$

where $a, b, c, d > 0$, then the value of the assignment problem is—

(A) $a + b + c + d$

(B) $\min \{a, b, c, d\}$

(C) $\max \{a, b, c, d\}$

(D) $4 \min \{a, b, c, d\}$

146. The unit cost C_{ij} of producing product i at plant j is given by the matrix

$$\begin{bmatrix} 14 & 12 & 16 \\ 21 & 9 & 17 \\ 9 & 7 & 5 \end{bmatrix}$$

The total cost of optimal assignment is—

(A) 20 (B) 22

(C) 25 (D) 28

147. Which of the following is not the real part of an analytic function ?

(A) $x^2 - y^2$ (B) $\frac{1}{1 + x^2 + y^2}$

(C) $\cos x \cos hy$ (D) $x + \frac{x}{x^2 + y^2}$

148. Let $u(x, y) = f(xe^y) + g(y^2 \cos(y))$,

where f and g are infinitely differentiable functions. Then the particle differential equation of minimum order satisfied by u is—

(A) $u_{xy} + xu_{xx} = u_x$ (B) $u_{xy} + xu_{xx} = xu_x$

(C) $u_{xy} - xu_{xx} = u_x$ (D) $u_{xy} - xu_{xx} = xu_x$

149. Consider the following L.P.P. :

Max $z = x_1$

subject to $x_1 + x_2 \leq 1$

$x_1 - x_2 \geq 1$

$-2x_1 + x_2 \geq 1$

$x_1 \geq 0$

and $x_2 \geq 0$

The solution to its dual is—

(A) Unbounded (B) Infeasible

(C) Degenerate (D) Bounded

150. The quadratic equation $x^2 - ax + b = 0$ is known to have two real roots, x_1 and x_2 ($x_1 > x_2$), but the coefficient b is a positive unknown and can be assumed to have a uniform distribution in the permissible range of variation. The expected value of x_1 is ....

(A) 1 (B) 3
(C) 4 (D) 5

151. The residue of $f(z) = \cot z$ at any of its poles is—

(A) 0 (B) 1
(C) $\sqrt{3}$ (D) None of these

152. A random variable X has position distribution. If 2P(X = 2) = P (X = 1) + 2P (X = 0)
then the variance of X is—

(A) $\frac{3}{2}$ (B) 2
(C) 1 (D) $\frac{1}{2}$

153. If $y(t) = 1 + \int_0^t y(v)\, e^{-(t+v)}\, dv$ then $y(t)$ at $t =$ 1 equals

(A) 0 (B) 1
(C) 2 (D) 3

154. The connected subset of the real line with the usual topology are—

(A) All intervals
(B) Only bounded intervals
(C) Only compact intervals
(D) Only semi-infinite intervals

155. Pick out the largest of the sets given below on which the sequence of functions $\{e^{-n\cos^2 x}\}_{n=1}^{\infty}$ converges uniformly—

(A) $\left(0, \frac{9\pi}{20}\right) \cup \left(\frac{11\pi}{20}, \pi\right)$

(B) $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

(C) $\left(0, \frac{\pi}{2} - \delta\right) \cup \left(\frac{\pi}{2} + \delta, \pi\right), 0 < \delta < \frac{\pi}{100}$

(D) $(0, \pi)$

156. Consider the matrix $M = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ and let S_M be the set of 3×3 matrices N such that MN = 0. Then the dimension of the real vector space S_M is equal to—

(A) 0 (B) 1
(C) 2 (D) 3

157. Extremals for the variational problem $v[y(x)] = \int_1^2 (y^2 + x^2 y'^2)\, dx$ satisfy the differential equation—

(A) $x^2y'' + 2xy' - y = 0$
(B) $x^2y'' - 2xy' + y = 0$
(C) $2xy' - y = 0$
(D) $x^2y'' - y = 0$

158. The eigenvalues λ of the integral equation $y(x) = \lambda \int_0^{2\pi} \sin(x + t)\,(t)\,dt$ are—

(A) $\frac{1}{2\pi}, -\frac{1}{2\pi}$ (B) $\frac{1}{\pi}, -\frac{1}{\pi}$
(C) $\pi, -\pi$ (D) $2\pi, -2\pi$

159. The radius of convergence of \a(∞,Σ,n = 0) $\frac{\left(1 + \frac{1}{n}\right)^{n^2}}{n^3} z^n$ is—

(A) e (B) $\frac{1}{e}$
(C) 1 (D) ∞

160. Let C be the boundary of the triangle formed by the points (1, 0, 0), (0, 1, 0), (0, 0, 1). Then the value of the line integral $\oint -2y\,dx + (3x - 4y^2)dy + (z^2 + 3y)dz$ is—

(A) 0 (B) 1
(C) 2 (D) 4

161. Let V be a finite dimensional vector space over R

(i) There exists a linear map $f : V \rightarrow$ such that $f^2 = -Id$
(ii) For all linear maps $f : V \rightarrow V$, there exists a proper 1-dim.
(iii) V is finite union of 1-dim sub spaces
(iv) There exists a unique $f : V \rightarrow V$ such that $f^2 = f$ of dim (V) is even

(v) There exists a proper 1-dim. subspace W such (*e*) dim (V) = 0 that $\frac{V}{W}$ is connected.

(A) dim (V) $\geq$ 3 (B) dim (V) = 1
(C) dim (V) is odd (D) dim (V) is even
(E) dim (V) = 0

162. Let T be the linear operator defined on a finite dimensional vector space V. If rank (T^2) = rank (T), find R(T) $\cap$ N(T), where R(T), N(T) denote respectively the range and the null space of T—

(A) 1 (B) 6
(C) 4 (D) None of these

163. The harmonic conjugate of $U(x, y) = x^2 - y^2 + xy$ is—

(A) $x^2 - y^2 - xy$
(B) $x^2 + y^2 - xy$
(C) $2xy + \frac{1}{2}(y^2 - x^2)$
(D) $\frac{1}{2}xy + 2(y^2 - x^2)$

164. The joint probability density function of the random variables x, y and z is

$$f(x, y, z) = 8xyz, \; 0 < x,y,z < 1$$
$$= 0. \quad \text{otherwise}$$

Then $P(x < y < z)$ is—

(A) $\frac{1}{8}$ (B) $\frac{1}{3}$
(C) $\frac{1}{6}$ (D) $\frac{3}{8}$

165. Expand the function $f(z) = \frac{1}{3 - 2z}$ in powers of $(z - 3)$ and find the radius of convergence of the series so obtained—

(A) $|z - 3| < \frac{3}{2}$ (B) $|z - 4| < \frac{1}{2}$
(C) $|z - 3| > \frac{3}{2}$ (D) None of these

166. Let $f : [a, b] \to R$ be a bounded function where $-\infty < a < b < \infty$. Then is Riemann integrable if and only if f is continuous every- where on $[a, b]$ except on set

(A) The empty on
(B) A set of measure zero
(C) A finite number of points
(D) A countabily infinite number of points

167. Let $f : R^2 \to R$ be a smooth function with positive definite Hession at very point. Let $(a, b) \in R^2$ be a critical point of f. Then

(A) f has a global minimum at (a, b)
(B) f has a local, but not a global minimum at (a, b)
(C) f has a local, but not a global maximum at (a, b)
(D) f has a global maximum at (a, b)

168. Let $M = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & -4 & 0 & 0 \end{bmatrix}$. Then

(A) $MM^T = I$ where M^T is the transpose of M and I is the identify matrix.
(B) Column vectors of M form an orthogonal system of vectors
(C) Column vectors of M form an orthonormal system of vectors
(D) (MX, MY) = (X,Y) for all X, Y in R^4 where (,) is the standard inner product on R^4.

169. Let V be the subspace of R^3 spanne by $u = (1, 1, 1)$ and $v = (1, 1, -1)$. The orthonormal basis of V obtained by the Gram-Schmidt process on the ordered basis (u, v) of V is—

(A) $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right)\right\}$
(B) $\{(1, 1, 0), (1, 0, 1)\}$
(C) $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)\right\}$
(D) $\left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)\right\}$

170. Let S and T be two linear operators on R^3 defined by $S(x, y, z) = (x, x + y, x - y - z)$, $T(x, y, z) = (x + 2z, y - z, x + y + z)$. Then

(A) S is invertible but not T
(B) T is invertible but not S
(C) Both S and T are invertible
(D) Neither S nor T is invertible

171. Let S, T $\subseteq R^2$ be given by

$$S = \left\{\left(x, \sin\frac{1}{x}\right) : 0 < x \leq 1\right\} \cup \{(0, 0)\}$$

and T $= \left\{ \left(x, x \sin \frac{1}{x} \right) : 0 < x \leq 1 \right\} \cup \{(0, 0)\}$. Then under the usual metric on R^2.

(A) S is compact but not T
(B) T is compact but not S
(C) Both S and T are compact
(D) Neither S nor T is compact

172. Let X be a complete metric space and let $E \subseteq X$. Consider the following statements :

(S_1) E is compact,

(S_2) E is closed and bounded,

(S_3) E is closed and totally bounded,

(S_4) Every sequence in E has a subsquence converging in E.

Which one of the above statement does not imply all the other statements ?

(A) S_1 (B) S_2
(C) S_3 (D) S_4

Write the correct pair in your answerbook.

173. (i) l^1
(ii) $(c^1 [a, b], \|\cdot\|_\infty)$
(iii) l^2
(iv) l^∞
(v) $(R'', \|\cdot\|_1)$

(A) Norm satisfies parallelogram law
(B) Complete, non-reflexive, separable
(C) Non-separable, complete
(D) Finite basis
(E) Not complete
(F) Denumerable (Hamel) basis

174. Which of the following is true—

(A) $\mu \equiv \frac{1}{2}(E^{1/2} + E^{-1/2}]$
(B) $\mu \equiv [E^{1/2} + E^{-1/2}]$
(C) $\mu \equiv [E^{1/2} - E^{-1/2}]$
(D) None of these

175. Consider the following primal problem :

Minimize $z = 10x_1 + x_2 + 5x_3$,
Subject to $x_1 - 7x_2 + 3x_3 \geq 50$,
$x_1, x_2, x_3 \geq 0$

The optimal value of the primal is—

(A) $\frac{50}{3}$ (B) $\frac{10}{3}$
(C) $\frac{250}{3}$ (D) $\frac{100}{3}$

176. The singularity of $e^{\sin z}$ at $z = \infty$ is—

(A) A pole
(B) A removable singularity
(C) Nonisolated essential singularity
(D) Isolated essential singularity

177. Let h be the finite difference, then sift operator is defined as—

(A) $Ef(x) = f(x)$
(B) $Ef(x) = f(x - h)$
(C) $Ef(x) = f(x + h)$
(D) None of these

178. The general solution of the differential equation $\frac{dy}{dx} + \tan y \tan x = \cos x \sec y$ is—

(A) $2 \sin y = (x + c - \sin x \cos x)\sec x$
(B) $\sin y = (x + c) \cos x$
(C) $\cos y = (x + c) \sin x$
(D) $\sec y = (x + c) \cos x$

179. Let G be a group of order 30. Let A and B be normal subgroups of orders 2 and 5 respectively. Then the order of group $\frac{G}{AB}$ is—

(A) 10 (B) 3
(C) 2 (D) 5

180. Let $M = \begin{bmatrix} 1 & 1+i & 2i & 9 \\ 1-i & 3 & 4 & 7-i \\ -2i & 4 & 5 & i \\ 9 & 7+i & -i & 7 \end{bmatrix}$. Then

(A) M has purely imaginary eigen values
(B) M is not diagonalizable
(C) M has eigen values which are neither, real nor purely imaginary
(D) M has only real eigen values

181. In R^2, $<(x_1, y_1), (x_2, y_2)> = x_1x_2 - \alpha (x_2y_1 + x_1y_2) + y_1y_2$ is an inner product—

(A) For all $\alpha \in R$
(B) If and only if $\alpha = 0$
(C) If and only if $\alpha < 1$
(D) If and only if $|\alpha| < 1$

182. Let V, W and X be three finite dimensional vector spaces such that dim V = dim X. Suppose $S : V \to W$ and $T : W \to X$ are two linear maps such that $T \cdot S : V \to X$ is injective. Then

(A) S and T are surjective

(B) S is surjective and T is injective
(C) S and T are injective
(D) S is injective and T is surjective.

183. Let S, T $\subseteq$ R be given by

$$S = \left\{ x \in R : 2x^2 \cos \frac{1}{x} = 1 \right\}$$

$$\text{and} T = \left\{ x \in R : 2x^2 \cos \frac{1}{x} \le 1 \right\} \cup \{0\}$$

Then, under the usual metric on R,
(A) S is complete but not T
(B) T is complete but not S
(C) Both S and T are complete
(D) Neither S nor T is complete

184. Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \sin(nx)$
Then the series—
(A) Converges uniformly on R
(B) Converges pointwise but not uniformly on R
(C) Converges in L^1 norm to an integrable function on $[0, 2\pi]$ but does not converge uniformly on R
(D) Does not converge pointwise

185. Which of the following is true—
(A) $E^{-1} \equiv 1 + \nabla$ (B) $E^{-1} \equiv \nabla$
(C) $E^{-1} \equiv 1 - \nabla$ (D) None of these

186. Relation between cartesian and polar coordinate system—
(A) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$
(B) $\begin{cases} x = r\tan\theta \\ y = r\sin\theta \end{cases}$
(C) $\begin{cases} x = r\sin\theta \\ y = r\cos\theta \end{cases}$
(D) None of these

187. Let $d < x, y \ge |x_1 - y_1| + |x_2 - y_2|$ be a metric defined on R^2, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then the set
$S = \{x \in R^2 : d < x, 0 \ge 1\}$ is
(A) A circle (B) An ellipse
(C) A rhombus (D) A rectangle

188. Consider the linear programming problem—
Maximize $z = 2x_1 - 4x_2$
Subject to
$$x_1 + 2x_2 \le 3,$$
$$3x_1 + 4x_2 \le 5,$$
$$x_1, x_2 \ge 0.$$
The total number of basic solution is—
(A) 6 (B) 2
(C) 4 (D) ∞

189. The topology τ on the real line R generated by the class J of all closed intervals $[d, d + l]$ with length l is—
(A) Indescrete
(B) Descrete
(C) Standard topology
(D) Neither descrete nor Housdorff

190. The eigenvalues of the sturm Liouville system
$$y'' + \lambda y = 0, \quad 0 \le x \le \pi$$
$$y(0) = 0, \quad y'(\pi) = 0 \text{ are}$$
(A) $\frac{n^2}{4}$ (B) $\frac{(2n-1)^2 \pi^2}{4}$
(C) $\frac{(2n-1)^2}{4}$ (D) $\frac{n^2\pi^2}{4}$

191. Let m and n be coprime natural numbers. Then the kernel of the ring homomorphism $\phi : z \to z_m \times z_n$, defined by $\phi(x) = (\bar{x}, \bar{x})$, is—
(A) 1 (B) $\frac{\pi}{2}$
(C) π (D) 2π

192. Consider the matrix $M = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$
where a, b and c are non zero real numbers. Then the matrix has—
(A) Three non-zero real eigen values
(B) Complex eigen values
(C) Two non-zero eigen value
(D) Only one non-zero eigen value

193. Let $\{v_1, v_2, v_3, v_4\}$ be a basis of R^4 and $v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$, where $a_i \in R$, $i = 1, 2, 3, 4$. Then $\{v_1 - v, v_2 - v, v_3 - v, v_4 - v\}$ is a basis of R^4 if and only if—
(A) $a_1 = a_2 = a_3 = a_4$
(B) $a_1a_2a_3a_4 = 1$
(C) $a_1 + a_2 + a_3 + a_4 \ne 0$
(D) $a_1 + a_2 + a_3 + a_4 = 0$

194. If a square matrix of order 10 has exactly 4 distinct eigenvalues then the degree of its minimal polynomial is—
(A) At least 4 (B) At most 4
(C) At least 6 (D) At most 6

195. Let $f : \mathrm{R} \to \mathrm{R}$ be defined by
$$f(x) = \begin{cases} n, \text{ if } x = n, x \in \mathrm{N} \\ 0, \text{ otherwise} \end{cases}$$
and $\mathrm{T} = \mathrm{N} \cup \left\{ n + \frac{1}{n} : n \in \mathrm{N} \right\}$

Then under the usual metric on R, f is uniformly continuous on
(A) N but not T (B) T but not N
(C) Both N and T (D) Neither N nor T

196. Let $f(z)$ be an analytic function. Then the value of $\int_0^{2\pi} f(e^{it}) \cos(t)\, dt$ equals—
(A) 0 (B) $2\pi f(0)$
(C) $2\pi f'(0)$ (D) $\pi f'(0)$

197. A rigid body moving freely in space has degrees of freedom—
(A) Three (B) Six
(C) Nine (D) Tweleve

198. A subset A of topological space X is closed set if—
(A) (X – A) is open (B) A is closed
(C) A is open (D) None of these

199. Let $A \in R^{n \times n}$ $B \in R^n$. Consider the two systems of linear equation.
(I) $Ax = b$ and (II) $A'Az = A'b$ Then
(A) Both (I) and (II) have the same solution
(B) (I) has more solutions than (II)
(C) Both (I) and (II) may not have any solution
(D) (I) may not have any solution but (II) always has atleast one solution.

200. The solution of the integral equation
$$g(s) = s + \int_0^1 su^2\, g(u) du$$
is given by
(A) $g(t) = \frac{3t}{4}$ (B) $g(t) = \frac{4t}{3}$
(C) $g(t) = \frac{2t}{3}$ (D) $g(t) = \frac{3t}{2}$

Answers with Explanations

1. (A)
2. (A) C [0, 1] is the space of all continuous real valued functions on [0, 1]. Then identity map I : (C [0, 1], $\|\cdot\|_\infty$) $\to$ (C [0, 1], $\|\cdot\|_1$) is continuous but not open.
3. (A) 4. (B)
5. (A) A real value function f defined on an interval I is said to be monotonically non-increasing if f
$$x > y \Rightarrow f(x) \geq f(y)\ \forall\, x, y \in \mathrm{I}$$
and monotonically non-increasing iff
$$x > y \Rightarrow f(x) \leq f(y)\ \forall\, x, y \in \mathrm{I}$$
so f must be continuous function.
6. (A) $P(A \cap B) + P(A \cap \bar{B}) = P(A)$
$$P(A \cap \bar{B}) = P(A) - P(A \cap \bar{B})$$
$$\frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{P(A) - P(A \cap B)}{P(\bar{B})}$$
$$\therefore \frac{P(A) - P(A \cap B)}{P(\bar{B})} = \frac{P(A \cap \bar{B})}{P(\bar{B})}$$
$$= P(A \mid \bar{B})$$
7. (A)
8. (A) The error term for linear interpolation is defined as,
$$\frac{h^{n+1}}{8} |f''(z)| \leq \epsilon$$
For $n = 1$,
$$\frac{h^2}{8} 1 \leq 5 \times 10^{-5} \quad \ldots(1)$$
We know, $f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$\therefore \max |f''(x)| \leq 1$
From equation (i), we have
$h^2 \leq 4 \times 10^{-4}$
or, $h \leq 2 \times 10^{-2}$
or, $h \leq 0{\cdot}02$
9. (B) 10. (D) 11. (B)
12. (A) Let $z = \frac{1}{x+y}$
Take $h = k = 0{\cdot}2$

$x \quad y$	$y_0 = 1$	$y_1 = 1{\cdot}2$	$y_2 = 1{\cdot}4$	$y_3 = 1{\cdot}6$	$y_4 = 1{\cdot}8$	$y_5 = 2{\cdot}0$	A_x
1	$\frac{1}{2}$	$\frac{1}{2{\cdot}2}$	$\frac{1}{2{\cdot}4}$	$\frac{1}{2{\cdot}6}$	$\frac{1}{2{\cdot}8}$	$\frac{1}{3}$	0·4059
1·2	$\frac{1}{2{\cdot}2}$	$\frac{1}{2{\cdot}4}$	$\frac{1}{2{\cdot}6}$	$\frac{1}{2{\cdot}8}$	$\frac{1}{3}$	$\frac{1}{3{\cdot}2}$	0·3751
1·4	$\frac{1}{2{\cdot}4}$	$\frac{1}{2{\cdot}6}$	$\frac{1}{2{\cdot}8}$	$\frac{1}{3}$	$\frac{1}{3{\cdot}2}$	$\frac{1}{3{\cdot}4}$	0·3486
1·6	$\frac{1}{2{\cdot}6}$	$\frac{1}{2{\cdot}8}$	$\frac{1}{3}$	$\frac{1}{3{\cdot}2}$	$\frac{1}{3{\cdot}4}$	$\frac{1}{3{\cdot}6}$	0·3257
1·8	$\frac{1}{2{\cdot}8}$	$\frac{1}{3}$	$\frac{1}{3{\cdot}2}$	$\frac{1}{3{\cdot}4}$	$\frac{1}{3{\cdot}6}$	$\frac{1}{3{\cdot}8}$	0·3056
2·0	$\frac{1}{3}$	$\frac{1}{3{\cdot}2}$	$\frac{1}{3{\cdot}4}$	$\frac{1}{3{\cdot}6}$	$\frac{1}{3{\cdot}8}$	$\frac{1}{4}$	0·2878

Applying trapezoidal rule to each row for

$$h = 0{\cdot}2$$

We find $A_1, A_{1\cdot2}, A_{1\cdot4}, A_{1\cdot6}, A_{1\cdot8}$ and A_2 as given in the above table. Then, again by Trapezoidal rule, we get

$$\int_1^2\int_1^2 \frac{dx\,dy}{x+y} = 0{\cdot}2\left[\frac{1}{2}(0{\cdot}4059 + 0{\cdot}2878) + (0{\cdot}3751 + 0{\cdot}3486 + 0{\cdot}3257 + 0{\cdot}3056)\right]$$

$$= 0{\cdot}3404$$

$$\approx \frac{17}{18}$$

13. (A) 14. (D) 15. (B) 16. (C) 17. (A)

18. (B) 19. (B)

20. (B)

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$y_1 = y_0 + (k_1 + k_2)/2$$

$$y' = -y$$

$$f(x, y) = -y$$

$$y(0) = y_0$$

$$k_1 = -hy_0$$

$$k_2 = h[-(y_0 + k_1)]$$

$$= h[-(y_0 - hy_0)]$$

$$= -hy_0(1 - h)$$

$$y_1 = y_0 + \left[\frac{-hy_0 - hy_0(1-h)}{2}\right]$$

$$= y_0 - \frac{hy_0}{2}[1 + 1 - h]$$

$$= y_0 - \frac{hy_0}{2}(2 - h)$$

$$= y_0 - hy_0 + \frac{h^2y_0}{2}$$

$$\therefore \quad y_1 = \frac{y_0}{2}(h^2 - 2h + 2)$$

21. (B) 22. (D)

23. (D) Heat equation is given by

$$\frac{\partial U}{\partial t} = C^2\frac{\partial^2 U}{\partial x^2};\ C^2 = 1$$

$$U^2 = \frac{e^{-x^2}}{\sqrt{t}}$$

$$\therefore \quad \frac{\partial U}{\partial t} = \frac{\sqrt{t}\,.\,e^{-x^2/4t}\left(\frac{x^2}{4t^2}\right) - e^{-x^2/4t}\frac{1}{2\sqrt{t}}}{t}$$

$$= \frac{e^{-x^2/4t}}{\sqrt{t}}\left[\frac{x^2}{4t^2} - \frac{1}{2t}\right]$$

$$= 4\left(\frac{x^2}{4t^2} - \frac{1}{2t}\right)$$

$$\frac{\partial U}{\partial x} = \frac{1}{\sqrt{t}}.$$

$$e^{-x^2/4t}\left(\frac{2x}{4t}\right) = U\left(\frac{-x}{2t}\right)$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial x}\left(\frac{-x}{2t}\right) - U\left(\frac{1}{2t}\right)$$

$$= U\left\{\frac{x^2}{4t^2} - \frac{1}{2t}\right\}$$

$$\therefore \quad \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

24. (B) 25. (B)

26. (A)

$$x_{k+1} = \frac{1}{2}\left[x_k + \frac{a}{x_k^{\,n-1}}\right]$$

$$x_{1+1} = \frac{1}{2}\left[x_1 + \frac{a}{x_1^{n-1}}\right]$$

$$x_{2+1} = \frac{1}{2}\left[x_2 + \frac{a}{x_2^{n-1}}\right]$$

$n = 1$, $$x_{3+1} = \frac{1}{2}\left[x_3 + \frac{a}{x_3^{n-1}}\right]$$

But for $x_{k+1} = \frac{1}{2}[x_k + a]$

Hence $n = 1$, this fails to converge.

27. (A) 28. (D) 29. (D) 30. (B) 31. (B)
32. (A) 33. (B) 34. (B) 35. (A) 36. (C)
37. (A) 38. (D) 39. (B) 40. (A)

41. (B) $$x^2 - y^2 = c_1$$

Differentiating, $$2x - 2y\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = \frac{x}{y}$$

For orthogonal trajectory, $$\frac{-dx}{dy} = \frac{x}{y}$$

$\Rightarrow$ $$\frac{dx}{x} + \frac{dy}{y} = 0$$

Integrating, $$\log xy = \log c_2$$

$\Rightarrow$ $$xy = c_2$$

42. (A) $P^{-1}\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}.P = \begin{bmatrix}e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{bmatrix}$

Let $P = \begin{bmatrix}a & b \\ c & d\end{bmatrix}$, where a, b, c, d are complex numbers and

$$ad - bc \neq 0$$

$$\therefore \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}a & b \\ c & d\end{bmatrix} = \begin{bmatrix}a & b \\ c & d\end{bmatrix}\begin{bmatrix}e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{bmatrix}$$

write down the equation $a = 1$, $c = -i$, $b = 1$, $d = i$

$$\therefore \quad P = \begin{bmatrix}1 & 1 \\ -i & i\end{bmatrix}$$

43. (A) None of the answer is correct

$$\int_c \vec{F}\,dr = \int_0^1 5x^2\,dx + (xz - y)\,dy + 3z\,dz$$

The parametric from,

$$x(t) = t,$$
$$y(t) = t$$

and $$z(t) = t$$

$$\therefore \int_c \vec{F}\,dr = \int_0^1 5t^2\,dt + \int_0^1 (t^2 - t)\,dt + 3\int_0^1 t\,dt$$

$$= \left[\frac{5t^3}{3}\right]_0^1 + \left[\frac{t^3}{3} - \frac{t^2}{2}\right]_0^1 + 3\left[\frac{t^2}{2}\right]_0^1$$

$$= \frac{5}{3} + \left(\frac{1}{3} - \frac{1}{2}\right) + \frac{3}{2} = 3$$

44. (C) 45. (A) 46. (B) 47. (A) 48. (A)
49. (A)

50. (A) Let $$u(x, y) = \frac{y}{x}$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2},$$

and $$\frac{\partial u}{\partial y} = \frac{1}{x}$$

$\Rightarrow$ $$x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Also, $$u(1, y) = \frac{y}{1} = y$$

51. (B) 52. (C) 53. (D)

54. (A) Let $A = \begin{bmatrix}0 & 1 \\ -1 & 0\end{bmatrix}$, $B = \begin{bmatrix}0 & i \\ i & 0\end{bmatrix}$

Note : $$A^4 = B^4 = e;$$
$$A^2 = B^2$$
$$AB = -BA$$

Thus the elements of the group are 1, A, A^2, A^3, B, AB, A^2B, A^3B.

Note : BA^2, BA^3 are all elements from the above collection for

$$BA^2 = BAA = -ABA = +AAB = A^2B$$

Similarly others.

Hence order of group generated by A and B is 8.

55. (A)

56. (A) For the initial value problem,

$$\text{PDE} = u_{tt} - c^2 u_{xx} = 0,$$
$$-\infty < x < \infty, t \geq 0$$
$$\text{ICS} = u(x, 0) = \eta(x)$$
$$u_t(x, 0) = v(x)$$

D′ Alembert solution is given by

$$u(x, t) = \frac{1}{2}[\eta(x+ct) + \eta(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} u(x)\,dx)$$

For the given problem, $u(x) = 0,$

$\eta(x) = x(1-x),$

$c = 1$

$\therefore$ The solution is of the form,

$$u(x, t) = \frac{1}{2}[\eta(x+t) + \eta(x-t)]$$

$$= \frac{1}{2}[(x+t)(1-x-t) + (x-t)(1-x+t)]$$

$$\therefore \quad u\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{2}\left[\frac{3}{4}, \frac{1}{4} + \frac{1}{4}, \frac{3}{4}\right]$$

$$= \frac{3}{16}$$

57. (A) 58. (C)

59. (A) Rate of the convergence of Newton's formula when there exists multiple roots—

We have $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

i.e. $$a + \epsilon_{n+1} = a + \epsilon_n - \frac{f(a+\epsilon_n)}{f'(a+\epsilon_n)}$$

$$= \epsilon_n - \frac{f(a) + \epsilon_n f'(a) + \frac{1}{2}\epsilon_n^2 f''(a) + \ldots}{f'(a) + \epsilon_n f''(a) + \frac{1}{2}\epsilon_n^2 f'''(a) + \ldots}$$

$$= \epsilon_n - \frac{\frac{1}{2}\epsilon_n^2 f''(a) + \ldots}{\epsilon_n f''(a) + \ldots}$$

$$= \epsilon_n - \epsilon_n \frac{\frac{1}{2} f''(a) + \ldots}{f''(a)}$$

$$\approx \frac{1}{2}\epsilon_n$$

If $x = a$ is the root of multiplicity $m > 1$, then the speed of convergence is given by

$$\epsilon_{n+1} \approx \frac{m-1}{m}\epsilon_n$$

Here $m = 3$

$$\therefore \quad \epsilon_{n+1} \approx \frac{2}{3}\epsilon_n,$$

which is also linear.

60. (C) 61. (B)

62. (C) Let $u = f(x)\,g(y)$

$$\therefore \quad \frac{\partial u}{\partial x} = f'(x)\,g(y),$$

and $$\frac{\partial u}{\partial y} = f(x)\,g'(y)$$

and $$\frac{\partial^2 u}{\partial x \partial y} = f'(x)\,g'(y)$$

$$\Rightarrow \quad u\frac{\partial^2 u}{\partial x\,\partial y} - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} = f(x)\,g(y)\,.\,f(x)\,g'(y) - f(x)\,g(y)\,.\,f'(x)\,g'(y) = 0$$

63. (A) 64. (C) 65. (D)

66. (A) Let $f(z) = u + iv,$

u and v are functions of x, y

where $z = x + iy$

$v = ex\,(y \cos y + x \sin y)$

Since $f(z)$ is analytic, $v_y = u_x$

$$\therefore \quad u_x = e^x[-y \sin y + \cos y + x \cos y]$$

Integrating with respect to x, we get,

$$u = e^x(\cos y - y \sin y + \cos y\,(xe^x - e^x) + g(y)$$

$$= -e^x y \sin y + xe^x \cos y + g(y)$$

But $$u_y = -v_x$$

$$= -e^x(y \cos y + x \sin y + \sin y)$$

$$\therefore \quad g'(y) = 0$$

$$\therefore \quad g(y) = \text{constant}$$

Hence, $$f(z) = e^x(-y \sin y) + \cos y\,(x)\,e^x + i\,\{e^y(y \cos y + x \sin y)\}$$

$$= e^x(x \cos y - y \sin y) + e^x\,i\,(y \cos y + x \sin y) + c$$

$$= xe^x[\cos y + i \sin y] + e^x iy\,[\cos y + i \sin y]$$

$$= xe^x e^{iy} + e^x\,iye^{iy}$$

$$= x\,e^{x+iy} + iy\,e^{x+iy}$$

$$= e^{x+iy}(x + iy)$$

$$= z\,e^z$$

67. (B) 68. (A) 69. (B) 70. (B) 71. (C)

72. (A) 73. (C) 74. (A) 75. (B) 76. (D)

77. (A) $$\frac{dX}{dt} = MX + b$$

$$\frac{dX}{dt} - MX = b$$

where, $$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix};\ M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore \quad \text{I.F.} = e^{-\int M dt}$

$= e^{-Mt}$

$\therefore \quad Xe^{-Mt} = \int be^{-Mt}\,dt + c$

$= -b\dfrac{e^{-Mt}}{M} + c$

$\Rightarrow \quad x = \dfrac{-b}{M} + ce^{Mt}$

78. (A) The characteristic equation is

$$\xi^2 - \left(1 \div \frac{23}{12}\bar{h}\right)\xi^2 + \frac{16}{12}\bar{h}\,\xi - \frac{5}{12}\bar{h} = 0$$

Put $\quad \xi = \dfrac{1+z}{1-z}$

The equation becomes

$$b_0 z^3 + b_1 z^2 + b_2 z + b_3 = 0$$

where $\quad b_0 = 2 + \dfrac{44}{12}\bar{h}$

$b_1 = 4 - \dfrac{8}{12}\bar{h},$

$b_2 = 2 - 2\bar{h}$

and $\quad b_3 = -\bar{h}$

Hence, Hurwtiz criterion is $b_0 \geq 0$, $b_1 \geq 0$, $b_2 \geq 0$, $b_3 \geq 0$ and $b_1 b_2 - b_0 b_3 \geq 0$. This gives $b_0 \geq -\dfrac{6}{11}, \bar{h} \leq 6; \bar{h} \leq 1; \bar{h} \leq 0$ and $15\bar{h}^2 - 22\bar{h}^2 + 24 \geq 0$

Since its disc ≤ 0

$\therefore$ The range is $\left(\dfrac{-6}{11}, 0\right)$

79. (A)

80. (D) None of the given answer is correct

$4\ddot{x} = -8x$

$\therefore \quad 4m^2 + 8 = 0$

or $\quad m^2 + 2 = 0$

or $\quad m = \pm\sqrt{2}\,i$

$\therefore \quad x = A\cos\sqrt{2}t + B\sin\sqrt{2}t$

But $\quad t = 0$ At $x = 10$

$\therefore \quad A = 10$

$x = -2A\sin\sqrt{2}.t + 2B\cos\sqrt{2}.t$

At $\quad x = 10, t = 0$

$\therefore \quad 0 = B$

Hence $\quad x = 10\cos\sqrt{2}.t = a\cos nt$

$T = \dfrac{2\pi}{n} = \dfrac{2\pi}{\sqrt{2}}$

$\therefore \quad T = \sqrt{2}\pi$

$\therefore \quad W = \dfrac{1}{T}$

$= \dfrac{1}{\sqrt{2}\pi}$

Hence none of the above answer is correct.

81. (C) 82. (A) 83. (D) 84. (C) 85. (A)

86. (A) 87. (B)

88. (A) Given, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$D^2 + D'^2 = 0$

A. E. is $\quad D^2 + 1 = 0$

or $\quad D = \pm i$

$\therefore \quad u = f(x+iy) + g(x-iy)$

89. (B) $\quad I = -\int \vec{F}\,.\,d\vec{r}$

$= \int (zdx + xdy + ydz)$

$= \int_0^{2\pi} a^2\cos^2\theta\,d\theta$

Setting $\quad x = a\cos\theta,$

$y = a\sin\theta,$

$z = 6$

$= -\pi a^2$

90. (A) $\quad \displaystyle\int_E \phi(x)\,dx = \sum_{k=1}^{n} \frac{1}{k} m(E_k)$

$= \displaystyle\sum_{k=1}^{n} \frac{1}{k} k^3$

$= \displaystyle\sum_{k=1}^{n} k^2$

$= \dfrac{n(n+1)(2n+1)}{b}$

91. (B) 92. (C) 93. (B) 94. (B) 95. (B)

96. (C) 97. (A) 98. (D) 99. (C)

100. (C) Weak duality theorem : Let $x0$ be a feasible solution of the primal problems.

$$f(x) = cx$$

Subject to $\quad Ax \leq b; x \geq 0$

where x^T and $C \in R^n$, $b^T \in R^m$ and A is $m \times n$ real matrix. If w_0 be a feasible solution of the dual of the primal namely.

Mini $\quad g(w) = b^T w$

Subject to $\quad A^T w \geq C^T,$

$$w \geq 0$$

where $\quad w^T \in R^m$

then $\quad cx_0 \leq b^T w_0$

Proof : Since x_0 and w_0 are the feasible solution to the primal and its dual respectively, we must have

$$Ax_0 \leq b, x_0 \geq 0$$

and $\quad A^T w_0 \geq C^T, w_0 \geq 0$

Thus $\quad c \leq {w_0}^T A$

or $\quad cx_0 \leq {w_0}^T$

or $\quad Ax_0 \leq {w_0}^T b$

$$cx_0 = b^T w_0$$

Equality sign holds only if both one optimal solutions. Hence if x be a non-optimal feasible solution of a L.P. maximum problem and y a dual feasible solution, then $cx_0 < b^T w_0$ *i.e.* the primal objective value at x is less than the dual objective value at y.

101. (D) We have

$$\begin{aligned} P(P \cap Q) &= P(P) - P(P \cap Q) \\ &= P(P) - P(P)\,P(Q) \\ &= P(P)\,[1 - P(Q)] \\ &= P(P)\,P(Q^C) \end{aligned}$$

$\Rightarrow$ P and Q^C are independent events.

102. The required Green's function is obtained by finding the non-trivial solution of the system

$$\frac{\partial^2 y}{\partial x^2} = 0, u(0) = 0,$$

$$0 \leq x \leq 5 \qquad \ldots(i)$$

$$\frac{d^2 v}{dx^2} = 0,$$

$$v(\pi) = 0,$$

$$\xi \leq x \leq \pi \qquad \ldots(ii)$$

Subject to the conditions

$$u(\xi) = r(\xi) \qquad \ldots(iii)$$

$$v'(\xi) - u'(\xi) = -1 \qquad \ldots(iv)$$

Thus we get $\quad u = c_1 x,$

$$v = c_2(\pi - x)$$

Using equations (iii) and (iv), we get

$$c_1 = \frac{\pi - \xi}{\pi}$$

and $\quad c_2 = \dfrac{\xi}{\pi}$

Hence the required Green's function is

$$c_1(x, \xi) = \left(1 - \frac{\xi}{\pi}\right) x,$$

$$0 \leq x \leq \xi$$

$$= \xi\left(1 - \frac{x}{\pi}\right),$$

$$5 \leq x \leq \pi$$

103. (A) 104. (C) 105. (D) 106. (D)

107. (D) None of these

Characteristic equation is

$$\begin{aligned} \lambda(\lambda - 1)(\lambda + 1) &= 0 \\ \Rightarrow \quad \lambda(\lambda^2 - 1) &= 0 \\ \Rightarrow \quad \lambda^3 - \lambda &= 0 \\ \Rightarrow \quad A^3 - A &= 0 \\ \Rightarrow \quad A^3 &= A \end{aligned}$$

So, $\quad A^{100} = A^{99} A$

$$\begin{aligned} &= (A^3)^{33}.A \\ &= A^{33}.A \\ &= (A^3)''.A \\ &= A^{11}.A \\ &= A^{12} \\ &= (A^3)^4 \\ &= A^4 \\ &= A^3.A \\ &= A^2 \end{aligned}$$

Now $\quad A^3 = A$

$$A^2 = I$$

So $\quad A^{100} = A^2 = I$

So $\quad I + I = 2I$

$\therefore \quad |2I| = 8$

108. (C) 109. (C) 110. (C) 111. (D) 112. (A)

113. (A) Let X_i be the number obtained on the i^{th} dice (i = 1, 2), when thrown. Then the sum of the number of points on two discs is given by

$$S = X_1 + X_2$$

$$E(S) = E(X_1) + E(X_2)$$

$$\Rightarrow \quad = \frac{7}{2} + \frac{7}{2} = 7$$

$$\text{Because } E(X) = \frac{1}{6}\times 1 + \frac{1}{6}\times 2 + \frac{1}{6}\times 3 + \ldots + \frac{1}{6}\times 6$$

$$= \frac{1}{6}(1+2+3+\ldots+6)$$

$$= \frac{1}{6}\times\frac{6\times 7}{2}$$

$$= \frac{7}{2}$$

In General, sum of the points obtained in a random throw of n dice is given by

$$E(S) = \sum_{i=1}^{n} E(X_i)$$

$$= \sum_{i=1}^{n}\left(\frac{7}{2}\right)$$

$$= \frac{7n}{2}$$

114. (A) The characteristic of

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x\partial y} + \cos^2 x\frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = 0$$

are given by the differential equations.

$$\frac{dy}{dx} = \frac{h \pm \sqrt{h^2 - ab}}{a}, \text{ with usual notation}$$

i.e. $a = 1, h = 1, b = \cos^2 x$

$$\therefore \frac{dy}{dx} = 1 \pm \sin x$$

Integrating, the characteristics are

$$y - x + \cos x = c_1$$

and $\quad y - x - \cos x = c_2$

115. (A) 116. (B) 117. (A) 118. (B) 119. (C)

120. (B) 121. (D) 122. (B) 123. (C) 124. (C)

125. (A)

126. (A) The relation between Lagrangian L and Hamiltonian H is

$$L = \dot{q}P - H$$

Here, $H = \frac{1}{2}(q^{-2} + p^2q^4)$

$$\therefore \quad \dot{q} = \frac{\partial H}{\partial P} = pq^4 \quad \ldots(i)$$

and $\quad L = p^2q^4 - \left(\frac{1}{2q^2} + \frac{p^2q^4}{2}\right)$

$$= \frac{p^2q^4}{2} - \frac{1}{2q^2}$$

$$= \frac{\dot{q}^2}{2q^4} - \frac{1}{2q^2} \text{ from equation (i)}$$

The Lagrangen's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \text{ yields}$$

$$\frac{d}{dt}\left(\frac{\dot{q}}{q^4}\right) + \frac{2\dot{q}^2}{q^5} - \frac{1}{q^3} = 0$$

or, $\quad f(q, \dot{q}, \ddot{q}) = 0$

where $\quad f(q, \dot{q}, \ddot{q}) = \ddot{q} - \frac{2\dot{q}^2}{q} - q$

127. (D)

128. (A) Given,

$$P(E \cap F) = \frac{1}{6}$$

$$P(E^C \cap F^C) = \frac{1}{3}$$

$$P(E) > P(F),$$

$$P(E \cap F) = \frac{1}{6}$$

$$P[(E \cap F)^C] = \frac{1}{3}$$

$$P(E \cup F) = \frac{2}{3}$$

Also $\quad P(E) + P(F) = P(E \cap F) + P(E \cup F)$

$$P(E) + P(F) = \frac{1}{6} + \frac{2}{3} = \frac{5}{6} \quad \ldots(i)$$

$$P(E \cap F) = \frac{1}{6}$$

Since E and F are independent events

$$\therefore \quad P(E)\,P(F) = \frac{1}{6} \quad \ldots(ii)$$

or, $\quad P(F) = \frac{1}{6P(E)}$

Substituting equation (ii) in equation (i), we get

$$P(E) + P(F) + \frac{5}{6}$$

or, $\quad P(E) + \frac{1}{6P(E)} = \frac{5}{6}$

Let $\quad P(E) = x$

Then $6x^2 - 5x + 1 = 0$

$$x = \frac{1}{2}, \frac{1}{3}$$

$$P(E) = \frac{1}{3}$$

$\Rightarrow$ $P(F) = \frac{1}{2}$

Also $P(E) = \frac{1}{2}$

$\Rightarrow$ $P(F) = \frac{1}{3}$

Given, $P(E) > P(F)$

Hence $P(E) = \frac{1}{2}$

129. (A) 130. (C) 131. (B) 132. (B) 133. (C)
134. (C) 135. (D) 136. (A) 137. (B) 138. (A)
139. (A)

140. (D) P(Getting ahead in a fair coin) $= \frac{1}{2}$

P(Getting ahead in two headed coin) $= 1$

Total Probability = [1st coin (head) (head) + 2nd coin (head) (head)]

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times 1\times 1$$

$$= \frac{5}{8}$$

Required Probability $= \frac{1/2}{5/8}$

$$= \frac{4}{5}$$

141. (D) 142. (C)

143. (D) Given, $f(z) = \sin\frac{1}{z}$

zeros of $f(z)$ are given by

$$\sin\frac{1}{z} = 0$$

$\therefore$ $\frac{1}{z} = \pi x$

$\Rightarrow$ $z = \frac{1}{n\pi}$;

where $n = 0, \pm 1, \pm 2, \ldots$

144. (A) 145. (D) 146. (C) 147. (A) 148. (A)

149. (B) Max $z = x_1$

$\Rightarrow$ Max $z_x = 1 . x_1 + 0 . x_2$

Sub to
$$\begin{array}{rcl|rcl} x_1 + x_2 & \le & 1 & x_1 + x_2 & \le & 1 \\ x_1 - x_2 & \ge & 1 & -x_1 + x_2 & \le & -1 \\ -2x_1 + x_2 & \ge & 1 & 2x_1 - x_2 & \le & -1 \end{array}$$

Its dual is given by

Min. $z_w = w_1 - w_2 - w_3$

Sub to $w_1 + w_2 - w_3 \ge 1$

$w_1 + w_2 - w_3 \ge 0$

$w_1, w_2, w_3 \ge 0$

Introducing surplus variables $S_1 \ge 0$, $S_2 \ge 0$ and artificial variables

$a_1 \ge 0$

$a_2 \ge 0$

The problem becomes

Max $zw' = -(z_w)$

$= -w_1 + w_2 + w_3 + 0.S_1 + 0.S_2 - a_1M - a_2M$

Sub. to constraints

$w_1 - w_2 + 2w_3 - S_1 + a_1 = 0$

$w_1 + w_2 - w_3 - S_2 + a_2 = 0$

ω_i and $S_i \ge 0$

Use simplex method :

		$C_{ij} \rightarrow$	-1	1	1	0	0	$-M$	$-M$		
Basic Variable	C_B	X_B	W_1	W_2	W_3	S_1	S_2	A_1	A_2	Mini. Ratio	X_B/WK
a_1	$-M$	1	$\boxed{1}$	-1	2	-1	0	1	0	1/1	$= 1 \leftarrow$
a_2	$-M$	0	$\boxed{1}$	1	-1	0	1	0	1		
	$Z_W = -M$		$-2M+1$	-1	$-M-1$	M	$-M$	0	0	$\leftarrow \Delta$	
W_1	-1	1	0	-2	3	-1	-1	1	-1		
a_2	$-M$	0	1	1	-1	0	1	0	1		
	$Z_w = 1$		$-M+1$	$-M+1$	$M-4$	1	$-(M+1)$	$(-1+M)$	1	$-\Delta_j = z_j - c_j$	

150. (A) $x^2 - ax + b = 0$

b is + ve and has uniform distribution

$$x_1, x_2 = \frac{a \pm \sqrt{a^2 - 4b}}{2}$$

Since the roots are real, $a^2 - 4b > 0$

or $\quad b < \frac{a^2}{4}$

Since, $x_1 > x_2$

$$\therefore \quad x_1 = \frac{a + \sqrt{a^2 - 4b}}{2}$$

$$x_2 = \frac{a - \sqrt{a^2 - 4b}}{2}$$

b has a uniform distribution in $\left(0, \frac{a^2}{4}\right)$

Let $\quad X = \frac{a + \sqrt{a^2 - 4b}}{2}$

The distribution function of X is

Now,
$$\begin{aligned} F(x) &= P\,(X < x) \\ &= P\,(b > ax - x^2) \\ &= \int_{ax-x^2}^{a^2/4} \frac{4}{a^2}\,dx \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} b &> ax - x^2 \\ &= \frac{4}{a^2}\left(\frac{a^2}{2} - ax + x^2\right) \end{aligned}$$

$$\therefore \quad f(x) = \frac{4}{a^2}\,[2x - a)$$

Also $\quad 0 < b < \frac{a^2}{4}$

$$\Rightarrow \quad \frac{a}{2} < x < a$$

$$\begin{aligned} E(x) &= \int_{a/2}^{a} \frac{4}{a^2}\,(2x - a)\,dx \\ &= \frac{4}{a^2}\,(x^2 - ax)_{a/2}^{a} \\ &= \frac{4}{a^2} \cdot \frac{a^2}{4} \\ &= 1 \end{aligned}$$

151. (B) Singularities of $f(z)$ are

$$z = n\pi, n = 0, \pm 1, \pm 2, \ldots$$

$f(z)$ has a simple pole

$$\text{Res} \quad f(z) = \lim_{z \to n\pi} (z - n\pi)\frac{\cos z}{\sin z}$$

$$= \frac{(-1)^n}{(-1)^n}$$

$$= 1$$

152. (D) Given, $\quad 2P(X = 2) = P(X = 1) + 2P(X = 0)$

When a random variable X is poisson distributed

$$P(X = n) = \frac{e^{-\lambda}\lambda^n}{n_0^1}$$

$$\therefore \quad \frac{2e^{-\lambda}\lambda^2}{2_0^1} = \frac{e^{-\lambda}\lambda^1}{1_0^1} + 2e^{-\lambda}$$

For a Poisson distribution with Parameter λ,

$$\lambda^2 - \lambda - 2 = 0$$

$$\therefore \quad \lambda = 2, -1$$

$$\text{Mean} = \text{variance} = \frac{1}{\lambda}$$

Thus, variance $= \frac{1}{\lambda} = \frac{1}{2}$

153. (B) 154. (A) 155. (A) 156. (D) 157. (D)

158. (B) 159. (B) 160. (D)

161. (C) (i) $\rightarrow$ (C)
(ii) $\rightarrow$ (A)
(iii) $\rightarrow$ (B)
(iv) $\rightarrow$ (D)
(v) $\rightarrow$ (E)

162. (D)

163. (C)
$$u_x = 2x + y,$$
$$u_y = -2y + x$$

$$\begin{aligned} f(z) &= \int [u_x(z, 0) + iv_x(z, 0)]\,dz \\ &= \int [u_x(z, 0) - i\,u_y(z, 0)]\,dz \\ &= \int (2z - iz)\,dz \\ &= z^2 - \frac{iz^2}{2} \\ &= \left(1 - \frac{i}{2}\right)z^2 \end{aligned}$$

$$I_m\,|f(z)| = \frac{y^2 - x^2}{2} + 2xy$$

164. (C)

165. (A) We have

$$f(z) = \frac{1}{3-2z}$$
$$= \frac{1}{2\left[\frac{3}{2}-z\right]}$$
$$= \frac{1}{2\left[\frac{3-6}{2}-(z-3)\right]}$$
$$= \frac{1}{2\left[\frac{-3}{2}-(z-3)\right]}$$
$$= \frac{1}{2\left(\frac{-3}{2}\right)\left[1+\frac{2}{3}(z-3)\right]}$$
$$= -\frac{1}{3}\left[1+\frac{2}{3}(z-3)\right]^{-1}$$
$$= -\frac{1}{3}\left[1-\frac{2}{3}(z-3)+\frac{4}{9}(z-3)^2\ldots\right]$$

and the region of convergence is

$$|z-3| < \frac{3}{2}$$

166. (B) 167. (C)

168. (B) Let A_1 and A_2 be two complex n-vectors Λ, then A_1 is said to be orthogonal to A_2, if

$$(A_1, A_2) = 0,$$

i.e., $A_1^T A_2 = 0$

Here, $A_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$;

$$A_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \end{bmatrix}$$

$$A_1^T = [0, 2, 0, 0]$$

$$(A_1A_2) = [0, 2, 0, 0]\begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \end{bmatrix}$$
$$= 0$$

$\therefore$ Similarly $(A_2A_3) = (A_3A_4) = 0$

Thus column vectors form the orthogonal set of system.

169. (D) 170. (A) 171. (C) 172. (B)

173. (i) $\rightarrow$ (B)
(ii) $\rightarrow$ (A)
(iii) $\rightarrow$ (D)
(iv) $\rightarrow$ (C)
(v) $\rightarrow$ (F)

174. (A)

175. (C) Optimal value of the Primal

The corner points, $x_1 = \frac{50}{5} = 10$

$$x_2 = \frac{-50}{7}$$
$$x_3 = \frac{50}{3}$$

Take x_1 and x_2,

At (10, 0, 0), $z = 100$

At $\left(0, 0, \frac{50}{3}\right)$, $z = \frac{250}{3}$

$\therefore$ Min $z = \frac{250}{3}$

176. (D) 177. (C) 178. (B)

179. (B) A and B are normal subgroup of G

and $O(A) = 2$
$O(B) = 5$

Since 2 and 5 are relatively prime so

$$O(A \cap B) = 1$$

and $O(AB) = \frac{O(A).O(B)}{O(A \cap B)} = 10$

Now, A and B are norml subgroups

So, AB is normal subgroup in G.

So, $O\left(\frac{G}{AB}\right) = O\left(\frac{G}{AB}\right)$

$$= \frac{30}{10}$$
$$= 3$$

180. (D) 181. (B) 182. (C) 183. (D) 184. (C)

185. (C) 186. (A) 187. (A)

188. (D) None or the answer is correct.

The region of intersection between the two linen is bounded, since the solution occurs only at the vertices of the intersecting region.

We conclude that it has an infinite number of solution.

189. (B) 190. (C) 191. (B) 192. (A) 193. (A)

194. (A) 195. (A) 196. (C) 197. (B) 198. (A)

199. (C)

200. (B)

$$\phi(x) = f(x) + \lambda \int_a^1 \mathrm{I}(x, s, \lambda) f(s)\, ds$$

$$y(x) = x + \int_0^1 x t^2 y(t)\, dt$$

$$y(x) = \phi(x),\ t = s^2,$$

$$k_1(x, s) = x s^2$$

$$k_2(x, s) = \int_0^1 k(x, t)\, k(x, t)\, k_1(t, s)\, dt$$

●●

LEVEL OF DIFFICULTY–3

OBJECTIVE TYPE QUESTIONS

1. The function $f(x) = e^x$ has the Fourier expansion $e^x = \sum_1^{\infty} b_n \sin(nx)$ in the interval $(0, \pi)$. Then $\sum_1^{\infty} (b_n)^2$ converges to—
   (A) $\frac{1}{\pi}(e^{\pi} - 1)$ (B) $\frac{1}{\pi}(e^{\pi} + 1)$
   (C) $\frac{1}{\pi}(e^{2\pi} - 1)$ (D) $\frac{1}{\pi}(e^{2\pi} + 1)$

2. Let W be an m-dimensional subspace of an n-dimensional vector space V, where $m < n$. Then the dimension of V/W is—
   (A) $\frac{n}{m}$ (B) $n - m$
   (C) $n + m$ (D) 0

3. Each of the following subset Φ, R, (0, 1), [0, 1], of R, with the usual metric, is—
   (A) Complete (B) Compact
   (C) Connected (D) Bounded

4. The value of the Wronskian of the functions x^2, $3x + 2$ and $2x + 3$ is—
   (A) 0 (B) – 10
   (C) – 5 (D) 8

5. Let Z_{10} denote the ring of integers modulo 10. Then the number of ideals in Z_{10} is—
   (A) 2 (B) 3
   (C) 4 (D) 5

6. A particle is placed on the top of a sphere in a gravitational field and allowed to slide without friction. Then the motion has—
   (A) No constraint
   (B) A holonomic constraint
   (C) A non-holonomic constraint
   (D) A rheonomic constraint

7. The order of a differential equation is the order of …… derivative that appears in the differential equation.
   (A) Highest (B) Lowest
   (C) Any (D) None of these

8. The degree of a differential equation is the power of the ……… derivative term.
   (A) Highest
   (B) Lowest
   (C) Any
   (D) None of these

9. The differential equation whose linearly independent solutions are cos $2x$, sin $2x$ and e^{-x} is—
   (A) $(D^3 + D^2 + 4D + 4)\, y = 0$
   (B) $(D^3 - D^2 + 4D - 4)\, y = 0$
   (C) $(D^3 + D^2 - 4D - 4)\, y = 0$
   (D) $(D^3 - D^2 - 4D + 4)\, y = 0$
   where $D = \frac{d}{dx}$

10. Let $(z, +)$ denote the group of all integers under addition. Then the number of all automorphisms of $(z, +)$ is—
    (A) 1 (B) 2
    (C) 3 (D) 4

11. Let G be a finite group of order 200. Then the number of subgroups of G of order 25 is—
    (A) 1 (B) 4
    (C) 5 (D) 10

12. If p is prime, and Z_{p^4} denote the ring of integers of modulo p^4, then the number of maximal ideals in Z_{p^4} is—
    (A) 4 (B) 2
    (C) 3 (D) 1

13. All norms on a normed vector space X are equivalent provided—
    (A) X is reflexive
    (B) X is complete
    (C) X is finite dimensional
    (D) X is an inner product space

14. The space lp is a Hilbert space if and only if—
    (A) $p > 1$
    (B) p = even
    (C) $p = \infty$
    (D) $p = 2$

15. The least squares approximation of first degree to the function $f(x) = -\sin x$ over the interval $-\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is—

(A) $\frac{24x}{\pi^3}$ (B) $\frac{24x}{\pi^2}$

(C) $\frac{24x}{\pi}$ (D) $24x$

16. The order of the numerical differentiation formula $f''(x_0) = \frac{1}{12h^2}\{-\{f(x_0-2h)+f(x_0+2h)\} + 16\{f(x_0-h)+f(x_0+h)\} - 30f(x_0)]$ is—

(A) 2 (B) 3

(C) 4 (D) 1

17. The general integral of the partial differential equation $(y + zx)\, z_x - (x + yz)\, z_y = x^2 - y^2$ is—

(A) $F(x^2 + y^2 + z^2, xy + z) = 0$

(B) $F(x^2 + y^2 - z^2, xy + z) = 0$

(C) $F(x^2 - y^2 - z^2, xy + z) = 0$

(D) $F(x^2 + y^2 + z^2, xy - z) = 0$

where F is an arbitrary function.

18. A metric space is always—

(A) First countable

(B) Second countable

(C) Lindelof

(D) Separable

19. Let x be the indiscrete space and y a To space. If $f : x \to y$ is continuous, then—

(A) x must be a one-point space

(B) y must be a discrete

(C) f must be a constant

(D) y must be a one-point space

20. Let (x, y) be the co-ordinates of a point chosen at random inside the disc $x^2 + y^2 \leq r^2$ where $r > 0$. The probability that $y > mx$ is—

(A) $\frac{1}{2^r}$ (B) $\frac{1}{2}$

(C) $\frac{1}{2^m}$ (D) $\frac{1}{2^{m+r}}$

21. Let Z* denote the optimal value of LPP

max $Z = 4x_1 + 6x_2 + 2x_3$

Such that

$3x_1 + 2x_2 + x_3 = 12$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

Then—

(A) $10 \leq Z^* \leq 20$

(B) $20 < Z^* \leq 30$

(C) $30 < Z^* \leq 40$

(D) $Z^* \geq 40$

22. The topology on the real line R generated by left-open right closed intervals $(a, b]$ is—

(A) Strictly coarser than the usual topology

(B) Strictly finer than the usual topology

(C) Not comparable with the usual topology

(D) Same as the usual topology

23. Let x, y be topological spaces and $f : x \to y$ be a continuous and bijective map. Then f is a homomorphism, if—

(A) x and y are compact

(B) x is a Hausdorff and y is compact

(C) x is compact and y is a Hausdorff

(D) x and y are Hausdorff

24. If the integrating factor $(x^7 y^2 + 3y)\, dx + (3x^8 y - x)\, dy = 0$ is $x^m y^n$, then—

(A) $m = -7, n = 1$

(B) $m = 1, n = -7$

(C) $m = n = 0$

(D) $m = n = 1$

25. The initial value problem $(x^2 - x)\frac{dy}{dx} = (2x - 1)\, y$, $y(x_0) = y_0$ has a unique solution if (x_0, y_0) equals—

(A) (2, 1) (B) (1, 1)

(C) (0, 0) (D) (0, 1)

26. If u is harmonic on $\{(x, y) \mid x^2 + y^2 \leq 1|\}$, then $\int_0^{2\pi} \frac{\partial u}{\partial n}\, d\theta$ equals—

(where $\frac{\partial u}{\partial n}$ is the normal derivative of u on the boundary of the unit disc)

(A) 2π (B) 1

(C) π (D) 0

27. Let u be a solution of the initial value problem $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0;\ u(x, 0) = x^2,\ \frac{\partial u}{\partial t}(x, 0) = 0$

Then $u(0, 1)$ equals—

(A) 1 (B) 0

(C) 2 (D) $\frac{1}{2}$

28. The fourth divided difference of the polynomial $3x^3 + 11x^2 + 5x + 11$ over the points $x = 0, 1, 4, 6$ and 7 is—

(A) 18 (B) 11

(C) 0 (D) 3

29. The polynomial of least degree interpolating the data (0, 4), (1, 5), (2, 8), (3, 13) is—

(A) 4 (B) 3

(C) 2 (D) 1

30. For the matrix $\begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix}$ the bound for the eigenvalues predicted by Gershgorin's theorem is—

(A) 3 (B) 1

(C) 2 (D) 4

31. An extremal of the functional

$I[y(x)] = \int_a^b F\left(x,\ y,\ \frac{dy}{dx}\right) dx : y(a) = y_1, y(b) = y_2$ satisfies Euler's equation, which in general—

(A) Is a second order linear ordinary differential equation (ODE)

(B) Is a non linear ODE of order greater than two

(C) Admits a unique solution satisfying the conditions $y(a) = y_1$, $y(b) = y_2$

(D) May not admit a solution satisfying the conditions $y(a) = y_1$, $y(b) = y_2$

32. A lot of 1000 screws contains 1% with major defect and ·5% with minor defects. If 50 screws are picked at random and inspected, then the orderd pair (expected number of major defectives, expected number of minor defectives) is—

(A) (1, 5)

(B) (2·5, 0·5)

(C) (0·5, 2·5)

(D) (5, 1)

33. The sample correlation of the transformed random variables $aX + b$ and $cY + d$ is same as that of x and y provided—

(A) $ac < 0; b, d \in (0, \infty)$

(B) $ac < 0; b, d \in (-\infty, 0)$

(C) $ac > 0; b, d \in R$

(D) $ac < 0; b, d \in R$

34. The system $Ax \leq 0$, where A is an $n \times n$ matrix—

(A) May not have non zero solution

(B) Always has a non zero solution

(C) Always has atleast two linearly independent solutions

(D) Always has atleast n linearly independent solutions

35. The minimal polynomial of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ is—

(A) $(x-1)^2 (x-2)$

(B) $(x-1)(x-2)^2$

(C) $(x-1)(x-2)$

(D) $(x-1)^2 (x-2)^2$

36. Let r be the curve : $r = 2 + 4 \cos \theta$, $(0 \leq \theta \leq 2\pi)$. If $I_1 = \int_r \frac{dz}{z-1}$ and $I_2 = \int_r \frac{dz}{z-3}$, then—

(A) $I_1 = 2I_2$ (B) $I_1 = I_2$

(C) $2I_1 = I_2$ (D) $I_1 = 0, I_2 \neq 0$

37. An example of a function with a non-isolated essential singularity at $z = 2$ is—

(A) $\tan \frac{1}{z-2}$ (B) $\sin \frac{1}{z-2}$

(C) $e^{-(z-2)}$ (D) $\tan \frac{z-2}{z}$

38. Let $f(z) = u(x, y) + iv(x, y)$ be an entire function having Taylor's series expansion as $\sum_{n=0}^{\infty} a_n z^n$. If $f(x) = u(x, 0)$ and $f(iy) = iv(0, y)$, then—

(A) $a_{2n} = 0$ for all n

(B) $a_0 = a_1 = a_2 = a_3 = 0, a_4 \neq 0$

(C) $a_{2n+1} = 0$ for all n

(D) $a_0 \neq 0$ but $a_2 = 0$

39. Let $I = \int_C \frac{\cot(\pi z)}{(z-i)^2} dz$, where C is the contour $4x^2 + y^2 = 2$ (contour clock-wise). Then I is equal to—
(A) 0
(B) $-2\pi i$
(C) $2\pi i \left(\frac{\pi}{\sinh^2\pi} - \frac{1}{\pi}\right)$
(D) $-\frac{2\pi^2 i}{\sinh^2\pi}$

40. Let $X = \{x \text{ in } Q \mid 0 < x < 1\}$ be the metric square with standard metric from R. The completion of x is—
(A) $\{x \text{ in } Q \mid 0 < x < 1\}$
(B) $\{x \text{ in } R \mid 0 < x < 1\}$
(C) $\{x \text{ in } Q \mid 0 \le x \le 1\}$
(D) $\{x \text{ in } R \mid 0 \le x \le 1\}$

41. The function $f(x, y) = (e^x \cos y, e^x \sin y)$ from R^2 to R^2 is—
(A) One-one on all of R^2
(B) One-one on some neighbourhood of any point in R^2
(C) An onto map
(D) Such that some neighbourhood of any point surjects on to R^2

42. In the interval [–1, 1], the series
$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^2 + n^2}{n^3} \text{ is—}$$
(A) Uniformly and absolutely convergent
(B) Absolutely convergent but not uniformly convergent
(C) Neither uniformly nor absolutely convergent
(D) Uniformly convergent but not absolutely convergent

43. The maximum magnitude of the directional derivative for the surface V at the point (1, 2, 3) is along the direction $x^2 + xy + yz = 9$—
(A) $\hat{i} + \hat{j} + \hat{k}$
(B) $2\hat{i} + 2\hat{j} + \hat{k}$
(C) $\hat{i} + 2\hat{j} + 3\hat{k}$
(D) $\hat{i} - 2\hat{j} + 3\hat{k}$

44. Let $B = \{(x, y, z) \mid x, y, z \in R \text{ and } x^2 + y^2 + z^2 \le 4\}$. Let $v(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$ be a vector-valued function defined on B. If $r^2 = x^2 + y^2 + z^2$, the value of the integral $\iiint_B \nabla . (r^2 v(x, y, z))\, dV$ is—
(A) 16π (B) 32π
(C) 64π (D) 128π

45. All real solutions of the differential equation $y'' + 2ay' + by = \cos x$ (where a and b are real constants) are periodic if—
(A) $a = 1$ and $b = 0$
(B) $a =$ and $b = 1$
(C) $a = 1$ and $b \ne 0$
(D) $a = 0$ and $b \ne 1$

46. Let $y = \psi(x)$ be a bounded solution of the equation : $(1 - x^2)y'' - 2xy' + 30y = 0$. Then—
(A) $\int_{-1}^{1} x^3\psi(x)\, dx \ne 0$
(B) $\int_{-1}^{1} (1 + x^3 + x^4)\psi(x)\, dx \ne 0$
(C) $\int_{-1}^{1} x^5\psi(x)\, dx =)0$
(D) $\int_{-1}^{1} x^{2n}\psi(x)\, dx = 0$ for all $n \in N$

47. $\int x^3 J_0(x)\, dx$ is equal to (upto a constant)—
(A) $xJ_0(x) - x^3J_1(x)$
(B) $x^2J_0(x) + J_1(x)$
(C) $x^3J_1(x) - 2x^2J_2(x)$
(D) $2x^2y_1(x) + xJ_2(x)$

48. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions $xy'' + y' + x^2y = 0$, in the neighbourhood of $x = 0$. If $y_1(x)$ is a power series around $x = 0$, then—
(A) $y_2(x)$ is bounded around $x = 0$
(B) $y_2(x)$ is unbounded around $x = 0$
(C) $y_2(x)$ has power series solutions
(D) $y_2(x)$ has solution of the form $\sum_{n=1}^{\infty} b_n x^{n+r}$, where $r \ne 0$, and $b_0 \ne 0$

49. Any subgroup of Q (the group of rational numbers under addition) is—
(A) Cyclic and finitely generated but not Abelian and normal
(B) Cyclic and Abelian but not finitely generated and normal

(C) Abelian and normal but not cyclic and finitely generated

(D) Finitely generated and normal but not cyclic and Abelian

50. Upto isomorphism, the number of Abelian groups of order 10^5 is—

(A) 2 (B) 5

(C) 7 (D) 49

51. Let r be a simple closed curve in the complex. Then the set of all possible values of $\oint_r \frac{dz}{z(1-z^2)}$ is—

(A) $\{0, \pm \pi i\}$

(B) $\{0, \pi i, 2\pi i\}$

(C) $\{0, \pm \pi i, \pm 2\pi i\}$

(D) $\{0\}$

52. The principle value of the improper integral $\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx$ is—

(A) $\frac{\pi}{e}$ (B) πe

(C) $\pi + e$ (D) $\pi - e$

53. The number of roots of the equation $z^5 - 12z^2 + 14 = 0$ that lie in the region $\left\{ z \in C : 2 \le |z| < \frac{5}{2} \right\}$ is—

(A) 2 (B) 3

(C) 4 (D) 5

54. Let $f : (0, 2) \to R$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational} \\ 2x - 1, & \text{if } x \text{ is irrational} \end{cases}$$

then—

(A) f is differentiable exactly at one point

(B) f is differentiable exactly at two points

(C) f is not differentiable at any point in $(0, 2)$

(D) f is differentiable at every point in $(0, 2)$

55. Let $f : R^2 \to R$ be defined by

$$f(x, y) = \begin{cases} x^2 + y^2, & \text{if } x \text{ and } y \text{ are rational} \\ 0, & \text{otherwise} \end{cases}$$

Then—

(A) f is not continuous at $(0, 0)$

(B) f is continuous at $(0, 0)$ but not differentiable at $(0, 0)$

(C) f is differentiable only at $(0, 0)$

(D) f is differentiable everywhere

56. Let $f, g : R^2 \to R$ be defined by $f(x, y) = x^4 + y^2 : g(x, y) = x^4 + y^2 - 10x^2y$. Then at $(0, 0)$—

(A) f has a local minimum but not g

(B) g has a local minimum but not f

(C) Both f and g have a local minimum

(D) Neither f nor g has a local minimum

57. Suppose c_1 is the boundary of $\{(x, y) \in R^2 : 0 \le x \le 1, 0 \le y \le 1\}$ and c_2 is the boundary of $\{(x, y) \in R^2 : -1 \le x \le 0, -1 \le y \le 0\}$. Let

$$\alpha_i = \int_{c_i} xy^2 dx + (x^2 y + 2x)\, dy, \; i = 1, 2$$

be evaluated in the counterclockwise direction. Then—

(A) $\alpha_1 = 1, \alpha_2 = -1$

(B) $\alpha_1 = \alpha_2 = 1$

(C) $\alpha_1 = 2, \alpha_2 = -2$

(D) $\alpha_1 = \alpha_2 = 2$

58. Let $f : R \to R$ be a non-zero function such that $|f(x)| \le \frac{1}{1+2x^2}$ for all $x \in R$. Define real valued functions f_n on R for all $n \in N$ by $f_n(x) = f(x + n)$. Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly—

(A) On $[0, 1]$ but not on $[-1, 0]$

(B) On $[-1, 0]$ but not on $[0, 1]$

(C) On both $[-1, 0]$ and $[0, 1]$

(D) Neither on $[-1, 0]$ nor on $[0, 1]$

59. Let $\in$ be a non-measurable subset of $(0, 1)$. Define two functions f_1 and f_2 on $(0, 1)$ as follows :

$$f_1(x) = \begin{cases} 1/x & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \Big/ \text{ and}$$

$$f_2(x) = \begin{cases} 0 & \text{if } x \in E \\ 1/x & \text{if } x \notin E \end{cases} \text{. Then—}$$

(A) f_1 is measurable but not f_2

(B) f_2 is measurable but not f_2

(C) Both f_1 and f_2 are measurable

(D) Neither f_1 nor f_2 are measurable

60. Consider the following improper integral

$$I_1 = \int_1^{\infty} \frac{dx}{(1+x^2)^{1/2}} \quad \text{and} \quad I_2 = \int_1^{\infty} \frac{dx}{(1+x^2)^{3/2}}$$

then—

(A) I_1 converges but not I_2

(B) I_2 converges but not I_1

(C) Both I_1 and I_2 converge

(D) Neither I_1 nor I_2 converges

61. Let $P_n(x)$ denote the Legendre polynomial of degree n. If

$$f(x) = \begin{cases} x, & -1 \le x \le 0 \\ 0, & 0 \le x \le 1 \end{cases}$$

and $f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \dots$ then—

(A) $a_0 = \frac{1}{4}, a_1 = \frac{1}{2}$

(B) $a_0 = \frac{-1}{4}, a_1 = \frac{1}{2}$

(C) $a_0 = \frac{1}{2}, a_1 = \frac{-1}{4}$

(D) $a_0 = \frac{1}{2}, a_1 = \frac{1}{4}$

62. If $J_n(x)$ and $Y_n(x)$ denote Bessel functions of order n of the first and the second kind, then the general solution of the differential equation $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + xy = 0$ is given by—

(A) $y(x) = \alpha x J_1(x) + \beta x Y_1(x)$

(B) $y(x) = \alpha J_1(x) + \beta Y_1(x)$

(C) $y(x) = \alpha J_0(x) + \beta Y_0(x)$

(D) $y(x) = \alpha x J_0(x) + \beta x Y_0(x)$

63. The general solution of the system of differential equations

$$y + \frac{dz}{dx} = 0$$

$$\frac{dy}{dx} - z = 0$$

is given by—

(A) $y = \alpha e^x + \beta e^{-x}$
$z = \alpha e^x - \beta e^{-x}$

(B) $y = \alpha \cos x + \beta \sin x$
$z = \alpha \sin x - \beta \cos x$

(C) $y = \alpha \sin x - \beta \cos x$
$z = \alpha \cos x + \beta \sin x$

(D) $y = \alpha e^x - \beta e^{-x}$
$z = \alpha e^x + \beta e^{-x}$

64. It is required to find the solution of the differential equation $2x(2+x)\frac{d^2y}{dx^2} + 2(3+x)\frac{dy}{dx} - xy = 0$ around the point $x = 0$. The roots of the indicial equation are—

(A) $0, \frac{1}{2}$

(B) $0, 2$

(C) $\frac{1}{2}, \frac{1}{2}$

(D) $0, -\frac{1}{2}$

65. S : Every non-Abelian group has a nontrivial Abelian sub-group T : Every nontrivial Abelian group has a cyclic sub-group. Then—

(A) Both S and T are false

(B) S is true and T is false

(C) T is true and S is false

(D) Both T and S are true

66. Let S_{10} denote the group of permutations on ten symbols $\{1, 2, \dots, 10\}$. The number of elements of S_{10} commuting with the element $\sigma = (1, 3, 5, 7, 9)$ is—

(A) 5!

(B) 5·5!

(C) 5! 5!

(D) $\frac{10!}{5!}$

67. Match the following in an integral domain

U. The only nilopotent element (S) $a \cdot 0$

V. The only idempotent elements (S) $b \cdot 1$

W. The only unit and idempotent element (S) $c \cdot 0, 1$

(A) $U \to a : V \to b : W \to c$

(B) $U \to b : V \to c : W \to a$

(C) $U \to c : V \to a : W \to b$

(D) $U \to a : V \to c : W \to b$

68. Let z be the ring of integers under the usual addition and multiplication. Then every nontrivial ring homomorphism $f : z \to z$ is—
(A) Both injective and surjective
(B) Injective but not surjective
(C) Surjective but not injective
(D) Neither injective nor surjective

69. Let $X = (C[0, 1], \|\cdot\|_P)$, $1 \le p \le \infty$ and
$$f_n(t) \begin{cases} n(1 - nt), & \text{if } 0 \le t \le 1/n \\ 0, & \text{if } 1/n < t \le 1 \end{cases}$$
if $S = \{f_n \in X ; n > 1\}$, then S is—
(A) Bounded if $p = 1$
(B) Bounded if $p = 2$
(C) Bounded if $p = \infty$
(D) Unbounded for all p

70. Suppose the iterates x_n generated by $x_{n+1} = x_n - \dfrac{2f(x_n)}{f'(x_n)}$ where f' denotes the derivative of f, converges to a double zero $x = a$ of $f(x)$. Then the convergence has order—
(A) 1 (B) 2
(C) 3 (D) 1·6

71. Suppose the matrix $M = \begin{bmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$ has a unique Cholesky decomposition of the form $M = LL^T$, where L is a lower triangular matrix. The range of values of α is—
(A) $-2 < \alpha < 2$
(B) $\alpha > 2$
(C) $-2 < \alpha < 3/2$
(D) $3/2 < \alpha < 2$

72. The Runge-Kutta method of order four is used to solve the differential equation $\dfrac{dy}{dx} = f(x), y(0) = 0$ with step size h. The solution of $x = h$ is given by—
(A) $y(h) = \dfrac{h}{6}\left[f(0) + 4f\left(\dfrac{h}{2}\right) + f(h)\right]$
(B) $y(h) = \dfrac{h}{6}\left[f(0) + 2f\left(\dfrac{h}{2}\right) + f(h)\right]$
(C) $y(h) = \dfrac{h}{6}[f(0) + f(h)]$
(D) $y(h) = \dfrac{h}{6}\left[2f(0) + f\left(\dfrac{h}{2}\right) + 2f(h)\right]$

73. The values of the constants, α, β, x_1 for which the quadrature formula $\int_0^1 f(x)\,dx = \alpha f(0) + \beta f(x_1)$ is exact for polynomials of degree as high as possible, are—
(A) $\alpha = \dfrac{2}{3}, \beta = \dfrac{1}{4}, x_1 = \dfrac{3}{4}$
(B) $\alpha = \dfrac{3}{4}, \beta = \dfrac{1}{4}, x_1 = \dfrac{2}{3}$
(C) $\alpha = \dfrac{1}{4}, \beta = \dfrac{3}{4}, x_1 = \dfrac{2}{3}$
(D) $\alpha = \dfrac{2}{3}, \beta = \dfrac{3}{4}, x_1 = \dfrac{1}{4}$

74. The partial differential equation
$$x\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial x} = 0$$
is—
(A) Elliptic in the region $x < 0, y < 0, xy > 1$
(B) Elliptic in the region $x > 0, y > 0, xy > 1$
(C) Parabolic in the region $x < 0, y < 0, xy > 1$
(D) Hyperbolic in the region $x < 0, y < 0, xy > 1$

75. A function $u(x, t)$ satisfies the wave equation
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \; 0 < x < 1, t > 0$$
If $u\left(\dfrac{1}{2}, 0\right) = \dfrac{1}{4}$, $u\left(1, \dfrac{1}{2}\right) = 1$ and $u\left(0, \dfrac{1}{2}\right) = \dfrac{1}{2}$, then $u\left(\dfrac{1}{2}, 1\right)$ is—
(A) $\dfrac{7}{4}$ (B) $\dfrac{5}{4}$
(C) $\dfrac{4}{5}$ (D) $\dfrac{4}{7}$

76. The kernel of the laplace transform is given by—
(A) e^{-st} (B) $\dfrac{e^{ist}}{\sqrt{2\pi}}$
(C) $\sqrt{\dfrac{2}{\pi}}\sin st$ (D) $\sqrt{\dfrac{2}{\pi}}\cos st$

77. The kernel of the fourier transform is given by—

(A) e^{-st} (B) $\frac{e^{ist}}{\sqrt{2\pi}}$

(C) $\sqrt{\frac{2}{\pi}}\sin st$ (D) $\sqrt{\frac{2}{\pi}}\cos st$

78. Lower and upper limits for Laplace Transform is given by—

(A) 0 and ∞ (B) $-\infty$ and ∞

(C) 0 and $-\infty$ (D) None of these

79. Lower and upper limits for Fourier Transform is given by—

(A) 0 and ∞ (B) $-\infty$ and ∞

(C) 0 and $-\infty$ (D) None of these

80. The Laplace Transform of te^{at} is—

(A) $\frac{1}{s-a}$

(B) $\frac{1}{(s-a)^2}$

(C) $\frac{1}{(s-a)^n}(n = 1, 2...)$

(D) None of these

81. The function, who's Laplace Transform is $\frac{1}{s^2+\omega^2}$ is—

(A) $\frac{1}{\omega}\sin \omega t$ (B) $\sin \omega t$

(C) $\omega \sin \omega t$ (D) None of these

82. Every …… image of a group G is isomorphic to some quotient group of G.

(A) Homomorphic (B) Automorphic

(C) Isomorphic (D) None of these

83. Every …… image of an abelian group is abelian but converse is not true.

(A) Homomorphic (B) Automorphic

(C) Isomorphic (D) None of these

84. If f is ……… mapping of a group G into a group G′ and f(G) is homomorphic image of G in G′, then f(G) is a subgroup of G′.

(A) Homomorphic

(B) Automorphic

(C) Isomorphic

(D) None of these

85. The necessary and sufficient condition for a ……… f of a group G into G′ with kernel K to be isomorphism is that K = $\{e\}$.

(A) Homomorphic

(B) Automorphic

(C) Isomorphic

(D) None of these

86. A …… from a simple group is either trivial or one-to-one.

(A) Homomorphism

(B) Automorphism

(C) Isomorphism

(D) None of these

87. The product of even and odd funtion is—

(A) Odd

(B) Even

(C) Constant

(D) None of these

88. The graph of even function $y = f(x)$ is symmetric about—

(A) x-axis

(B) y-axis

(C) Both (A) and (B)

(D) None of these

89. The Fourier series of an even function of period 2L is a—

(A) Fourier cosine series

(B) Fourier sine series

(C) Fourier complex series

(D) None of these

90. The Fourier series of an odd function of period 2L is a—

(A) Fourier cosine series

(B) Fourier sine series

(C) Fourier complex series

(D) None of these

91. The following represents the cartesian coordinate—

(A) (x, y, z)

(B) (r, θ, ϕ)

(C) (r, θ, z)

(D) None of these

92. The following represents the cylindrical coordinates—
    (A) (x, y, z)
    (B) (r, θ, ϕ)
    (C) (r, θ, z)
    (D) None of these

93. The following represents the spherical coordinates—
    (A) (x, y, z)
    (B) (r, θ, ϕ)
    (C) (r, θ, z)
    (D) None of these

94. Generalized coordinates—
    (A) depend on each other
    (B) are independent of each other
    (C) are spherical coordinates
    (D) None of these

95. A rigid body moving freely in space has degrees of freedom—
    (A) Three (B) Six
    (C) Nine (D) Twelve

96. The general displacement of a rigid body is a translation with a rotation—
    (A) Euler's theorem
    (B) Chasle's theorem
    (C) Low of inertia
    (D) Low of force

97. The following $I_{yy} = \sum_{i=1}^{n} m_i \, (z_i^2 + y_i^2)$ represents—
    (A) Moment of inertia about x-axis
    (B) Moment of inertia about y-axis
    (C) Moment of inertia about z-axis
    (D) None of these

98. There exist zero-divisors in—
    (A) The ring of integers of modulo a prime p
    (B) The ring of real matrices of order p
    (C) The ring of polynomials and a field of characteristics
    (D) The ring of entire functions

99. Let G be a group such that $a^2 = e$ for each $a \in$ G, where e is the identity element of G. Then—
    (A) G is cyclic
    (B) G is finite
    (C) G is Abelian
    (D) G has a subgroup which is not normal

100. The following matrix admits a Cholesky (also called LL*) decomposition—
    (A) $\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 2i \\ -2i & 1 \end{bmatrix}$
    (C) $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

101. The value of the line integral $\oint_C \frac{-ydx + xdy}{x^2 + y^2}$, where C is the unit circle centre at 0, equals—
    (A) 2π (B) -2π
    (C) 0 (D) None of these

102. If $(f(x, y, z))$ is a harmonic function in a domain D containing the region T : $x^2 + y^2 + z^2 \le 100$, then the surface integral $\int_S \left(x\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \right) dS$, where S : $x^2 + y^2 + z^2 = 100$, equals—
    (A) 0 (B) 1
    (C) 10 (D) 100

103. Let R be the image of the triangular region S with vertices (0, 0), (1, 0) and (0, 1) is uv-plane under the transformation $x = 2u - 3v$, $y = u + v$, then $\iint_R x \, dA$ equals—
    (A) $\iint_S (2u - 3v) \, dS$
    (B) $3 \iint_S (2u - 3v)(u + v) \, dS$
    (C) $-\iint_S (2u + 3v) \, dS$
    (D) $5 \iint_S (2u - 3v) \, dS$

104. If $y_1(x)$ and $y_2(x)$ are solutions of $y'' + xy' + (1 - x^2) y = y \sin x$, then which of the following is also its solution ?
    (A) $y_1(x) + y_2(x)$
    (B) $y_1(x) - y_2(x)$

(C) $2y_1(x) - y_2(x)$

(D) $y_1(x) - 2y_2(x)$

105. For the differential equation $xy' - y = 0$ which of the following function is not an integrating factor ?

(A) $\frac{1}{x^2}$ (B) $\frac{1}{y^2}$

(C) $\frac{1}{xy}$ (D) $\frac{1}{x+y}$

106. The partial differential equation $y^3 u_{xx} - (x^3 - 1) u_{yy} = 0$, is—

(A) Parabolic in $\{(x, y) : x < 0\}$

(B) Hyperbolic in $\{(x, y) : y > 0\}$

(C) Elliptic in $\Re^2$

(D) Parabolic in $\{(x, y) : x > 0\}$

107. The partial differential equation of the family of surfaces $z = (x + y) + A(xy)$ is—

(A) $xp - yq = 0$

(B) $xp - yq = x - y$

(C) $xp + yq = x + y$

(D) $xp + yq = 0$

108. Which of the following statements is the strongest ?

(A) X_n converges to X in probability

(B) X_n converges to X in distribution

(C) A subsequence of $\{X_n\}$ converges to X with probability 1

(D) $E(X_n - X^2) \to 0$ as $n \to \infty$.

109. $P(A/B) + P(A/B^C)$ is—

(A) P (A)

(B) 1

(C) Greater than P (A)

(D) None of these

110. Let X be a set for which a topology T is defined, then—

(A) Only X is in T

(B) Only X is not in T

(C) ϕ and X are in T

(D) None of these

111. Let X be a set for which a topology T is defined, then—

(A) The union of the elements of finite subcollection of T is in T

(B) The union of the elements of any subcollection of T is in T

(C) The union of the elements of any subcollection of T is not in T

(D) None of these

112. Let X be a set for which a topology T is defined, then—

(A) The intersection of the elements of any finite subcollection of T is in T

(B) The intersection of the elements of any sub collection of T is in T

(C) The intersection of the elements of any finite subcollection of T is not in T

(D) None of these

113. Let (X, T) is a topological space, $U \subset X$ is open set—

(A) U is a subset of T

(B) U is an element of T

(C) U is a super set of T

(D) None of these

114. If B is the collection of all intervals in the real line $(a, b) = \{x \mid a < x < b\}$, the topology generated by B is called the—

(A) Standard topology on the real line

(B) Discrete topology on real line

(C) Hausdorff topology on real line

(D) None of these

115. Let (X, T) is a given topological space $A \subset X$. Then interior of A—

(A) is the union of all open sets contained in A

(B) is the intersection of all open sets contained in A

(C) is the union of all closed sets contained in A

(D) None of these

116. Neighborhood of x is—

(A) An open set U containing x

(B) An elosed set U containing x

(C) An Null set

(D) None of these

117. Given (X, T) a topological space and $A \subset X$, a point $x \in x$ is a limit point of A—
   (A) If every neighborhood of x intersects A in some point other than x
   (B) If every neighborhood of x intersects A in all point other than x
   (C) If every neighborhood of x intersects A in some point including x
   (D) None of these

118. If X is any set, T is a collection of all subsets of X then topology (X, T) is—
   (A) A discrete topology
   (B) Indiscrete topology
   (C) Trivial topology
   (D) None of these

119. If X is any set, $T = \{\phi, X\}$, then topology (X, T)—
   (A) A discrete topology
   (B) Indiscrete topology
   (C) Trivial topology
   (D) Both (A) and (B)

120. If (X, T) is a topological space, then—
   (A) ϕ is open
   (B) ϕ is closed
   (C) ϕ is discrete
   (D) None of these

121. If (X, T) is a topological space, then—
   (A) X is open
   (B) X is closed
   (C) X discrete
   (D) None of these

122. If (X, T) is a topological space, then—
   (A) Arbitrary intersections of closed sets are open
   (B) Arbitrary intersections of closed sets are closed.
   (C) Arbitrary intersections of open sets are closed
   (D) None of these

123. If (X, T) is a topological space, then—
   (A) Finite unions of closed sets are closed
   (B) Finite unions of closed sets are open
   (C) Finite unions of open sets are closed
   (D) None of these

124. If Y is a subspace of X, then set A is closed in Y iff—
   (A) It is equal to the intersection of a open set of X with Y
   (B) It is equal to the union of a closed set of X and Y
   (C) It is equal to the intersection of a closed set of X with Y
   (D) None of these

125. If Y is a subspace of X. If A is closed in Y and Y is closed in X—
   (A) Then A is semi closed in X
   (B) Then A is open in X
   (C) Then A is closed in X
   (D) None of these

126. If Y is a subspace of X, $A \subset Y$ and $\bar{A}$ is a closure of A in X then closure of A in Y—
   (A) is equal to $A \cap Y$
   (B) is equal to $A \cup Y$
   (C) is equal to Y
   (D) None of these

127. If A is a subset of topological space, X. Let $X \in \bar{A}$, closure of A in X, iff—
   (A) Every open set U containing x, intersects A
   (B) Every open set U that does not contain x, intersects A
   (C) Every closed set U containing x, intersects A
   (D) None of these

128. If A is a subset of topological space X, suppose B is a basis for X, then $x \in \bar{A}$, iff—
   (A) Every basis element $B \in B$, containing x does not intersects A
   (B) Every basis element $B \in B$, containing x intersects A
   (C) Every basis element $B \in B$, containing x is disjoint to A
   (D) None of these

129. Let A be a subset of topological space X and A' be the set of all limit points of A. Then closure of A—
   (A) $\bar{A} = A \cup A'$

(B) $\bar{A} = A - A'$

(C) $\bar{A} = A \cap A'$

(D) None of these

130. A subset of a topological space is closed iff—

(A) If contains none of its limit points

(B) It contains all its limit points

(C) It contains some of its limit points

(D) None of these

131. Let X be a Hausdorff space, $A \subset X$. Then a point x is a limit point of A iff—

(A) Every neighborhood of x contains infinitely many points of A

(B) Every neighborhood of x contains finitely many points of A

(C) Every neighborhood of x contains no points of A

(D) None of these

132. Every simple ordered set is a Hausdorff space in the—

(A) Order topology

(B) Non-ordered topology

(C) Discrete topology

(D) None of these

133. The product of two Hausdorff spaces is a—

(A) Hausdorff space

(B) Discrete space

(C) Closed set

(D) None of these

134. A subspace of a Hausdorff space is a—

(A) Hausdorff space

(B) Discrete space

(C) Closed space

(D) None of these

135. Let X and Y are topological spaces. A function $f : X \to Y$ is a continuous function—

(A) If for each open subset V of Y, the set f^{-1} (V) is anclosed subset of X

(B) If for each closed subset V of Y, the set f^{-1} (V) is an open subset of X

(C) If for each open subset V of Y, the set f^{-1} (V) is an open subset of X

(D) None of these

136. Let X and Y be topological spaces. A map $f : X \to Y$ is open map if—

(A) For every open set U of X, the set $f(U)$ is open in Y

(B) For every closed set U of X, the set $f(U)$ is open in Y

(C) For every open set U of X, the set $f(U)$ is closed in Y

(D) None of these

137. Let X and Y be topological spaces. A map $f : X \to Y$ is closed map if—

(A) For every closed set A of X, the set $f(A)$ is closed in Y

(B) For every closed set U of X, the set $f(U)$ is open in Y

(C) For every open set U of X, the set $f(U)$ is closed in Y

(D) None of these

138. If X is a topological space and A is a set if $P : X \to A$ is surjective map, then there exists exactly one topology T on A relative to which P is a quotient map and this topology is called—

(A) Quotient topology induced by P

(B) Hausdorff topology

(C) Discrete topology

(D) None of these

139. Let X and Y are topological spaces. A function $f : X \to Y$ is continuous at $x \in X$—

(A) If for every neighborhood N of $f(x)$, there is a neighborhood M of x such that $f(N) = M$

(B) If for every neighborhood N of $f(x)$, there is a neighborhood M of x such that $f(N) \subset M$.

(C) If for some neighborhood N of $f(x)$, there is a neighborhood M of x such that $f(N) = M$

(D) None of these

140. Let X and Y be topological spaces and $f : X \to Y$. If f is continuous, then—

(A) For every closed set $B \subset Y$, set f^{-1} (B) is closed in X

(B) For every open set $B \subset Y$, set $f^{-1}(B)$ is closed in X

(C) For every closed set $B \subset Y$, set $f^{-1}(B)$ is open in X

(D) None of these

141. Let X and Y be topological spaces the constant function $f : X \rightarrow Y$, is—
(A) Continuous function
(B) Discontinuous function
(C) Inverse function
(D) None of these

142. If A is a subspace of topological space X, the inclusion function $j : A \rightarrow X$ is continuous—
(A) Continuous function
(B) Discontinuous function
(C) Inverse function
(D) None of these

143. If Y is a subspace of topological space X—
(A) Y is connected if there exist no separation of Y
(B) Y is connected if there exist at least one separation of Y
(C) Y is connected if there exist separation of Y
(D) None of these

144. If sets C and D from a separation of topological space X and Y is connected subset of X, then—
(A) Y lies entirely with in C
(B) Y lies entirely with in D
(C) Y lies entirely with in either C or D
(D) None of these

145. The union of a collection of connected sets that have a point in common is—
(A) Connected (B) Separable
(C) Disconnected (D) None of these

146. Let A be a connected subset of topological space X. If $A \subset B \subset \overline{A}$, then B is—
(A) Connected
(B) Separable
(C) Disconnected
(D) None of these

147. If set B is formed by adjoining to the connected set A of topological space X, some or all its limit points, then B is—
(A) Connected
(B) Separable
(C) Disconnected
(D) None of these

148. Every closed subset of a compact sapce is—
(A) Compact sapce
(B) Open set
(C) Null set
(D) None of these

149. Every compact subset of a Hausdorff space is—
(A) Closed set
(B) Open set
(C) Null set
(D) None of these

150. The image of a compact topological space under continuous map is—
(A) Continuous
(B) Discontinuous
(C) Constant
(D) None of these

Answers with Explanation

1. (C) 2. (B)

3. (C) $\mathbb{R}$ is not compact and not bounded (0, 1) in $\mathbb{R}$ is not complete, hence connected.

4. (B)	5. (A)	6. (B)	7. (A)	8. (A)
9. (A)	10. (B)	11. (A)	12. (B)	13. (D)
14. (D)	15. (A)	16. (A)	17. (B)	18. (A)
19. (C)	20. (C)	21. (B)	22. (D)	23. (C)

24. (A) $(x^7y^2 + 3y)\,dx + (3x^8y - x)\,dy = 0$ ...(1)

$\approx x^{\alpha}y^{\beta}(mydx + nxdy)$

$+ x^{\alpha'}y^{\beta'}(m'ydx + n'xdy) = 0$ ...(2)

$x^7y\,(ydx + 3xdy) + (3ydx - xdy) = 0$ ...(3)

Comparing equation (2) and (3), we get

$\alpha = 7$	$\alpha' = 0$
$\beta = 1$	$\beta' = 0$
$m = 1$	$m' = 3$
$n = 3$	$n' = -1$

$\therefore$ I. F. of first term of (3) is

$$x^{km-1-\alpha}\, y^{kn-1-\beta} = x^{k-8}\, y^{3k-2}$$

I. F. of second terms of (3) is

$$x^{k'm'-1-\alpha'}\, y^{k'n'-1-\beta'} = x^{3k'-1}\, y^{-k'-1}$$

$$\therefore \quad k-8 = 3k'-1$$

and $\quad 3k-2 = -k'-1$

$$k = 3k'+7$$

$$k = \frac{-k'+1}{3}$$

$$3k'+7 = \frac{-k'+1}{3}$$

$$9k'+21 = -k'+1$$

$$10k' = -20$$

$$k' = -2$$

$$k = 1$$

$$\therefore \quad \text{I. F.} = x^{-7}y = x^m y^n$$

$$\Rightarrow \quad m = -7, n = 1$$

25. (B) $\quad (x^2-x)\dfrac{dy}{dx} = (2x-1)\,y$

Put $x^2 - x = t,\ 2x-1 = \dfrac{dt}{dx}$

$$\therefore \quad t\frac{dy}{dx} = y\frac{dt}{dx}$$

$$\Rightarrow \quad ydt = tdy$$

$$\Rightarrow \quad \frac{dt}{t} = \frac{dy}{y}$$

Integrating $\quad \log\dfrac{y}{t} = \log c$

$$\Rightarrow \quad y = ct = c\,(x^2-x)$$

$$y\,(x_0) = c\,(x^2{}_0 - x_0) = y_0$$

$$\Rightarrow \quad c = \frac{y_0}{x^2{}_0 - x_0}$$

$$y = \frac{y_0}{x^2{}_0 - x_0}(x^2-x)$$

It has unique solution if x_0, y_0 equals (1, 1)

26. (D) u is harmonic on $\{(x, y) | x^2 + y^2 \leq 1\}$

We know that $\displaystyle\iint_S \frac{\partial \vec{u}}{\partial n}\, d\vec{S}$

$$= \iint_S \left(\frac{\partial u}{\partial n}\hat{n}\right)\cdot \hat{n}\, dS$$

$$= \iint_S (\Delta u)\cdot \hat{n}\, dS$$

$$= \iiint_v \nabla\cdot(\nabla u)\, dV, \text{by divergenic theorem}$$

$$= \iiint_v \nabla^2 u\, dV$$

$= 0$ since $\nabla^2 u = 0$ in $v \cdot u$ is harmonic

27. (A) $\quad \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = 0$

A. E. is $\quad D^2 - 1 = 0$

$$\Rightarrow \quad D = \pm 1$$

$$\text{C. F.} = \phi_1(x+t) + \phi_2(x-t)$$

It's solution is, $u\,(x, 0) = \phi_1(x) + \phi_2(x) = x^2 \quad \ldots(1)$

$$\frac{\partial u}{\partial t} = \phi'_1(x+t) - \phi'_2(x-t)$$

$$\frac{\partial u}{\partial t} = (x, 0) = \phi'_1(x) - \phi'_2(x) = 0 \quad \ldots(2)$$

$$\Rightarrow \quad \phi'_1(x) = \phi'_2(x)$$

Integrating, we get

$$\phi_1(x) = \phi_2(x) + c \quad \ldots(3)$$

$$\phi_1(x) + \phi_2(x) = x^2$$

Adding $\quad \dfrac{\phi_1(x) - \phi_2(x) = c}{2\phi_1(x) = c + x^2}$

$$\phi_1(x) = \frac{c+x^2}{2}$$

$$\Rightarrow \quad \phi_2(x) = \phi_1(x) - c$$

$$\Rightarrow \quad = \frac{x^2+c}{2} - c$$

$$= \frac{x^2-c}{2}$$

$$u = \phi_1(x+t) + \phi_2(x-t)$$

$$= \frac{(x+t)^2}{2} + \frac{(x-t)^2 - c}{2}$$

$$= \frac{1}{2}[x^2 + 2xt + t^2 + c + x^2 - 2xt + t^2 - c]$$

$$= x^2 + t^2$$

$$\therefore \quad u\,(0, 1) = 0 + 1 = 1$$

28. (D) None of these

$$f(x) = 3x^3 + 11x^2 + 5x + 11$$

Newtons's divided difference table of fourth order—

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	11	$\frac{30-11}{1-0} = 9$			
1	30	$\frac{399-30}{4-1} = 123$	$\frac{123-9}{4-0} = 28{\cdot}5$	$\frac{44-28{\cdot}5}{6-0} = 2{\cdot}58$	
4	399	$\frac{1085-399}{6-4} = 343$	$\frac{343-123}{6-1} = 44$	$\frac{62-44}{7-1} = 3$	$\frac{3-2{\cdot}58}{7} = 0{\cdot}59$
6	1085	$\frac{1614-1085}{7-6} = 529$	$\frac{529-343}{7-4} = 62$		
7	1614				

Fourth divided difference = 0·059

29. (B) 30. (C) 31. (A) 32. (B) 33. (B)

34. (D)

35. (C) The monic polynomial of lowest degree that anihilates a matrix A is called the minimal polynomial of A.

Here

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2 (2-\lambda)^2$$

$\therefore$ Roots of equation $|A - \lambda I| = 0$ are 1, 1, 2, 2

Each characteristic root of A is also a root of minimal polynomial. If $m(x)$ is the minimal polynomial of A, then both $(x-1)$ and $(x-2)$ are factors of $m(x)$.

36. (B) Both singularities are within the curve and are equal. So by Cavely Residue theorem

$$I_1 = I_2$$

37. (A)
$$\tan\frac{1}{z-2} = \frac{\sin\frac{1}{z-2}}{\cos\frac{1}{z-2}}$$

For singularities, $\cos\frac{1}{z-2} = 0$

$$\Rightarrow \quad \frac{1}{z-2} = (2n+1)\frac{\pi}{2}$$

$$z - 2 = \frac{2}{\pi(2n+1)}$$

Hence, $z = 2$ is non-isolated essential singu-`larity.

38. (B)

39. (D)
$$\text{Residue} = \operatorname*{Lt}_{z \to i} \frac{d}{dt} \cot \pi z$$
$$= -\frac{\pi}{\sin h^2 \pi}$$

For Cauchy residue theorem,

$$I = 2\pi i\,(R)$$
$$= 2\pi i \times \frac{-\pi}{\sinh^2 \pi}$$
$$= \frac{-2\pi^2 i}{\sinh^2 \pi}$$

40. (C) 41. (A) 42. (A)

43. (B)
$$f = x^2 + xy + yz - 9 = 0$$
$$\text{grad.} f = 2xi + (yi + xj) + (zj + yk)$$
$$= i(2x + y) + j(x + z) + yk$$

grad. f at point (1, 2, 3) is $4i + 4j + 2k$.

The directional derivative of f is maximum in the direction of grad

$$f = 4i + 4j + 2k$$
$$= 2i + 2j + k$$

44. (D)
$$I = \int \nabla (r^2.r)\, dV \qquad (\because\ V = r)$$
$$= \int 5r^2 dV$$
$$= \int_0^2 \int_0^{\pi} \int_{\phi=0}^{2\pi} 5r^2 \cdot r^2 \sin\theta\, d\theta\, d\phi\, dr$$
$$= 128\pi$$

45. (B) 46. (D)

47. (C) Since $\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$

so $\int x^n J_{n-1}(x)\, dx = x^n J_n(x)$ ...(1)

$$\therefore \quad \int_0^x x^3 J_0(x)\, dx = \int_0^x x^2 [x J_0(x)]\, dx$$
$$= [x^2 \{x J_1(x)\}]_0^x - \int_0^x 2x \{x J_1(x)\}\, dx$$

(Integrating by parts and using (1) for $n = 1$)

$= x^3 J_1(x) - 2\int_0^x x^2 J_1(x)\,dx$

$= x^3 J_1(x) - 2[x^2 J_2(x)]$

48. (C) Where $u = 1 - \frac{x^3}{3^2} + \frac{x^6}{3^4(\lfloor 2)^2} - \frac{x^9}{3^6(\lfloor 3)^2} + \ldots$

and $v = u \log x + 2\left[\frac{x^3}{3^3} - \frac{1}{3^5(2!)^2}\left(1+\frac{1}{2}\right)x^6 + \ldots\right]$

49. (C) 50. (B) 51. (A)

52. (A) $I = \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx$

$= 2\int_0^{\infty} \frac{\cos x}{1+x^2}\,dx$ ...(1)

$\left[\because \frac{\cos x}{1+x^2} \text{ is even}\right]$

Let $I_1 = \int_0^{\infty} \frac{\cos x}{1+x^2}\,dx$

Let suppose $\int_0^{\infty} 2z \exp\{-(a^2+x^2)z^2\}\,dz$

$= \frac{1}{a^2+x^2}$

If $a = 1$, then $\int_0^{\infty} 2z \exp\{-(1+x^2)z^2\}\,dz$

$= \frac{1}{1+x^2}$

Now multiplying both sides by $\cos x$ and integrating from 0 to ∞ with respect to x, we have

$I_1 = \int_0^{\infty} \frac{\cos x}{1+x^2}\,dx$

$= \int_0^{\infty}\int_0^{\infty} \cos x \times 2z \exp[-(1+x^2)z^2]\,dz\,dx$

$= \int_0^{\infty} 2ze^{-z^2}\left[\int_0^{\infty} e^{-x^2z^2} \cos x\,dy\right] dz$

$= \int_0^{\infty} 2ze^{-z^2} \frac{\sqrt{\pi}}{2z} \exp\left(-\frac{1}{4z^2}\right) dz$

$= \sqrt{\pi}\int_0^{\infty} \exp\left(-z^2 - \frac{1}{4z^2}\right) dz$

$= \sqrt{\pi}\int_0^{\infty} \exp\left(-a^2\left(z^2 + \frac{1}{4z^2}\right)\right) dz$

$= \sqrt{\pi}\frac{\sqrt{\pi}}{2} \exp\left(-2\frac{1}{2}\cdot 1\right) = \frac{\pi}{2e}$

$\Rightarrow I = 2I_1 = 2\frac{\pi}{2e} = \frac{\pi}{e}$

53. (B)

54. (A) $f(0, 2) \to R$

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 2x-1 & \text{if } x \text{ is irrational} \end{cases}$$

$$f'(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ 2 & \text{if } x \text{ in irrational} \end{cases}$$

$\Rightarrow$ $f(x)$ is differentiable only when $x = 1$

i.e., $f(x)$ is differentiable exactly at one point

55. (B)

56. (D) $f(x, y) = x^4 + y^2$

$\frac{\partial f}{\partial x} = 4x^3$

$\Rightarrow r = \frac{\partial^2 f}{\partial x^2} = 12x^2$

$\frac{\partial f}{\partial y} = 2y$

$\Rightarrow t = \frac{\partial^2 f}{\partial y^2} = 2$

$s = \frac{\partial^2 f}{\partial x \partial y} = 0$

$rt - s^2 = 0$

$\Rightarrow$ Not a local niminum at (0, 0).

$\Rightarrow g(x, y) = x^4 + y^2 - 10x^2y$

$\frac{\partial g}{\partial x} = 4x^3 - 20x$

$\Rightarrow r = \frac{\partial^2 g}{\partial x^2} = 12x^2 - 20$

$\frac{\partial g}{\partial y} = 2y - 10x^2$

$\Rightarrow t = \frac{\partial^2 g}{\partial y^2} = 2$

$s = \frac{\partial^2 g}{\partial x \partial y} = 0$

$\Rightarrow rt - s^2 = (12x^2 - 20)\,2 - 0 < 0$ at (0, 0) *i.e.*, max.

Hence neither f nor g has a local minimum.

57. (A)
$$\alpha_1 = \int_0^1 xy^2dx + \int_0^1 (x^2y+2x)dy$$
$$= \left[\frac{x^2y^2}{2}\right]_0^1 + \left[\frac{x^2y^2}{2} + 2xy\right]_0^1$$
$$= \frac{y^2}{2} + \frac{x^2}{2} + 2x$$
$$\alpha_2 = \int_{-1}^0 xy^2dx + \int_{-1}^0 (x^2y+2x)\,dy$$
$$= \left[\frac{x^2y^2}{2}\right]_{-1}^0 + \left[\frac{x^2y^2}{2} + 2xy\right]_{-1}^0$$
$$= \frac{y^2}{2} - \frac{x^2}{2} + 2x$$

58. (D)

59. (A) E is a non measurable subset of (0, 1)
$$f_1(x) = \begin{cases} 1/x & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases} \quad \ldots(1)$$
$$f_2(x) = \begin{cases} 0 & \text{if } x \in E \\ 1/x & \text{if } x \notin E \end{cases} \quad \ldots(2)$$
By these two equations f_1 is measurable but not f_2.

60. (C)
$$I_1 = \int_1^\infty \frac{dx}{(1+x^2)^{1/2}}$$
Here $\phi(x) = \dfrac{1}{(1+x^2)^{1/2}}$

Let $f(x) = \dfrac{1}{x}$

Now for $x \geq 1$, $\left|\dfrac{1}{(1+x^2)^{1/2}}\right| \leq \dfrac{1}{x}$, *i.e.* $|\phi(x) < f(x)$

But $\displaystyle\int_1^\infty \frac{dx}{x}$ is divergent.

Hence by comparison test it follows that $\displaystyle\int_1^\infty \frac{dx}{(1+x^2)^{1/2}}$ is convergent.

Similarly, $I_2 = \displaystyle\int_1^\infty \frac{dx}{(1+x^2)^{3/2}}$,

Here $\phi(x) = \dfrac{1}{(1+x^2)^{3/2}}$

Let $f(x) = \dfrac{1}{x^3}$

i.e., $x > 1$ and $\left|\dfrac{1}{(1+x^2)^{3/2}}\right| < \dfrac{1}{x^3}$

so $\displaystyle\int_1^\infty \frac{dx}{x^3}$ is convergent

Hence by comparison test, $\displaystyle\int_1^\infty \frac{dx}{(1+x^2)^{3/2}}$ is convergent.

61. (B) $f(x) = \begin{cases} x, & -1 \leq x \leq 0 \\ 0, & 0 \leq x \leq 1 \end{cases}$

and $f(x) = a_0P_0(x) + a_1P_1(x) + a_2P_2(x) + \ldots$

Now $f(x) = \displaystyle\sum_{r=0}^{\infty} a_rP_r(x)$

where, $a_r = \left(r + \dfrac{1}{2}\right)\displaystyle\int_{-1}^1 f(x)\,P_r(x)\,dx$
$$= \frac{2r+1}{2}\left[\int_{-1}^0 f(x)\,P_r(x)\,dx + \int_0^1 f(x)\,P_r(x)\,dx\right]$$
$$= \frac{2r+1}{2}\left[\int_{-1}^0 x\,P_r(x)\,dx + 0\right]$$
$$a_r = \frac{2r+1}{2}\int_{-1}^0 x\,P_r(x)\,dx$$
$$a_0 = \frac{1}{2}\int_{-1}^0 x\,(1)\,dx$$
$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^0 = -\frac{1}{4}$$
$$a_1 = \frac{3}{2}\int_{-1}^0 x\,(x)\,dx$$
$$= \frac{3}{2}\cdot\frac{1}{3}[x^3]_{-1}^0 = \frac{1}{2}$$
Hence, $a_0 = -\dfrac{1}{4},\ a_1 = \dfrac{1}{2}$

62. (A)

63. (C)
$$y = \alpha\sin x - \beta\cos x \quad \ldots(1)$$
$$z = \alpha\cos x + \beta\sin x \quad \ldots(2)$$
Differentiating eqn. (1) with respect to x, we get
$$\frac{dy}{dx} = \alpha\cos x + \beta\sin x = z$$
$$\frac{dy}{dx} - z = 0 \quad \ldots(3)$$
Diff. eqn. (2) with respect to x, we get
$$\frac{dy}{dx} = -\alpha\sin x + \beta\sin x = -y$$

$\Rightarrow \quad y + \frac{dy}{dx} = 0 \quad \ldots(4)$

Hence, equation (1) and (2) are general solution of differential equations (3) and (4).

64. (B) $2x(2+x)\frac{d^2y}{dx^2} + 2(3+x)\frac{dy}{dx} - xy = 0$

Let $\quad 2x(2+x) = v^2$

$\Rightarrow \quad 2x^2 + 4x - v^2 = 0$

$\Rightarrow \quad x = \frac{-4 \pm \sqrt{16+8v^2}}{4}$

$x = \frac{-2 \pm \sqrt{4+2v^2}}{2}$

65. (D) Every Non-Abelian group has nontrivial abelian subgroup and every nontrivial Abelian group has a cyclic subgroup. Both statements are true.

66. (C) S_{10} is the group of permutations on ten symbols $\{1, 2, \ldots, 10\}$. Then, the number of elements of S_{10} commuting with the element $\sigma = (1, 3, 5, 7, 9) = 5!\,5!$.

67. (A) The only nilopotent element $\to 0$

The only idempotent element $\to 1$

The only unit and idempotent element $\to 0, 1$

68. (C) z is the ring of integers under the usual addition and multiplication. Then every nontrivial ring homomorphim $f : z \to z$ must be onto but not one-one, *i.e.*, must be surjective but not injective.

69. (B) 70. (D)

71. (C) $\quad M = \begin{bmatrix} 2 & \alpha & -1 \\ \alpha & 2 & 1 \\ -1 & 1 & 4 \end{bmatrix}$

L is a lower triangular matrix

Then $\quad L = \begin{bmatrix} 2 & 0 & 0 \\ \alpha & 2 & 0 \\ -1 & 1 & 4 \end{bmatrix}$

and $\quad L^T = \begin{bmatrix} 2 & \alpha & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

$$\Rightarrow \quad LL^T = \begin{bmatrix} 2 & 0 & 0 \\ \alpha & 2 & 0 \\ -1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & \alpha & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2\alpha & -2 \\ 2\alpha & \alpha^2+4 & -\alpha+2 \\ -2 & -\alpha+2 & 18 \end{bmatrix}$$

But by a unique Cholesky decomposition of the form

$M = LL^T$

Then by this relation, the range of values of α is $-2 < \alpha < 3/2$.

72. (A) $\quad \frac{dy}{dx} = f(x)$

Step size $= h$, $y(0) = 0$, $x_0 = 0$

By Runge-Kutta of the fourth order

$$k_1 = hf(x_0, h_0) = h[f(x_0)] = hf(0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$= hf\left(\left(x_0 + \frac{1}{2}h\right) = hf\left(\frac{h}{2}\right)\right.$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= hf\left(\frac{h}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= hf(h)$$

Hence by the fourth order Runge-Kutta formulae

The solution at $x = h$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_1 + 2k_3 + k_4)$$

$$\Rightarrow \quad y(h) = 0 + \frac{1}{6}\left(hf(0) + 2hf\left(\frac{h}{2}\right) + 2hf\left(\frac{h}{2}\right) + hf(h)\right)$$

$$\Rightarrow \quad y(h) = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right]$$

73. (B) Given quadrature formula $\int_0^1 f(x)\,dx = \alpha f(0) + \beta f(x_1)$.

This is exact for polynomials of degree as high as possible, only for $\alpha = \frac{3}{4}$, $\beta = \frac{1}{4}$, $x_1 = \frac{2}{3}$.

74. (D) $x\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial x} = 0$

If $\quad r = \frac{\partial^2 u}{\partial x^2}, s = \frac{\partial^2 u}{\partial x \partial y}, t = \frac{\partial^2 u}{\partial y^2}$

$$p = \frac{\partial u}{\partial x},\ q = \frac{\partial u}{\partial y}$$

i.e., $xr + 2xys + yt + xq + yp = 0$

Compare with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$

$$R = x,\ S = 2xy,\ T = y$$

Now λ the quadratic

$$R\lambda^2 + S\lambda + T = 0$$

$$x\lambda^2 + 2xy\lambda + y = 0$$

$$\Rightarrow \quad \lambda = \frac{-2xy \pm \sqrt{4x^2y^2 - 4xy}}{2x}$$

$$\lambda = \frac{-xy \pm \sqrt{x^2y^2 - xy}}{x}$$

But $x^2y^2 - xy > 0$

$\Rightarrow$ $xy(xy - 1) > 0$

Because $\sqrt{x^2y^2 - xy}$ is always positive

Hence, function is hyperbolic in the region $x < 0, y < 0, xy > 1$.

75. (C)	76. (A)	77. (B)	78. (A)	79. (B)
80. (B)	81. (A)	82. (A)	83. (A)	84. (A)
85. (A)	86. (A)	87. (A)	88. (B)	89. (A)
90. (B)	91. (A)	92. (C)	93. (B)	94. (B)
95. (B)	96. (B)	97. (B)	98. (B)	

99. (C)

$$a^2 = e$$

$$a = a^{-1}$$

Let $a, b \in G$,

$ab = (ab)^{-1}a^{-1} = b^{-1}a^{-1} = ba \ \forall\, a, b \in G$

100. (C) Symmetric, eigen values $3 + 2\sqrt{2}, 3 - 2\sqrt{2}$ both are more than 0.

Hence $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

101. (C) 102. (B)

103. (D) $dA = |J|\,dS$, where $J = \dfrac{\partial(x, y)}{\partial(u, v)}$

$$J = \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = 5$$

$$\therefore \iint_R x\,dA = 5\iint_R (2u - 3v)\,dS$$

104. (C)

$$y_1'' + xy_1' + (1 - x^2)y_1 = \sin x$$

$$y_2'' + xy_2' + (1 - x^2)y_2 = \sin x$$

$$2y_1(x) - y_2 = y$$

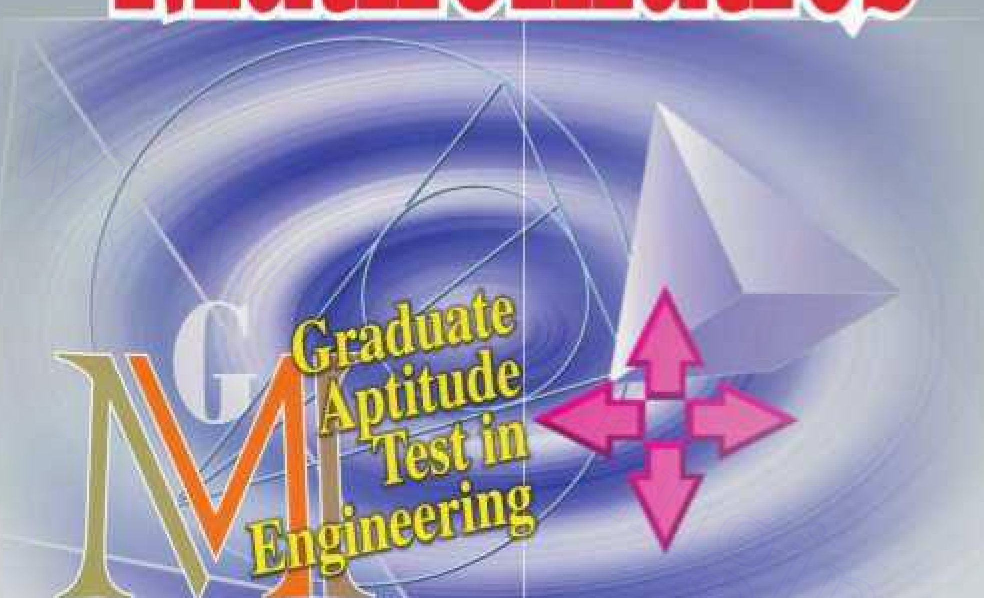
in LHS gives $\sin x$

Hence $2y_1(x) - y_2(x)$

105. (D)	106. (B)	107. (B)	108. (D)	109. (C)
110. (C)	111. (B)	112. (A)	113. (A)	114. (A)
115. (A)	116. (A)	117. (A)	118. (A)	119. (B)
120. (B)	121. (B)	122. (B)	123. (A)	124. (C)
125. (C)	126. (A)	127. (A)	128. (B)	129. (A)
130. (B)	131. (A)	132. (A)	133. (A)	134. (A)
135. (C)	136. (A)	137. (A)	138. (A)	139. (C)
140. (B)	141. (A)	142. (A)	143. (A)	144. (A)
145. (C)	146. (A)	147. (A)	148. (A)	149. (A)
150. (A)				

UPKAR'S

GATE Mathematics



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